Derivation – Sum of Geometric Series

Geometric Sequence is a sequence in which every term after the first is obtained by multiplying the preceding term by a fixed number, called the common ratio (r). To find the nth term of a geometric sequence, we know $a_n = a_1 r^{n-1}$

The first term is a_1, second term is $a_1 r$, third term is $a_1 r^2$, etc

This leads up to finding the sum of the geometric series, Sn, by starting with the first term and successively multiplying by the common ratio (r).

$$\begin{array}{cccccccc} & 1^{st} & & 2^{nd} & & 3^{rd} & & n^{th} \\ Sn = & a_1 & + & a_1 r & + & a_1 r^2 & + \ldots + & a_1 r^{n-1} \end{array}$$

To find a formula for the sum of a geometric series, Sn, we can multiply the preceding expansion by $(-r)$ and add the equations together by grouping like terms vertically – *all the terms except the first and last add out.*

$$\begin{array}{rccccccccccc} Sn = & a_1 & + & a_1 r & + & a_1 r^2 & + \ldots & + & a_1 r^{n-2} & + & a_1 r^{n-1} \\ -rSn = & & - & a_1 r & - & a_1 r^2 & - \ldots & - & a_1 r^{n-1} & - & a_1 r^n \end{array}$$

So

$$Sn - rSn = a_1 - a_1 r^n$$

Factoring out Sn $\quad Sn(1-r) = a_1 - a_1 r^n$

Divide both sides by $(1-r)$ $\quad Sn = \dfrac{a_1 - a_1 r^n}{1-r}$

We also know that the nth term of a geometric sequence is $a_n = a_1 r^{n-1}$.
If I multiply both sides by r, we have $r\,a_n = r\,(a_1 r^{n-1})$
And using the product rule for exponents $= a_1 r^n$

Therefore $ra_n = a_1 r^n$

Substituting ra_n into the formula $Sn = \frac{a_1 - a_1 r^n}{1-r}$ for $a_1 r^n$

We have $$Sn = \frac{a_1 - ra_n}{1-r}$$