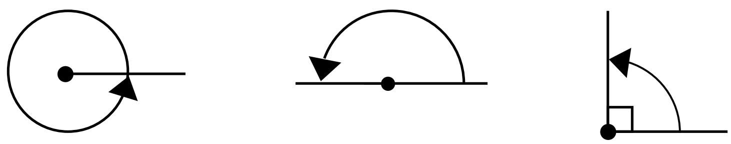
Naming Angles

What's the secret for doing well in geometry? Knowing all the angles.

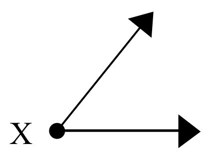
An angle can be seen as a rotation of a line about a fixed point. In other words, if I were mark a point on a paper, then rotate a pencil around that point, I would be forming angles.

One complete rotation measures 360º. Half a rotation would then measure 180º. A quarter rotation would measure 90º.



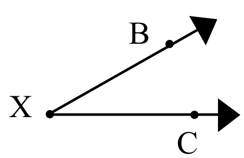
Let's use a more formal definition. An **angle** is the union of two rays with a common end point.

The common endpoint is called the **vertex**. Angles can be named by the vertex - X.



That angle is called angle X, written mathematically as $\angle X$.

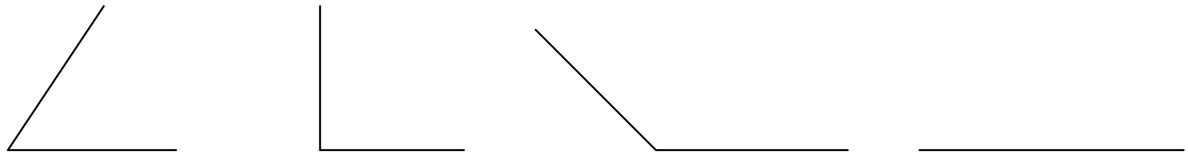
The best way to describe an angle is with three points. One point on each ray and the vertex always in the middle.



That angle could be NAMED in three ways: $\angle X$, $\angle BXC$, or $\angle CXB$.

Classifying Angles

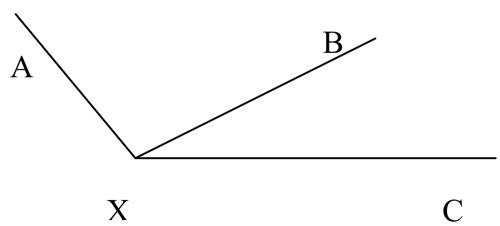
We classify angles by size. **Acute** angles are angles less than 90º. In other words, not quite a quarter rotation. **Right** angles are angles whose measure is 90º. O**btuse** angles are greater than 90º, but less than 180º. That's more than a quarter rotation, but less than a half turn. And finally, **straight** angles measure 180º.



Acute ∠ Right ∠ Obtuse ∠ Straight ∠

Angle Pairs

Adjacent angles are two angles that have a common vertex, a common side, and no common interior points.



∠AXB and ∠BXC are adjacent angles. They have a common vertex – X, they have a common side XB and no common interior points.

We also study angle pairs. We call two angles whose sum is 90º complementary angles. For instance, if ∠P= 40º and ∠Q = 50º, then ∠P and ∠Q are **complementary** angles. If ∠A = 30˚, then the complement of ∠A measures 60˚.

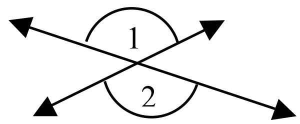
Two angles whose sum is 180º are called **supplementary** angles. If ∠M = 100˚ and ∠S = 80˚, then ∠M and ∠S are supplementary angles.

Example
Find the value of x, if ∠A and ∠B are complementary ∠s and ∠A = 3x and ∠B = 2x + 10.

$$\angle A + \angle B = 90^\circ$$

$$3x + (2x + 10) = 90^\circ$$
$$5x + 10 = 90^\circ$$
$$5x = 80$$
$$x = 16$$

The mathematical definition of **vertical** angles is: two angles whose sides form pairs of opposite rays. $\overrightarrow{ST}$ and $\overrightarrow{SR}$ are called opposite rays if S lies on $\overleftrightarrow{RT}$ between R and T



$\angle 1$ and $\angle 2$ are a pair of vertical angles.

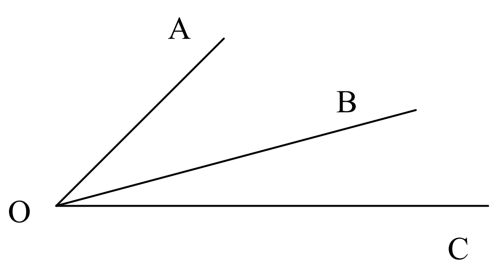
Before we continue with our study of angles, we'll need to introduce some more terms.

Axiom (postulate) is a basic assumption in mathematics.

A **theorem** is a statement that is proved. A **corollary** is a statement that can be proved easily by applying a theorem.

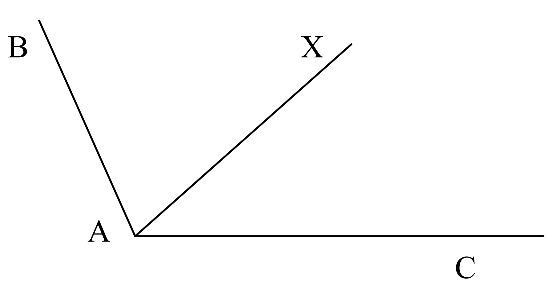
Angle Addition Postulate If point B lies in the interior of $\angle AOC$, then

$$m\angle AOB + m\angle BOC = m\angle AOC.$$

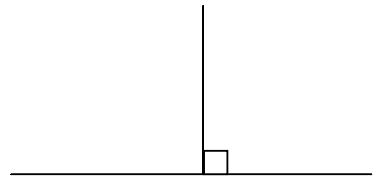


The Angle Addition Postulate just indicates the sum of the parts equal the whole.

Angle bisector; $\overrightarrow{AX}$ is said to be the bisector of $\angle BAC$ if X lies on the interior of $\angle BAC$ and $m\angle BAX = m\angle XAC$.

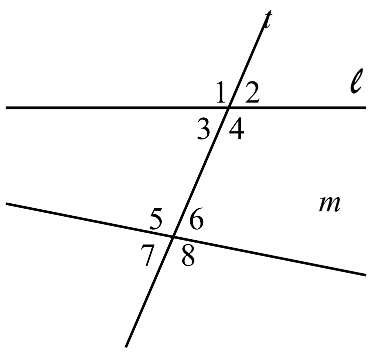


Perpendicular lines are two lines that form right angles.



Angles: Parallel Lines

Now we are going to name angles that are formed by two lines being intersected by another line called a transversal.



If I asked you to look at the figure above and find two angles that are on the same side of the transversal, one an interior angle (between the lines), the other an exterior angle that were not adjacent, could you do it?

$\angle 2$ and $\angle 4$ are on the same side of the transversal, one interior, the other is exterior – whoops, they are adjacent. How about $\angle 2$ and $\angle 6$?

Those two angles fit those conditions. We call those angles **corresponding** angles.

Can you name any other pairs of corresponding angles?

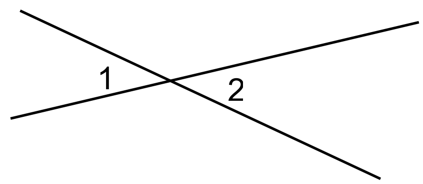
If you said $\angle 4$ and $\angle 8$, or $\angle 1$ and $\angle 5$, or $\angle 3$ and $\angle 7$, you'd be right.

Alternate Interior angles are on opposite sides of the transversal, both interior and not adjacent. $\angle 4$ and $\angle 5$ are a pair of alternate interior angles. Name another pair.

Proof- Vertical Angles are Congruent

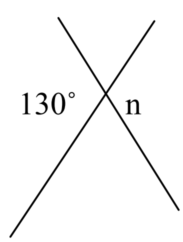
An observation we might make if we were to look at a number of vertical angles is they seem to be equal. We might wonder if they would always be equal.

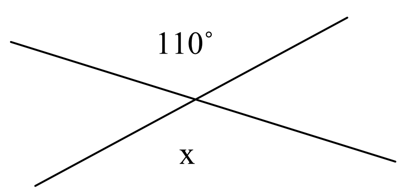
Well, I've got some good news for you. We are going to prove vertical angles are congruent.

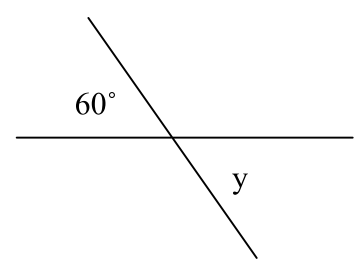


Proving something is true is different than showing examples of what we think to be true.

If you are going to be successful in geometry, then you have to have a body of knowledge to draw from to be able to think critically. What that means is you need to be able to recall definitions, postulates, and theorems that you have studied. Without that information, you are not going anywhere. So every chance you have, read those to reinforce your memory. And while you are reading them, you should be able to visualize what you are reading.





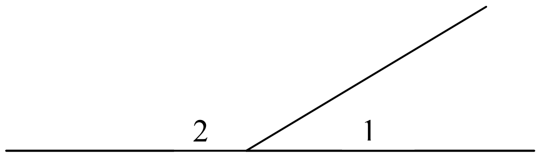


In order for me to prove vertical angles are congruent, I'd need to recall this information that we call theorems. Before we can prove vertical angles are congruent, I must be able to either accept the following theorem as true or prove the theorem.

Can you find the values of n, x, and y? How were you able to make those calculations? The next theorem formalizes that knowledge that led you to the answers.

Theorem

If the exterior sides of 2 adjacent angles are in a line, then the angles are supplementary.



Let's walk through this without proving it;

Angles 1 and 2 combined make a straight angle using the Angle Addition Postulate. A straight angle measures 180˚. Two angles whose sum is 180˚ are supplementary angles, so $\angle 1$ and $\angle 2$ are supplementary.

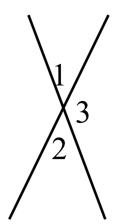
The next theorem is just as straight forward. See if you can draw the picture and talk your way through the theorem to convince other you are correct.

<u>Theorem</u>
If two angles are supplementary to the same angle, then the angles are congruent.

A proof has 5 parts, the statement, the picture, the given, the prove, and the body of the proof. Playing with the picture and labeling what you know will be crucial to your success. What's also crucial is bringing in your knowledge of previous definitions, postulates, and theorems.

Theorem - Vertical angles are congruent

To prove this theorem, we write the statement, draw and label the picture describing the theorem, write down what is given, write down what we are supposed to prove, and finally prove the theorem.



Given: $\angle 1$ and $\angle 2$ are vertical angles

Prove: $\angle 1 \cong \angle 2$

If I just labeled $\angle 1$ and $\angle 2$, I would be stuck. Notice, and this is important, by labeling $\angle 3$ in the picture, I can now use a previous theorem – If the exterior sides of 2 adjacent angles lie in a line, the angles are supplementary. That would mean $\angle 1$ and $\angle 3$ are supplementary and $\angle 2$ and $\angle 3$ are supplementary because their exterior sides lie in a line. If I didn't know my definitions and theorems, there is no way I could do the following proof.

After drawing the picture and labeling it, I will start by writing down what's given as Step 1. My second and third steps follows from the picture about supplementary angles, and my last step is what I wanted to prove.

	Statements	Reasons
1.	$\angle 1$ and $\angle 2$ are vert $\angle$'s	Given
2.	$\angle 1$ and $\angle 3$ are supp $\angle$'s	Ext sides, 2 adj $\angle$'s in a line
3.	$\angle 2$ and $\angle 3$ are supp $\angle$'s	Same as #2
4.	$\angle 1 \cong \angle 2$	Two $\angle$'s supp to same $\angle$

A proof is nothing more than an argument whose conclusion follows from the argument. Proofs can be done differently, all we care about is the conclusion follows from the argument.

Let's look at another way someone might use to prove vertical angles are congruent. I might suggest that as you begin to prove theorems, you write the statement, draw and label the picture, put more information into the picture based upon your knowledge of geometry, write down what is given, and what it is you are going to prove.

Now you are ready to go, make your T-chart. your first statement could be to write down what is given, the last step will always be what you wanted to prove.

	Statements	Reasons
1.	$\angle 1$ and $\angle 2$ are vert $\angle$'s	Given
2.	$\angle 1$ and $\angle 3$ are supp $\angle$'s $\angle 2$ and $\angle 3$ are supp $\angle$'s	Ext sides, 2 adj $\angle$'s in a line
3.	$\angle 1 + \angle 3 = 180^\circ$ $\angle 2 + \angle 3 = 180^\circ$	Def of supp $\angle$s
4.	$\angle 1 + \angle 3 = \angle 2 + \angle 3$	Substitution
5.	$\angle 1 = \angle 2$	Subtraction Prop of Equality
6.	$\angle 1 \cong \angle 2$	Def of congruence

This proof is clearly longer than the first way we proved it, but the conclusion still follows from the argument.

If you have not memorized previous definitions, postulates and theorems, you simply will not be able to do proofs.

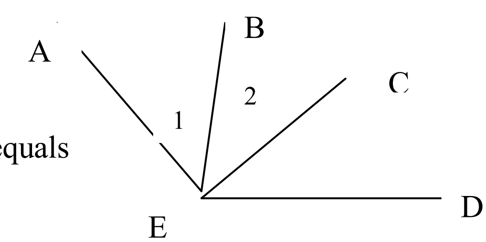
Angles

Review Questions

1. The vertex of ∠ RST is point

2. In the plane figure shown, ∠ 1 and ∠ 2 are ________________ angles.

3. How many angles are shown in the figure?



4. In the plane figure shown, m∠ AEC + ∠ CED equals

5. If $\overrightarrow{EC}$ bisects ∠ DEB and the m∠ DEC =28, then m∠ CEB equals

6. if m∠ 1 = 30 and the m∠ 2 = 60, then ∠ 1 and ∠ 2 are

7. If m∠ 1 = 3x and the m∠ 2 = 7x, and ∠ 1 is a supplement of ∠ 2, then x =

8. If the exterior sides of two adjacent angles lie in perpendicular lines, the angles are

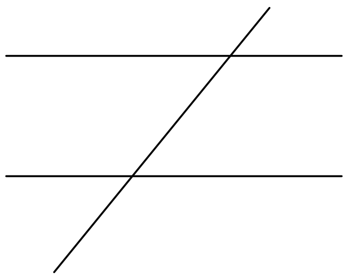
9. If ∠ 1 is complementary to ∠ 3, and ∠ 2 is complementary to ∠ 3, then

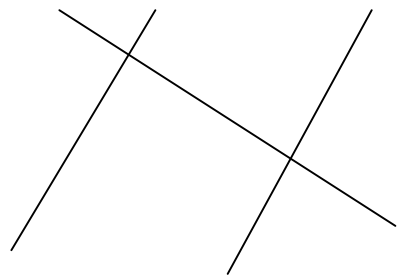
10. ∠ T and ∠ A are vertical angles. If m∠ T = 2x + 8 and m∠ A = x + 22, then x =

11. Name the 5 components of a proof.

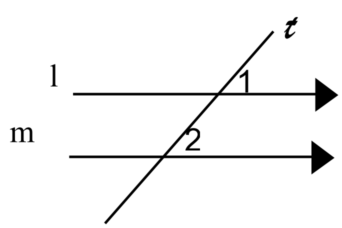
Angles Pairs; Parallel lines

Something interesting occurs if the two lines being cut by the transversal happen to be **parallel**. It turns out that every time I measure the corresponding angles, they turn out to be equal. You might use a protractor to measure the corresponding angles below. Since that seems to be true all the time and we can't prove it, we'll write it as an axiom – a statement we believe without proof.





Axiom
If two **parallel** lines are cut by a transversal, the corresponding angles are congruent.



$\angle 1 \cong \angle 2$

Now let's take this information and put it together and see what we can come up with.

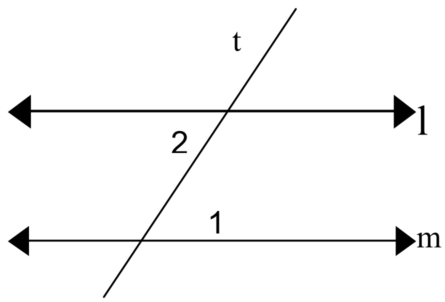
Proofs: Alternate Interior Angles

Let's see, we've already learned vertical angles are congruent and corresponding angles are congruent if they are formed by parallel lines. Using this information we can go on to prove alternate interior angles are also congruent if they are formed by parallel lines.

What we need to remember is drawing the picture will be extremely helpful to us in the body of the proof. Let's start.

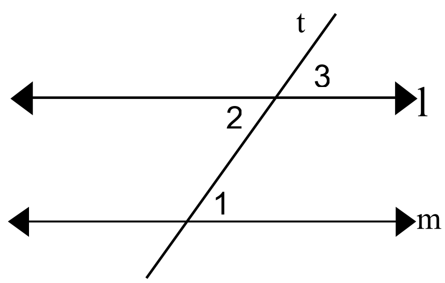
Theorem
If 2 parallel lines are cut by a transversal, the alternate interior angles are congruent.

By drawing the picture of parallel lines being cut by a transversal, we'll label the alternate interior angles.



The question is, how do we go about proving $\angle 1 \cong \angle 2$?

Now this is important. We need to list on the picture things we know about parallel lines. Well, we just learned that corresponding angles are congruent when they are formed by parallel lines.
Let's use that information and label an angle in our picture so we have a pair of corresponding angles.



Since the lines are parallel, $\angle 1$ and $\angle 3$ are congruent. Oh wow, $\angle 2$ and $\angle 3$ are vertical angles! They are congruent.

That means $\angle 1 \cong \angle 3$ because they are corresponding angles and $\angle 2 \cong \angle 3$ are congruent because they are vertical angles, that means $\angle 1$ must be congruent to $\angle 3$.

$$\angle 1 \cong \angle 3$$
$$\angle 2 \cong \angle 3$$

That would suggest that $\angle 1 \cong \angle 2$.

Now we have to write that in two columns, the statements on the left side, the reasons to back up those statements on the right side.

Let's use the picture and what we labeled in the picture and start with what has been given to us, line l is parallel to m.

	Statements	Reasons
1.	l ll m $\angle 1$ and $\angle 2$ are alt int $\angle$'s	Given
2.	$\angle 1$ and $\angle 3$ are corr. $\angle$s	Def of corr. $\angle$s
3.	$\angle 1 \cong \angle 3$	Two ll lines, cut by t, corr. $\angle$'s $\cong$
4.	$\angle 3 \cong \angle 2$	Vert $\angle$'s
5.	$\angle 1 \cong \angle 2$	Transitive Prop

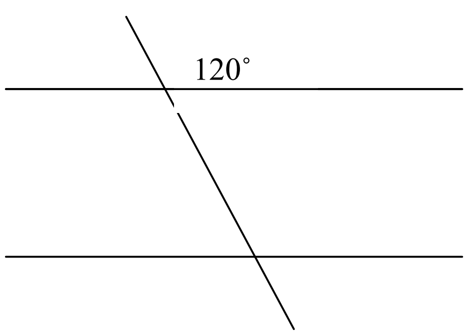
Is there a trick to this? Not at all. Draw your picture, label what's given to you, then fill in more information based on your knowledge. Start your proof with what is given, the last step will always be your conclusion.

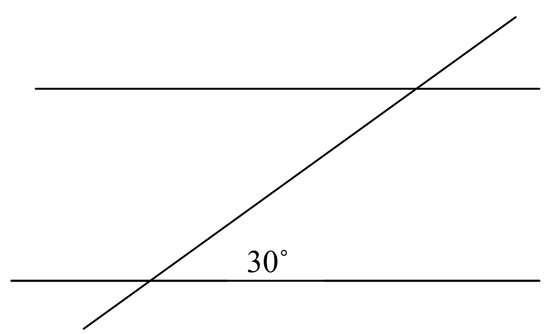
Now, we have proved vertical angles are congruent, we accepted corresponding angles formed by parallel lines are congruent, and we just proved alternate interior angles are congruent. Could you prove alternate exterior angles are congruent? Try it. Write the theorem, draw the picture, label the alternate exterior angles, add more information to your picture based on the geometry you know, identify what has been given to you and what you have to prove.

Let's write that as a theorem.

Theorem
If two parallel lines are cut by a transversal, the alternate exterior angles are congruent.

If we played some more in the world of angles being formed by parallel lines, we might find an interesting relationship between the same side interior angles. Let's take a look.





If you filled in all the angles formed by those parallel lines being cut by a transversal, what relationship do you see when looking at the same side interior angles?

Let's write that as a theorem.

Theorem
If two parallel lines are cut by a transversal, the same side interior angles are supplementary.

Summarizing, we have:

If two parallel lines are cut by a transversal:, then the corresponding ∠ 's are congruent.
then the alt. int. ∠ 's are congruent.
then the alt. ext. ∠ 's are congruent.
then the same side int. ∠ 's are congruent.

The good news is the converse of those statements are also true.

Showing lines are Parallel

Postulate
If two lines are cut by a transversal so that the corresponding angles are congruent, then the lines are parallel.

This is the converse of the postulate that read; if two parallel lines are cut by a transversal, the corresponding angles are congruent. Now what I will accept as true is if the corresponding angles are congruent, the lines must be parallel.

The converse of a conditional is not always true, so this development is fortunate. As it turns out, the other three theorems we just studied about having parallel lines converses' are also true.

Theorem
If two lines are cut by a transversal so that the alternate interior angles are congruent, then the lines are parallel.

Theorem
If two lines are cut by a transversal so that the alternate exterior angles are congruent, then the lines are parallel.

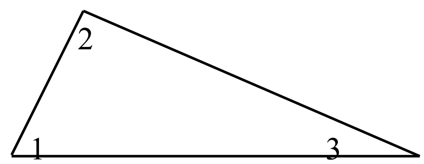
Theorem
If two lines are cut by a transversal so that the same side interior angles are supplementary, then the lines are parallel.

Now I have four ways to show lines are parallel, corresponding ∠ 's congruent, alternate interior ∠ 's congruent, alternate exterior ∠ 's congruent, same side interior ∠ 's supplementary.

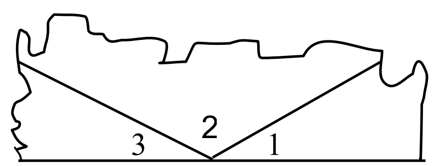
Angle Theorems: Polygons

If I asked an entire class to draw a triangle on a piece of paper, then had each person cut out their triangle, we might see something interesting happen.

Let's label the angles 1, 2, and 3 as shown.



By tearing each angle from the triangle, then placing them side by side, the three angles always seem to form a straight line. Neato!

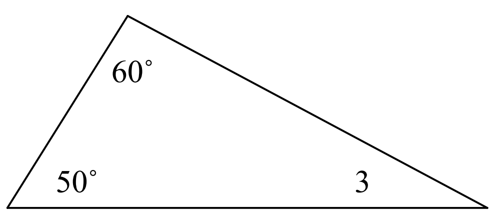


That might lead me to believe the sum of the interior angles of a triangle is 180º.

While that's not a proof, it does provide me with some valuable insights. The fact is, it turns out to be true, so we write it as a theorem.

<u>Theorem</u>
The sum of the interior angles of a triangle is 180º.

Find the measure of $\angle 3$.



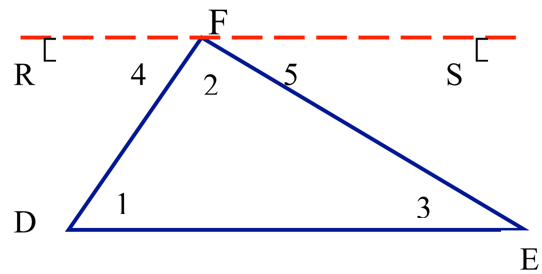
Since the sum of the two angles given is 110˚, $\angle 3$ must be 70˚

Drawing a triangle and cutting out the angles suggests the sum of the interior angles is 180˚ is not a proof. Let's see what that proof might look like.

Theorem Proof
The sum of the measures of the angles of a triangle is 180º.

G: Δ DEF

P: $\angle 1 + \angle 2 + \angle 3 = 180^\circ$



The most important part of this proof will be our ability to use the geometry we have already learned. If we just looked at the three angles of the triangle, we'd be looking for an awfully long time without much to show for it. What we will do is use what we just learned – we were just working with parallel lines, so what we will do is put parallel lines into our picture by constructing RS parallel to DE and labeling the angles formed. Now we have parallel lines being cut by transversals, we can use our knowledge of angle pairs being formed by parallel lines.

Statements	Reasons
1. Δ DEF	Given
2. Draw RS ∥ DE	Construction
3. $\angle 4$ ∧ $\angle$DFS are supp	Ext sides of 2 adj 2's
4. $\angle 4 + \angle DFS = 180^\circ$	Def Supp $\angle$'s
5. $\angle DFS = \angle 2 + \angle 5$	$\angle$ Add Postulate
6. $\angle 4 + \angle 2 + \angle 5 = 180^\circ$	Substitution
7. $\angle 1 = \angle 4$ $\angle 3 = \angle 5$	2 ∥ lines cut by t, alt int $\angle$'s =
8. $\angle 1 + \angle 2 + \angle 3 = 180^\circ$	Substitution

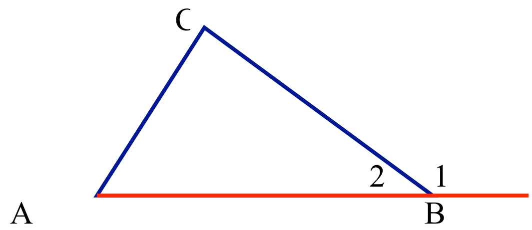
Notice how important it is to integrate our knowledge of geometry into these problems. Step 3 would not have jumped out at you. It looks like angles 4, 2, and 5 form a straight angle, but we don't have a theorem to support that so we have to look at $\angle 4$ and $\angle$DFS first.

A theorem that seems to follow directly from that theorem is one about the relationship between the exterior angle of a triangle and angles inside the triangle. If we drew three or four triangles and labeled in their interior angles, we would see a relationship between the two remote interior angles and the exterior angle.

Thm. The exterior $\angle$ of a Δ is equal to the sum of the 2 remote interior $\angle$'s

G: ΔABC

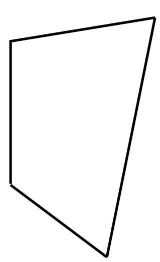
P: $\angle 1 = \angle A + \angle C$

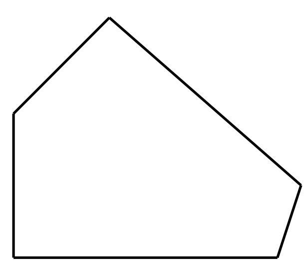


	Statements	Reasons
1.	$\angle A + \angle C + \angle 2 = 180^\circ$	Int $\angle$'s of $\Delta = 180^\circ$
2.	$\angle 1 \wedge \angle 2$ are supp $\angle$'s	Ext sides 2 adj $\angle$'s
3.	$\angle 1 + \angle 2 = 180^\circ$	Def Supp $\angle$'s
4.	$\angle A + \angle C + \angle 2 = \angle 1 + \angle 2$	Substitution
5.	$\angle A + \angle C = \angle 1$	Sub Prop Equality

Will that help us find the sum of the interior angles of any convex polygon? What's the sum of the interior angles of a quadrilateral? a pentagon? an octagon?

The answer is, I just don't know. But …, if I draw some pictures, that might help me discover the answer.

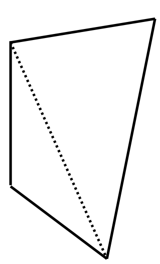


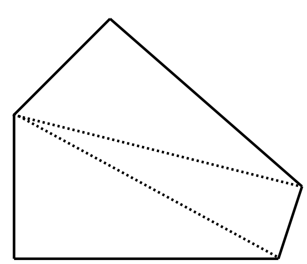


By drawing diagonals from a single vertex, I can form triangles.

4 sides

2 Δ's





5 sides

3 Δ's

Those observations might lead me to believe the sum of the interior angles of a quadrilateral is 360º because there are two triangles formed.

In a five sided figure, a pentagon, three triangles are being formed. Three times 180º is 540º.

The number of triangles being formed seems to be two less than the number of sides in the polygon. Try drawing an octagon and see if the number of triangles formed is two less than the number of sides.

So a polygon with **n** sides would have **(n – 2)** triangles formed. So, if I multiply the number of triangles by 180º, that should give me the sum of the interior angles. Sounds like a theorem to me.

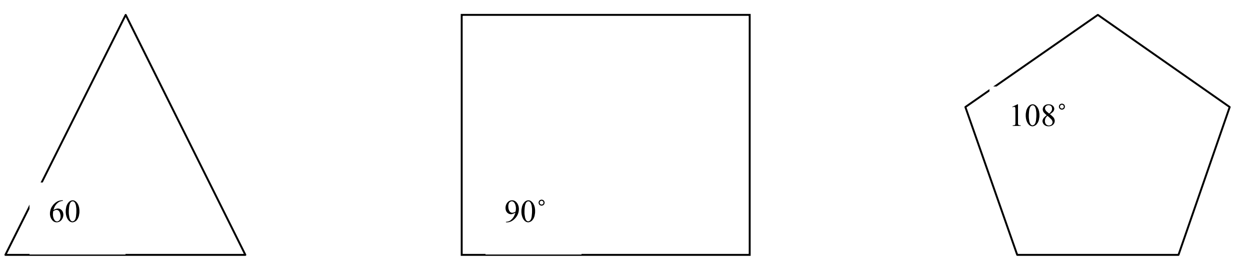
Theorem
The sum of the interior angles of a convex polygon is given by

$$(n - 2)\ 180^\circ$$

Using that, and since a hexagon has six sides, the sum of the interior angles should be (6 – 2) times 180º or 720º.

Exterior Angles – Polygons

If we played with these pictures longer, we'd find more good news that would lead to another theorem.



Let's say we drew a number of regular polygons, polygons whose sides and angles are congruent as we just did. We could find the measure of each exterior angle of the triangle, one angle at each vertex and we would find each exterior angle measures 120˚.

Looking at the square, each interior angles would measure 90˚, each exterior angle would also measure 90˚.

And finally, in the pentagon, each interior angle measures 108˚, then each exterior would measure 72˚.

Is there a pattern developing there?

Let's summarize, in the triangle there are 3 exterior angles each measuring 120˚, their sum is 360˚

In the square, there are four exterior angles each measuring 90˚, their sum is 360˚.

In the pentagon, there are five exterior angles each measuring 72˚, their sum is 360˚.

It would appear the sum of the exterior angles of the examples we used seem to be 360˚. That might lead us to the following theorem.

Theorem
The sum of the exterior angles of a convex polygon, one angle at each vertex, is 360º.

The sum of the interior angles of a polygon change with the number of sides. But the sum of the exterior angles always measure 360˚. That's important to know.

Assume I have a regular polygon whose exterior angle measured 40˚ and I wanted to know how many sides the polygon had, could I make that determination?

Since the polygon is regular, we know all angles must be congruent. If each exterior angle measured 40˚ and the sum of the exterior angles must be 360˚, then 360˚ ÷ 40˚ = 9.

The polygon must have 9 sides!

Let's make a minor change, let's say the interior angle of a regular polygon measured 150˚, how many sides would it have?

To make this determination, I would have to know the measure of the exterior angle. Since the interior angle measures 150˚, the exterior angle must measure 30˚since they would form a straight angle.

If each exterior angle measured 30˚, the 360˚ ÷ 30˚ = 12.

The polygon would have 12 sides.

Classification
Acute
Right
Obtuse

Adjacent ∠'s
Complementary ∠'s – 2 ∠'s whose sum is 90˚
Supplementary ∠'s – 2 ∠'s whose sum is 180˚

Angle Addition Postulate

If the exterior sides of two adj ∠'s lie in a line, they are supplementary
If the exterior sides of two adj ∠'s lie in perpendicular lines, they are complementary

All vertical ∠'s are equal

Angle bisector – $\overrightarrow{AX}$ is said to be the bisector or ∠ BAC if X lies in the interior and ∠BAX = ∠XAC

<u>Angles pairs formed by parallel lines</u>

If two parallel lines are cut by a transversal, then

The corresponding ∠'s are equal
The alt int ∠'s are equal
The alt ext ∠'s are equal
The same side interior ∠'s = 180˚

(and their respective converses)

<u>Angles formed by polygons</u>

The sum of the interior angles of a triangle is 180˚

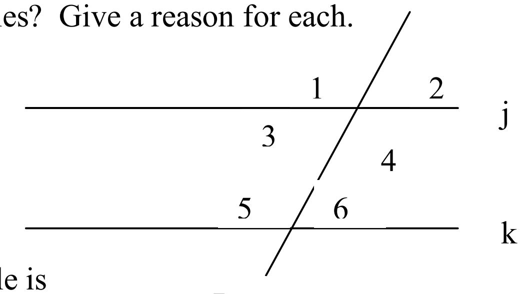
The exterior ∠ of a triangle is equal to the sum of the 2 remote int ∠'s

The sum of the interior angles of a convex polygon is given by (n – 2)180˚

The sum of the exterior angles of a convex polygon is 360˚

Angles; Parallel lines & Polygons

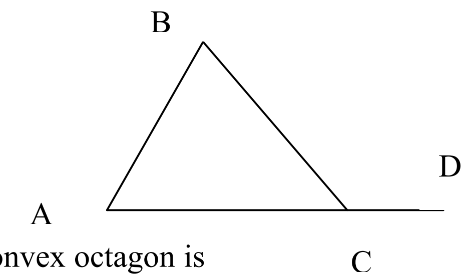
1. If j || k, then ∠ 5 is congruent to what other angles? Give a reason for each.



2. If j || k and m∠ 1 = 100˚, then ∠ 6 =

3. The number of obtuse angles in an obtuse triangle is

4. If m∠ A = 50˚ and m∠ B = 80˚, then ∠ BCD equals



5. The sum of the measures of the interior angles of a convex octagon is

6. The sum of the measures of the exterior angles of a convex pentagon is

7. The total number of diagonals that can be drawn from one vertex of a hexagon is

8. In Δ ABC, $\overrightarrow{BX}$ bisects ∠ ABC, m∠ A = 110˚ and m∠ C = 40˚, then m∠ ABX is

9. If the measure of the interior angle of a regular polygon is 140˚, how many sides does the polygon have?

10. If the m∠ 1 = 100˚, m∠ 2 = 80˚, and m∠ 3 = 100˚, then m∠ 4 is