EXPONENTIALS

In the number 2^3, read two to the third power, the three is called the exponent and the two is called the base.

$\mathbf{2}^3$ – exponent

base

The ***EXPONENT*** tells you how many times to write the base as a factor (how many times to multiply the base times itself).

Examples

1. $2^3 = 2 \times 2 \times 2 = 8$

2. $4^2 = 4 \times 4 = 16$

3. $6^4 = 6 \times 6 \times 6 \times 6 = 1{,}296$

4. $10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100{,}000$

Evaluate the following expressions.

1. 2^2	2. 3^2	3. 5^3
4. 6^3	5. 10^2	6. 3^5
7. 2^4	8. 5^2	9. 9^2

10. Can you find an easy way of finding the value of an exponential with base 10? (try a couple and look for a pattern.

So far, so good.

Normally, when we introduce new number sets, like fractions or decimals, etc., we show you how to work with the four basic operations of addition, subtraction, multiplication and division.

Let's start with multiplying exponentials.

Multiplying Exponentials

When we multiply exponentials with the same base, we see a pattern develop. Let's take a look.

***Example* 1** $4^2 \times 4^3$

$4^2 \times 4^3$ means $(4 \times 4) \times (4 \times 4 \times 4)$, writing the answer, we see we are using 4 as a factor 5 times. That is we are multiplying 4 times itself 5 times.

$$\begin{aligned} 4^2 \quad &\times \quad 4^3 \\ (4 \times 4) \times (4 \times 4 \times 4) &= 4 \times 4 \times 4 \times 4 \times 4 \\ &= 4^5 \end{aligned}$$

Example 2 $7^5 \times 7^3$

$7^5 \times 7^3$ means $(7 \times 7 \times 7 \times 7 \times 7) \times (7 \times 7 \times 7)$, writing the answer, we see we are using 7 as a factor 8 times.

$$\begin{aligned} 7^5 \qquad &\times \qquad 7^3 \\ (7 \times 7 \times 7 \times 7 \times 7) \times (7 \times 7 \times 7) &= 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \\ &= 7^8 \end{aligned}$$

What if I asked you to simplify the expression $3^{17} \times 3^8$, would you want to write all those 3's? No, then let's work smart.

If we were to look at the first two examples, you might see a pattern developing. Can you see it? Let's look

We said $4^2 \times 4^3 = 4^5$. We also said $7^5 \times 7^3 = 7^8$

Do you see anything interesting? Hopefully, you might notice when we multiply exponentials with the **same** base, a way we can find the answer is by adding the exponents. Kind of neat, huh?

If that works, then $3^{17} \times 3^8 = 3^{25}$. Yes, that's a little better than writing out all those three's.

Let's see another, you're saying $5^4 \times 5^6 = 5^{4+6}$
$= 5^{10}$

Seeing that does seem to work, we make a rule that will allow us to evaluate those quickly.

Rule 1. ***When you multiply exponentials with the same base, you add the exponents.***

Example Simplify in exponential form $8^4 \times 8^5$

$$8^4 \times 8^5 = 8^{4+5}$$
$$= 8^9$$

I know what you are thinking, this is easy, you are hoping to do dozens of these for homework, right?

Well, try some of them. With a little practice, you can almost simply all the problems in your head. But, since we want you to remember this rule, do some of them out as we did the last example so you'll be able to recall the rule later.

Simply In Exponential Notation

1) $5^2 \cdot 5^3$

2) $6^4 \cdot 6$

3) $7^5 \cdot 7^3$

4) $4^3 \cdot 4^2$

5) $10^8 \cdot 10^7$

6) $8^3 \cdot 8^2 \cdot 8^5$

7) $7^3 \cdot 7^4$

8) $3^8 \cdot 3^9$

9) $5^4 \cdot 5^2 \cdot 5^7$

10) $2^6 \cdot 2^5 \cdot 2^6$

11) $8^2 \cdot 8 \cdot 8^4$

12) $3^6 \cdot 3^{10}$

13) $7^{11} \cdot 7^4$

14) $8 \cdot 8$

15) $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

16) $27 \cdot 3^2$

17) $2^3 \cdot 4 \cdot 8$

18) $125 \cdot 5^2$

19) $16 \cdot 4^3$

20) $4^2 \cdot 3^2 \quad 5^3 \cdot 4^7$

What happens if you don't have the same base? Let's look problem 18. One way to approach that problem is say it does not fit the rule, the bases are not the same, so I can't simplify it. That sounds okay. But, if I played with this a little longer, I might realize I could write the number 125 as 5^3.

Rewriting the problem, I now have 5^3 x 5^2. That's a problem I can do! So, if you are asked to simplify exponentials that don't have the same base, you might try to change the base so you can use the rule.

Dividing Exponentials

If we divide exponentials with the same base, we might see another pattern develop.

***Example* 1** Simplify in exponential notation $\frac{4^5}{4^3}$

$$\frac{4^5}{4^3} = \frac{\cancel{4} \times \cancel{4} \times \cancel{4} \times 4 \times 4}{\cancel{4} \times \cancel{4} \times \cancel{4}}$$, dividing out the 4's, we have 4×4 or 4^2

Example 2 Simplify in exponential notation $\frac{7^6}{7^5}$

$$\frac{7^6}{7^5} = \frac{\cancel{7} \times \cancel{7} \times \cancel{7} \times \cancel{7} \times \cancel{7} \times 7}{\cancel{7} \times \cancel{7} \times \cancel{7} \times \cancel{7} \times \cancel{7}}$$, dividing out the 7's, we have 7 or 7^1

If we were to continue to do some more problems, again a pattern would seem to appear. Try a couple on your own, do you see a pattern?

What you might be noticing is when you divide exponentials with the same base, we can find the answer quickly by subtracting the exponents. That's good news, in math shortcuts are often turned into things we refer to as rules, formulas, algorithms and theorems.

Rule 2. ***When you divide exponentials with the same base, you subtract the exponents.***

Example $\frac{9^5}{9^3}$

$$\frac{9^5}{9^3} = 9^{5-3}$$

$$= 9^2$$

Yes, this is really a blast. You're hoping to do more work with exponentials, aren't you? Well, I've got some good news – we are.

Simplify In Exponential Notation

1) $\frac{4^8}{4^5}$ 2) $\frac{5^3}{5}$ 3) $\frac{10^8}{10^6}$

4) $\frac{3^5}{3^4}$ 5) $\frac{7^3}{7}$ 6) $\frac{9^8}{9^5}$

7) $\frac{12^6}{12^3}$ 8) $8^7 \div 8^3$ 9) $9^8 \div 9^3$

10) $10^7 \div 10^2$ 11) $20^6 \div 20^4$ 12) $15^{12} \div 15^9$

13) $\frac{6^8}{6^5}$ 14) $\frac{3^9}{3^7}$ 15) $3^{12} \div 3^7$

Let's say we want to divide 8^3 by 8^3, what should our answer be?

Zero Exponent

Let's look at that last problem. Using the rule we just developed, we'd have

$$\frac{8^3}{8^3} = 8^{3-3}$$

Oh wow, we have a slight problem. If we use Rule 2, we notice that we get 8 to the zero power; 8^0.

That does not make sense. Remember how we defined an exponent? It tells us how many times to write the base as a factor. How can you write the base zero times? That answer does not seem to make sense, which means the second rule we just developed for dividing does not seem to work.

If we continued to look at samples like that one, for instance;

$$\frac{4^5}{4^5} = 4^{5-5} \text{ or } 4^0$$

We'd also know that anytime we divide a number by itself, we should get one. $5 \div 5 = 1$;

$12 \div 12 = 1$. That would lead us to believe that $4^5 \div 4^5$ should also be equal to one. As would $8^3 \div 8^3$ be equal to one.

Let's see what we have. We are saying $4^5 \div 4^5 = 4^0$ and we are also saying $4^5 \div 4^5 = 1$. Shouldn't that mean they should equal to each other. Looking at $8^3 \div 8^3$ the same way things happens.

Using substitution in both of these examples leads us to believe that $4^0 = 1$ and $8^0 = 1$.

Oh yes, you can feel the excitement. We are seeing another pattern develop.

Rule 3. Any number to the zero power, except zero, equals 1.

Why the exception? Because you would never be able to divide by zero in the first place.

Examples

$12^0 = 1$ $\quad (1/2)^0 = 1$ $\quad 0^0 \rightarrow$ not defined

Evaluate The Following Expressions

	A	B	C
1)	$6+2^0$	8^0+12^0	13^1+5^0
2)	3^0+2^0	8^2+3^0	12^2-8^0
3)	3^2-5^0	7^0	128^0
4)	$(4+5)^0$	$(6+2)^0$	$(8-2)^0$
5)	$(4/3)^0$	$(1/2)^0$	$(6\ 2/3)^0$

We have developed three rules that will allow you to evaluate exponential expressions quickly. Now, instead of writing the rules as we did in word sentences, it might be helpful to learn how to write those same rules using algebraic notation.

For example, in ***Rule 1,*** we said: If you multiply ***exponentials*** with the same base, you add the exponents. Another way to say the same thing is $A^m \times A^n = A^{m+n}$. Notice the base is the same and we added the exponents m and n.

Looking at ***Rule 3,*** rather than writing; Any number to the zero power, except zero, equals one. We could write $A^0 = 1, A \neq 0$ Not letting $A = 0$ takes care of the exception.

Seeing this, how do you think we might rewrite ***Rule 2*** using Algebraic notation?
Rule 2 states: When you divide exponentials with the same base, you subtract the exponents. Remember, the bases have to be the same.

Hopefully you got $A^m \div A^n = A^{m-n}$

Learning how to use mathematical notation is important. It is supposed to make our lives easier. Clearly, using the notation for exponentials will save us some writing time. ***IMPORTANT* –** the rules we learned for multiplying and dividing exponentials only work when you have the **SAME** bases.

In other words $5^3 \times 7^2$ can't be simplified in exponential notation because they don't have the same base and I was not able to make them the same. However, I could evaluate it by multiplying 123×49.

Rule 1. $A^m \times A^n = A^{m+n}$

Rule 2. $A^m \div A^n = A^{m-n}$

Rule 3. $A^0 = 1, A \neq 0$

Evaluate In Exponential Notation

1)

$$\frac{4^2 \cdot 4^5 \cdot 4^3}{4^2 \cdot 4^5}$$

2)

$$\frac{7^3 \cdot 7^5}{7^2 \cdot 7^3}$$

3)

$$\frac{3^2 \cdot 5^3 \cdot 3^5 \cdot 5^4}{3^4 \cdot 5^2}$$

4)

$$\frac{6^7 \cdot 5^8 \cdot 6^4}{6^8 \cdot 5^3}$$

5)

$$\frac{5^7 \cdot 10^3 \cdot 5^6 \cdot 10^2}{5 \cdot 10^2 \cdot 5^2}$$

6)

$$\frac{6^4 \cdot 7^2 \cdot 7^3}{7^4 \cdot 6^3}$$

7)

$$\frac{12^7 \cdot 11^3 \cdot 12^3}{11^2}$$

8)

$$\frac{5^2 \cdot 6 \cdot 5^3 \cdot 6^4}{5^5 \cdot 6^3}$$

9)

$$\frac{8^2 \cdot 7^3 \cdot 8^4 \cdot 7^2}{8^3 \cdot 7^4}$$

10)

$$\frac{8^2 \cdot 3^4 \cdot 8^3}{8^4 \cdot 3^2}$$

Negative Exponents

Let's look at another division problem; $\frac{5^2}{5^5}$. Using the last rule since we are dividing exponentials with the same base, we should subtract the exponents.

Example Simplify in exponential notation $\frac{5^2}{5^5}$

$$\frac{5^2}{5^5} = 5^{2-5} \text{ which equals } 5^{-3}$$

Oh, oh. Having an exponent of -3 does not make sense. Using the definition of exponent, how can we write the base a negative three times? We either write it or don't.

Since we are running into difficulty, let's do the problem the long way (by definition) and see what's happening.

$$\frac{5^2}{5^5} = \frac{\cancel{5} \times \cancel{5}}{\cancel{5} x \cancel{5} \times 5 \times 5 \times 5}$$, dividing out the 5's, we have $\frac{1}{5 \times 5 \times 5} = \frac{1}{5^3}$

Another example might be in order.

Example: Simplify in exponential notation $\frac{7^4}{7^6}$

$\frac{7^4}{7^6} = 7^{4-6}$ which equals 7^{-2}. Doing it by the definition of exponent, we get,

$\frac{7^4}{7^6} = \frac{\cancel{7} \times \cancel{7} \times \cancel{7} \times \cancel{7}}{\cancel{7} \times \cancel{7} \times \cancel{7} \times \cancel{7} \times 7 \times 7}$, dividing out the 7's, we have $\frac{1}{7 \times 7} = \frac{1}{7^2}$

Looking at those answers, we see that 5^{-3} and $\frac{1}{5^3}$ are the answers to the first problem.

By the same token, we see 7^{-2} and $\frac{1}{7^2}$ are answers to the second example. Uh-huh, another pattern you're thinking. You've got it, another pattern, another rule.

Rule 4. $A^{-n} = \frac{1}{A^n}$

Notice a negative exponent does not mean the number is negative.

Write the answer using a negative exponent

1) $\frac{5^2}{5^6}$ 2) $\frac{3^4}{3^9}$ 3) $\frac{7^3}{7^5}$

4) $\frac{10^3}{10^7}$ 5) $\frac{8^4}{8^5}$ 6) $\frac{7^3}{7^{10}}$

7) $\frac{2^5}{2^8}$ 8) $\frac{4}{4^3}$ 9) $\frac{\times^4}{\times^7}$

Write the answer in fractional form using a positive exponent.

10) 4^{-2} 11) 5^{-3} 12) 10^{-1}

13) 10^{-2} 14) 3^{-4} 15) 7^{-5}

16) 10^{-3} 17) 2^{-3} 18) 6^{-5}

Raising a Power to a Power

Let's continue playing. I bet we can find more patterns that will allow us to simplify exponential expressions quickly.

Example Simplify in exponential notation $\left(5^2\right)^3$

The exponent 3 tells me how many times to write 5^2 as a factor.

$\left(5^2\right)^3 = 5^2 \times 5^2 \times 5^2$

Using Rule 1, when we multiply exponentials with the same base, we add the exponents,

$$5^2 \text{ x } 5^2 \text{ x } 5^2 = 5^6$$

we find see that $\left(5^2\right)^3 = 5^6$

Piece of cake, don't you think? Let's look at another one.

Example Simplify in exponential notation $\left(4^3\right)^5$

$$\left(4^3\right)^5 = 4^3 \times 4^3 \times 4^3 \times 4^3 \times 4^3$$
$$= 4^{15}$$

Looking at those two examples, how do you think $\left(2^{25}\right)^4$ can be written? If you look at the problem and the answer, the pattern almost jumps out at you. If you are thinking 2^{100}, then we have identified another pattern. Say Yes! to mathematics, this is really fun.

When you raise a power to a power, the pattern suggests we multiply the exponents. That leads us to another shortcut, another rule.

Rule 5. $\left(A^n\right)^m = A^{nxm}$

Example Simplify in exponential notation. $\left(6^4\right)^3$

Using **Rule 5** $\left(6^4\right)^3 = 6^{4x3}$
$= 6^{12}$

Simplify The Following Expression In Exponential Notation

1) $\left(5^2\right)^3$

2) $\left(4^3\right)^5$

3) $\left(7^4\right)^2$

4) $\left(10^2\right)4$

5) $\left(8^5\right)^2$

6) $\left(2^{10}\right)^8$

7) $\left(3^4\right)5$

8) $\left(3^5\right)4$

9) $\left(3^{-2}\right)5$

10) $\left(5^3\right)-4$

We have established 5 rules by identifying patterns. At first glance, if you didn't see the pattern develop, the rules (shortcuts) would not make sense. But since we saw those patterns develop, we are feeling pretty good about them. Aren't we?

Now because we also saw how much easier it was to write the rules using mathematical notation than in long hand, we are really starting to get into this.

Rule 1. $A^m \times A^n = A^{m+n}$

Rule 2. $A^m \div A^n = A^{m-n}$

Rule 3. $A^0 = 1, A \neq 0$

Rule 4. $A^{-n} = 1 / A^n$

Rule 5. $\left(A^n\right)^m = A^{n\times m}$

Next, let's see what happens when we have longer problems that require us to use more than one rule per problem. Excited?

Let's use a few of these rules in one problem.

Example Simplify in exponential notation $\dfrac{3^4 \times 4^3 \times 5^2 \times 3^6 \times 5^3}{3^7 \times 4^5 \times 5^5}$

Simplifying the numerator by using Rule 1, we get $3^{10} \times 4^3 \times 5^5$

$$\frac{3^4 \times 5^2 \times 3^6 \times 5^3}{3^7 \times 4^5 \times 5^5} = \frac{3^{10} \times 4^3 \times 5^5}{3^7 \times 4^5 \times 5^5}$$ Continuing using Rule 2

$$= 3^3 \times 4^{-2} \times 5^0$$
$$= 3^3 \times \frac{1}{4^2} \times 1$$
$$= \frac{3^3}{4^2}$$

When you simplify exponentials, you can not have a zero or negative exponent.

Okay, we'll let you try some of these combined problems on your own. We won't make you beg for more.

WRITE IN STANDARD NOTATION

1) $2^3 =$

2) $4^2 =$

3) $1^8 =$

4) $8^0 =$

5) $7^0 + 7^1 =$

6) $13^1 \times 52^0 =$

WRITE IN EXPONENTIAL NOTATION

7) $5^2 \ . \ 5^4 =$

8) $6^3 \ . \ 6^7 =$

9) $2^5 \ . \ 2^7 =$

10) $\frac{8^4}{8^3} =$

11) $10^7 + 10^5 =$

12) $\frac{7^3}{7} =$

13) $\frac{3^4 \ . \ 3^6}{3^8} =$

14) $\frac{4^7 \ . \ 4^5}{4^3} =$

15) $\frac{8^4 \ . \ 8^5}{8^6 \ . \ 8^2} =$

16) $3^2 \ .4^5 \ .3^7 =$

17) $8^2 \ .4^3 \ .8^4 \ .4^2 =$

18) $\frac{6^7 \ . \ 8^2 \ . \ 6^4}{8 \ . \ 6^8} =$

19) $3 \ . \ 3 \ . \ 3 \ . \ 3 \ . \ 3 =$

20) $10{,}000 =$

Evaluate In Exponential Notation

1)

$$\frac{4^2 \cdot 4^5 \cdot 4^3}{4^2 . 4^5}$$

2)

$$\frac{7^3 \cdot 7^5}{7^2 \cdot 7^3}$$

3)

$$\frac{3^2 \cdot 5^3 \cdot 3^5 \cdot 5^4}{3^4 \cdot 5^2}$$

4)

$$\frac{6^7 \cdot 5^8 \cdot 6^4}{6^8 \cdot 5^3}$$

5)

$$\frac{5^7 \cdot 10^3 \cdot 5^6 \cdot 10^2}{5 \cdot 10^2 \cdot 5^2}$$

6)

$$\frac{6^4 \cdot 7^2 \cdot 7^3}{7^4 \cdot 6^3}$$

7)

$$\frac{12^7 \cdot 11^3 \cdot 12^3}{11^2}$$

8)

$$\frac{5^2 \cdot 6 \cdot 5^3 \cdot 6^4}{5^5 \cdot 6^3}$$

9)

$$\frac{8^2 \cdot 7^3 \cdot 8^4 \cdot 7^2}{8^3 \cdot 7^4}$$

10)

$$\frac{8^2 \cdot 3^4 \cdot 8^3}{8^4 \cdot 3^2}$$

Simplify

1)

$$\frac{5^2 \cdot 3^4 \cdot 5^6 \cdot 3^8}{5^6 \cdot 3^{10}}$$

2)

$$\frac{4^5 \cdot 8^6 \cdot 4^3 \cdot 8^3}{4^5 \cdot 8^4}$$

3)

$$\frac{4^5 \cdot 10^6 \cdot 4^2 \cdot 10}{4^5 \cdot 10^4}$$

4)

$$\frac{5^3 \cdot 5^4}{5^1 \cdot 5^2}$$

5)

$$\frac{4^2 \cdot 5^3 \cdot 4^2 \cdot 5^5}{4^3 \cdot 5^7}$$

6)

$$\frac{9^2 \cdot 9^8 \cdot 10^7}{10^6 \cdot 9^4}$$

7)

$$\frac{6^2 \cdot 5^3 \cdot 6^4 \cdot 5^4}{5^2 \cdot 6^5 \cdot 5}$$

8)

$$\frac{2^4 \cdot 8^6 \cdot 2^5 \cdot 8^4}{8^9}$$

9)

$$\frac{10^7 \cdot 2^3 \cdot 10^8}{10^2 \cdot 2 \cdot 10^3}$$

10)

$$\frac{5^2 \cdot 4^8 \cdot 5^3 \cdot 4}{5^4 \cdot 4^6}$$

Evaluate In Exponential Notation

1)

$9^3 \cdot 9^2$

2)

$8^4 \cdot 8^5$

3)

$7^5 \times 7^6$

4)

$9\ \ 8/9^6$

5)

$\frac{10^7}{10^4}$

6)

$3^4 + 3^3$

7)

$\frac{8^2 \cdot 8^4}{8^3}$

8)

$\frac{10^6}{10^2 \cdot 10^3}$

9)

$\frac{8^2 \cdot 8^4}{8^6}$

10)

$10^6 \div 10^4$

(11)

$8^3 \cdot 8^5$

12)

$\frac{13^8}{13^2}$

13)

$\frac{7^6 \cdot 2^5}{2^3}$

14)

$\frac{8^2 \cdot 2^6 \cdot 8^3 \cdot 2^4}{8^4 \cdot 2^8}$

15)

$\frac{10^2 \cdot 3^4}{10 \cdot 3}$

16)

$\frac{4^2 \cdot 5^2}{4 \cdot 5}$

17)

$\left(3^2\right)^3$

18)

$\left(4^3\right)^2$