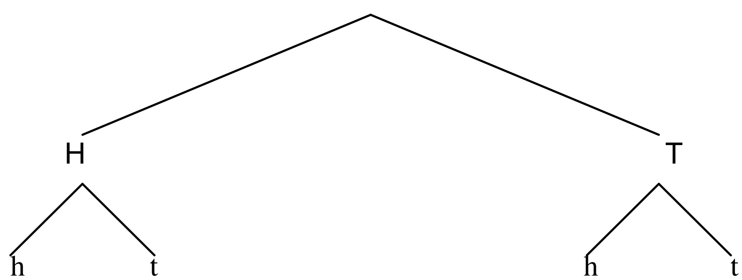
COUNTING METHODS

From our preliminary work in probability, we often found ourselves wondering how many different scenarios there were in a given situation. In the beginning of that chapter, we merely tried to list all the possible outcomes, hoping we didn't miss any. When we determined that was not good enough, we began to use a tree diagram to lend some order to the listing. While that worked out well for smaller sample spaces, we quickly saw its limitations when there were a great number of outcomes. For that reason, we will look at some counting methods that should make our work a lot easier

Methods Used for Counting

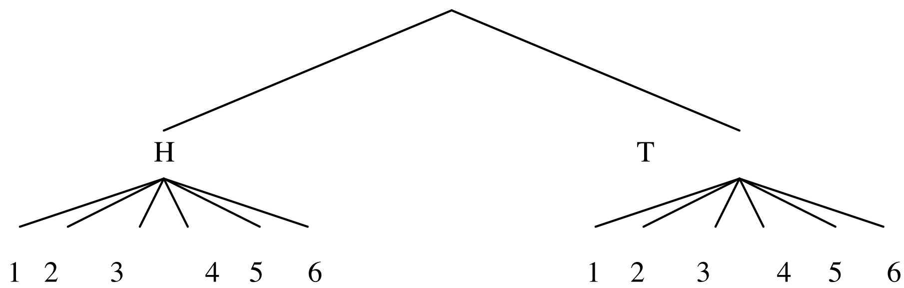
1. Listing
2. Cartesian product
3. Tree Diagram
4. Fundamental Counting Principle
5. Permutation
6. Combination

If we looked at the number of outcomes in a sample space being described using a tree diagram, we might notice a pattern that would suggest a counting method. For instance, if I drew the tree diagram for tossing 2 coins, I would see there would be four possible outcomes – Hh, Ht, Th, and Tt.



With a little investigation, I might also notice there were two possible outcomes throwing the first coin and two possible outcomes tossing the second.

If I looked at another example, say throwing a coin and rolling a die. Drawing a tree diagram I would quickly see there are twelve possible outcomes – H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, and T6. Again, you might notice there are two possible outcomes when tossing the coin and six outcomes when rolling the die.



By doing a few more of these problems, I might begin to see a pattern that would suggest a way of determining the total number of outcomes without listing and without using a tree diagram.

In the first example, we tossed two coins and discovered there were four possible outcomes. Each stage of the experiment having two outcomes, we notice that $2x2=4$.

In the second example, tossing a coin and rolling a die, we discovered using a tree diagram there were twelve possible outcomes. By looking at each stage of the experiment, we see that there are two outcomes possible for stage one, and rolling the die resulted in six possible outcomes. Notice that by multiplying 2 X 6 we end up with 12 possible outcomes. Getting excited?

That excitement will lead us to the Fundamental Counting Principle.

Fundamental Counting Principle – In general, if there are ***m*** choices for doing one thing, and after that occurs, there are ***n*** choices for doing another, then together they can be done in ***m x n*** ways.

This counting principle will allow me to determine how many different outcomes exist quickly in my head that could be verified using tree diagrams.

In the coin tossing example, since there were 2 things that could happen on the first toss, followed by two things that could happen on the second toss, the Fundamental Counting Principle states that there will be 2x2 or 4 possible outcomes.

Example
Suppose Jennifer has three blouses, two pairs of slacks, and four pairs of shoes. Assuming no matter what she wears, they all match, how many outfits does she have altogether?

She has three choices for a blouse, two choices for her slacks, and four choices for her shows. Using the Fundamental Counting Principle, she has 3 x 2 x 4 or 24 different outfits.

Example
Abe, Ben, and Carl are running a race, in how many ways can they finish?

I could draw a tree diagram to see all the possible outcomes or I could use the Fundamental Counting Principle. There are three ways I could choose the winner, and after that occurs, there are two ways to pick second place, and one way to pick the third place finisher. Therefore there are 3 x 2 x 1 or 6 different way these three boys could finish the race.

Before we go on, we need to learn a little notation.

Factorials

In the last example, we saw that we had to multiply 3 x 2 x 1. As it turns out, we will have a number of opportunities to multiply numbers like 5 x 4 x 3 x 2 x 1. A difficulty with this is that if I have to multiply 30 x 29 x 28 x …x 3 x 2 x 1, that is a lot of writing and a lot of space.

So we are going to abbreviate products that start at one number and work their way back to one by using an exclamation point (!). In math, however, that won't mean the number is excited. And we won't call it an exclamation point, we'll call it a factorial. So 5! Is read as five factorial.

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$3! = 3 \times 2 \times 1 = 6$$

$$2! = 2 \times 1 = 2$$

$$1! = 1$$

Later on, we will find it to our advantage to define 0! = 1.

Now that we have learned factorials, we can more efficiently determine the size of sample spaces.

One of the best strategies to use when trying to find out how many different outcomes are in a sample space is using the Fundamental Counting Principle. Find out how many ways the first event can occur, then multiply that by the number of ways events can occur in subsequent events.

Example
How many outcomes are there if two dice were thrown?

There are 6 outcomes on the first die, and after that occurs, there are 6 outcomes on the second die.

Using the Fundamental Counting Principle, there are 6 x 6 or 36 possible outcomes.

Example
How many different ways can the letters in the word "ACT" be arranged?

There are six different ways to write those letters. We can see that in the following list.

ACT
CTA
TAC
CAT
ATC
TCA

We could have determined there were six by using the Fundamental Counting Principle, three ways to pick the first letter, 2 ways to pick the second, then one way to pick the third. That could also have been described using 3!

Let's look at another example.

Example
If a different person must be selected for each position, in how many ways can we choose the president, vice president, and secretary from a group of seven members if the first person chosen is the president, the second the vice president, and the third is the secretary?

We have a total of 7 people taken three at a time. Using the Fundamental Counting Principle, the first person can be chosen 7 ways, the next 6, and the third 5, we have
7 x 6 x 5 or 210 ways of choosing the officers.

Another way of doing the same problem is by developing a formula. Let's see what that might look like and define it.

> ***Permutation*** is an arrangement of objects in which the order matters – without repetition.

Order typically matters when there is position or awards. Like first place, second place, or when someone is named president, or vice president. Various notations are used to represent the number of permutations of a set of **n** objects taken **r** at a time $n\mathrm{Pr}$ and $P(n,r)$ are the most popular.

In the last example, we would use the notation ${}_7P_3$ to represent picking three people out of the seven. We could then use the Fundamental Counting principle to determine the number of permutations.

$$
\begin{aligned}
{}_7P_3 &= 7 x 6 x 5 \\
&= 210
\end{aligned}
$$

Generalizing this algebraically, we could develop the following formula for a permutation.

$$n\mathrm{Pr} = \frac{n!}{(n-r)!}$$

Using that formula for the last example would give us

$$
\begin{aligned}
{}_7P_3 &= \frac{7!}{(7-3)!} \\
&= \frac{7!}{4!} \\
&= \frac{7x6x5x\cancel{4}x\cancel{3}x\cancel{2}x\cancel{1}}{\cancel{4}x\cancel{3}x\cancel{2}x\cancel{1}} \\
&= 7x6x5 \text{ or } 210
\end{aligned}
$$

Example
Three friends buy an all day pass to ride a two seated bike, if only two of them can ride at a time, how many possible seating arrangements are there?

Since order matters (sitting in front or back), using the Fundamental Counting Principle, the number of permutations of 3 friends taken 2 at a time is 3 x 2 or 6.

The front seat can be chosen three ways, and after that occurs, the person in the back seat can be chosen two ways. We could have used the formula.

$$
{}_3P_2 = \frac{3!}{(3-2)!}
$$

Example
There are 5 runners in a race. How many different permutations are possible for the places in which the runners finish?

By formula, we have a permutation of 5 runners being taken 5 at a time.

$$
\begin{aligned}
{}_5P_5 &= \frac{5!}{(5-5)!} \\
&= \frac{5!}{0!} \\
&= 5! \text{ Or } 5 \times 4 \times 3 \times 2 \times 1
\end{aligned}
$$

Notice, we could have just as easily used the Fundamental Counting Principle to solve this problem.

Using a permutation or the Fundamental Counting Principle, order matters. A permutation does not allow repetition. For instance, in finding the number of arrangements of license plates, the digits can be re-used. In other words, someone might have the license plate

333 333. To determine the possible number of license plates, I could not use the permutation formula because of the repetitions, I would have to use the Fundamental Counting Principle.

Since there are 10 ways to chose each digit on the license plate, the number of plates would be determined by – 10 x 10 x 10 x 10 x 10 x 10 or 1,000,000

Combination is an arrangement of objects in which the order does not matter – without repetition.

This is different from a permutation because the order does not matter. If you change the order, you don't change the group, you do not make a new combination.

So, a dime, nickel, and penny is the same combination of coins as a penny, dime, and nickel.

Example
Bob has four golf shirts. He wants to take two of them on his golf outing. How many different combinations of two shirts can he take?

Using the Fundamental Counting Principle, Bob could choose his first shirt four ways, his second shirt three ways – 12 ways.

But, hold on a minute. Let's say those shirts each had a different color, by using the Fundamental Counting Principle, that would suggest taking picking the blue shirt, then the yellow is different from picking the yellow first, then the blue.

We don't want that to happen since the order does not matter. To find the number of combinations, in other words, eliminating the order of the two shirts, we would divide the 12 permutations by 2! Or 2 x 1

There would be six different shirt combinations Bob could take on his outing.

In essence, a combination is nothing more than a permutation that is being divided by the different orderings of that permutation.

The notation we will use will follow that of a permutation, either nCr or $C(n,r)$

$$nCr = \frac{n\Pr}{r!}$$

Simplifying that algebraically, we have

$$nCr = \frac{n!}{(n-r)!r!}$$

In a permutation, A,B is different from B,A because order is important. In a combination, you would either have A,B or B,A – not both, they are the same grouping.

Example
From among 12 students trying out for the basketball team, how many ways can 7 students be selected?

Does the order matter? Is this a permutation or combination? Well, if you were going out for the team and a list was printed, would it matter if you were listed first or last? All you would care about is that your name is on the list. The order is not important, therefore this would be a combination problem of 12 students take 7 at a time.

$$_{12}C_7 = \frac{12!}{(12-7)!7!}$$

$$= \frac{12!}{5!7!}$$

$$= \frac{12\ 11\ 10\ 9\ 8\ \cancel{7}\ \cancel{6}\ \cancel{5}\ \cancel{4}\ \cancel{3}\ \cancel{2}\ \cancel{1}}{5\ 4\ 3\ 2\ 1\ \cancel{7}\ \cancel{6}\ \cancel{5}\ \cancel{3}\ \cancel{2}\ \cancel{1}}$$

$$= \frac{12\ 11\ 10\ 9\ 8}{5\ 4\ 3\ 2\ 1}$$

$$= 792$$

There would be 792 different teams that can be chosen.

Example
Ted has 6 employees, three of them must be on duty during the night shift, how many ways can he choose who will work?

Does order matter? Since it does not matter, this problem can be solved by using the Fundamental Counting Principle, then dividing out the same grouping or you could use the formula for combination of 6 people being taken three at a time.

There are 6 ways to choose the first person, 5 ways to choose the second, and 4 ways to choose the third, that's 120 permutations. Each group of three employees can be ordered 3! Or 6 ways.

So, we divide the number of permutations by the different ordering of the three employees.

$$\frac{6x5x4}{3!} = \frac{120}{6} \rightarrow 20 \text{ ways to pick the shifts}$$

By formula, we'd have $_6C_3 = \frac{6!}{(6-3)!x3!}$. Working that out, there would be 20 ways.

Doing these problems by hand can be very distracting, you would be able to concentrate on the problem more if you had a calculator that had permutations and combination on it. That way, when you had $_{12}C_7$, all you would do is plug those numbers in, press the appropriate buttons, and wala, you would have gotten 792. Don't you just love technology?

In summary, when order matters and there is no repetition, use a permutation. If order matters and there is repetition, then use the Fundamental Counting Principle. If order does not matter, use a combination.

In just about all cases, you can use the Fundamental Counting Principle to determine the size of the same space,

The formulas for permutation and combination just allow us to compute the answers quickly. However, if you read a problem and have trouble determining if it's a permutation or combination, then do it by the Fundamental Counting Principle.

Counting Problems

1. How many five digit numbers can be obtained given that the ten thousands place cannot be zero?
2. Ten cars are in a race. How many ways can we have first, second, and third place?
3. How many ways can a True-False test be answered if there are 4 questions? 6 questions?, n questions?
4. How many distinguishable words are in the word "grammar"?
5. In how many ways can six items be chosen from nine things?
6. License plates consist of two letters and five digits. How many different license plates can be made?
7. A shipment is received containing ten items, two of which are defective. Determine how many ways two items can be chosen so that a defective item is not chosen? One defective item is selected?
8. A man has a nickel, dime, quarter, and a half dollar. How many ways can the waitress be tipped if he gives her two coins? At least two coins?
9. There are five rotten apples in a crate of 25 apples: How many samples of three apples can be chosen? How many samples of three could be chosen if all three are rotten? If one is rotten and two are good?
10. How many two card hands can be dealt from a deck of 52 cards?
11. A city council is composed of 5 liberals and 4 conservatives. A delegation of three is to be selected to attend a convention. How many delegations are possible? How many could have all liberals? How many could have 2 liberals and one conservative?

12. How many ways can a basketball team of 5 players be selected from a squad of 12 players?
13. A man's wardrobe consists of 5 sport coats, 3 dress slacks, and 2 pairs of shoes. Assuming they all match, in how many ways can he select an outfit?
14. The school board consists of seven members, if the first person selected is the president, the second is the vice president and the third is the treasurer, how many ways can the officers of the board be chosen?
15. Ten people reached the finals of the contest, if the first one of the finalists chosen receives, $500, the second person receives $200, and the third person selected receives $100, how many ways can the prize money be distributed?