Ch. X4 Number Theory

Divisibility

The phrase *number theory* sounds impressive, but the chapter is just an acknowledgement of a great deal of observations, patterns, and logic that will help us in our work in mathematics.

Let's begin by looking at our rules of divisibility for twos, fives, and tens. When you learned your multiplication facts in third grade, you probably made some observations that helped you in your memorization. When you multiplied a number by 10, the answer always ended in zero. When you multiplied by 5, the answer either ended in a zero or a five. And when you multiplied by 2, the number was always even.

Using those observations, if we looked at a product (answer) in a multiplication computation and saw the answer ended in zero, we might surmise that 10 was a *factor* of that number. A factor is a number in a multiplication problem. Another way of saying that is that 10 would go into that number without a remainder (evenly). In other words, that number was divisible by ten.

"Ends-in" Rules of Divisibility

Ex. 1. Is 720 divisible by 10?

Since 720 ends in a zero, 720 is divisible by 10.

If a number ended on a five or a zero, using the same logic, we could surmise that 5 is a factor. Or that the number is divisible by 5.

Ex. 2. Is 720 divisible by 5?

Since 720 ends in a zero, 720 is divisible by 5.

Ex. 3. Is 435 divisible by 5?

Since 435 ends in a five, 435 is divisible by 5.

\Continuing, we could see from our multiplication facts that any number multiplied by two always results in an *even* number. That is, the last digit of a number ends in 0, 2, 4, 6, or 8.

Ex. 4 **Is 826 divisible by 2?**

Since 82<u>6</u> ends in an even number, 826 is divisible by 2.

By looking at those observations, we can come up with three rules of divisibility that will help us when working with larger numbers.

<u>“Ends-in” Rules of Divisibility</u>

Rule 1. Divisibility by 10. A number is divisible by 10 if it ends in a zero.

Rule 2. Divisibility by 5. A number is divisible by 5 if it ends in a zero or five.

Rule 3. Divisibility by 2. A number is divisible by 2 if it ends in an even number.

<u>“Sum” Rules of Divisibility</u>

The first three rules for divisibility were easy enough to see based on our observation of multiplication facts. Lets look at some other numbers, like 3 and 9.

If we picked numbers at random and multiplied them by 3, we might see some other patterns that would allow us to look at numbers and determine if they were divisible by those numbers.

We will start with 3. We will pick numbers at random like 11, 15, 20, 23, and 145 and multiply them by 3.

3 x 11 = 33 3 x 15 = 45 3 x 20 = 60 3 x 23 = 69 3 x 145 = 435

Upon first observation, nothing seems to jump out at me that would suggest those products are divisible by 3. But, if I were to look longer, think, and *try* to find a hint, I might begin to notice that the <u>sum of the digits in the products are divisible by 3</u>. I would know this from my memorization of the multiplication facts.

33 → 3 + 3 = 6

45 → 4 + 5 = 9

60 → 6 + 0 = 6

69 → 6 + 9 = 15

435 → 4 + 3 + 5 = 12

Using that observation, I might begin to think that if the sum of the digits of a number are divisible by 3, then the number is divisible by 3.

Ex. 5 **Is 111 divisible by 3?**

111 → 1 + 1 + 1 = 3, 3 is divisible by 3, so is 111 divisible by 3? Let's check to make sure that works by actually dividing 111 by 3. Yes, it works!

Ex. 6 **Is 864 divisible by 3?**

864 → 8 + 6 + 4 = 18, 18 is divisible by 3, so is 864 divisible by 3? Let's check by dividing to make sure that works. Again, it does.

These observations and previous examples lead me to another rule of divisibility.

Rule 4. Divisibility by 3. A number is divisible by 3 if the sum of the digits is divisible by 3.

Using that same type of observation and logic, we will look at numbers whose products were formed by multiplying by 9.

We will again pick numbers at random; 20, 34, 443, and 657, multiply those by 9, then look at their products.

9 x 20 = 180 9 x 34 = 306 9 x 443 = 3987 9 x 657 = 5913

180 → 1 + 8 + 0 = 9

306 → 3 + 0 + 6 = 9

3987 → 3 + 9 + 8 + 7 = 27

5913 → 5 + 9 + 1 + 3 = 18

Looking at these numbers, we see a pattern that suggests the if the sum of the digits of a number is divisible by 9, then the number is also divisible by 9.

Rule 5. Divisibility by 9. A number is divisible by 9 if the sum of the digits is divisible by 9.

The first three rules of divisibility for 2, 5, and 10 all deal with how numbers end, so in learning the rules of divisibility, you might want to group those three rules together. The next two rules, for 3 and 9, have to deal with the sum of the digits. So it might be helpful to group those two rules together.

The next rule is based more on logic than on observation. If a number is divisible by 2 and also divisible by 3, that means it has factors of 2 (even) and 3, thus the number would be divisible by 2 x 3 or 6.

"Combo" Rules of Divisibility

Rule 6. Divisibility by 6. A number is divisible by 6 if it is divisible by 2 and 3.

That means we can look at a number very quickly and determine if it is divisible by 6. First it would have to be even (divisible by 2) and second the sum of the digits would have to be divisible by 3.

Ex. 7. Is 354 divisible by 6?

354 is even so it is divisible by 2 and 3 + 5 + 4 = 12, 12 is divisible by 3 so 354 is divisible by 3. Since it is divisible by 2 and 3, it is divisible by 6.

You might ask, why would I want to know these rules for divisibility? Well, an immediate use would be for simplifying fractions. If you were asked to simplify 111/123, that might be considered more difficult than simplifying 3/12 by some. But knowing the rules for divisibility, I could determine very quickly that a common factor of the numerator and denominator of 111/123 is 3. Thus making the problem simpler.

We could extend the reasoning we used for divisibility by 6 for other numbers. For instance, if a number is divisible by 5 and by 3, do you think it might be divisible by 15? What do we know about how numbers end when they are multiplied by 25, could you come up with a rule for divisibility of 25?

"Last Digits" Rules of Divisibility

If we continued to look at patterns by multiplying by 4, then look at products, we would come up with other rules of divisibility.

Rule 8. Divisibility by 4. A number is divisible by 4 if the last two digits of the number is divisible by 4.

Ex. 8. **Is 512 divisible by 4?**

Since 12 in the number 5<u>12</u> is divisible by 4, then 512 is divisible by 4.

Ex. 9 **Is 1132 divisible by 4?**

Since the 32 in 11<u>32</u> is divisible by 4, then 1132 is divisible by 4.

There are other rules of divisibility. I skipped rules of divisibility for 7 and 8 because I believe it is just as easy to divide the numbers by those numbers than to memorize and use a rule.

Using the rules of divisibility, we can quickly look at numbers and determine if they are divisible by 2, 3, 4, 5, 6, 9, and 10.

With a little extra thought, we can construct numbers that are divisible by some or all of those numbers as well.

Ex. 10. **Write a 5-digit number that is divisible by 2, 3, 4, 5, 6, 9, and 10.**

Using logic, we know the number must end in zero if it is to be divisible by 10.

$$\_\ \_\ \_\ \_\ \underline{0}$$

If a number ends in 0, then it is divisible by 10, 5, and 2.

For a number to be divisible by 4, the last two digits have to be divisible by 4.

$$\_\ \_\ \_\ \underline{4}\ \underline{0}$$

What are other numbers could I have placed in that position so the last two numbers are divisible by 4?

Up to this point we have constructed a number that is divisible by 2, 5, 10, 6 and 4. If we continue with this construction so that the number is divisible by 9, then it is automatically divisible by 3.

$$\underline{2}\ \underline{0}\ \underline{3}\ \underline{4}\ \underline{0}$$

If the number is divisible by 3 and 2, then it is divisible by 6. Therefore we have constructed a number that is divisible by 2, 3, 4, 5, 6, 9, and 10.

If we continued to play with numbers and rules of divisibility, we might notice other patterns that will help us later with larger numbers. We know, for example, 3 is a factor of 12 and 3 is a factor of 21. The sum of 12 and 21 is 33, and 3 is a factor of 33. That is to say that 33 is divisible by 3. Another example might be 30 is divisible by 6, 24 is divisible by 6, and if we found their sum, 30 + 24 = 54, we see that is also divisible by 6.

We can see why that works by applying the Distributive Property. Both 12 and 33 are divisible by 3 (have a factor of 3). Rewriting 12 + 33 using the Distributive Property, we have 12 + 33 = 3(4 + 11). We can see if 3 is a factor of each number, then 3 is a factor of their sum.

Do you think that could be extended to subtraction? In other words, if 12 is a factor of 72 and 96, would 12 be a factor 96 –72 = 24? Would 24 be divisible by 12?

I know what you are thinking, this is just too easy. Let's write that observation as a theorem.

Theorem **For any integers *a, b* and *c*, if *a* is divisible by *c* and *b* is divisible by *c*, then *(a+b)* is divisible by *c* and *(a–b)* is divisible by *c*.**

Prime and Composite Numbers

The Natural (Counting) Numbers {1, 2, 3, 4, …} can be broken into three categories; prime, composite and neither.

By definition,

Prime numbers have exactly two factors, one and itself. Examples include 2, 3, 5, 7, 11, 13, 17, and so on.

Composite numbers are numbers that have more than two factors. Examples include 4, 6, 8, 9 10, 12, and so on.

Notice the number one (1) has only one factor. One is neither prime nor composite.

How can you determine if a number is prime? One way is to use the Sieve of Erostosthenes by writing all the numbers from 1 to however high you want to go, then underline all multiples of primes beginning with 2.. Underline the first prime number - 2, then color multiples of 2 in green. Next, go to the next prime number after 2 that has not been underlined , underline it and write those multiples of 3 in red. Continue this process with the next number not in color – 5 underline it and color multiples in blue. Continue to the next number not colored for 7, underline it and color multiples in orange. All the numbers underlined are prime. The numbers in color are composite.

1	<u>2</u>	<u>3</u>	4	<u>5</u>	6	<u>7</u>	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Fundamental Theorem of Arithmetic **All composite numbers can be written as a product of prime numbers in one and only one way (order does not matter).**

Ex. 1. **Write 12 as a product of primes.**

From our knowledge of the multiplication facts, we know that 12 can be written as 4 x 3. That's a product, but 4 is not prime. But 4 can be written as 2 x 2, so 12 = 2 x 2 x 3 - a product of primes.

What if 12 was written as 6 x 2 instead of 4 x 3? Well 6 x 2 is not a product of primes because 6 is not prime. But 6 can be written as 3 x 2, so 12 = 3 x 2 x 2. Notice those two answers are the same – just written in a different order!

$$2 \times 2 \times 3 = 3 \times 2 \times 2$$

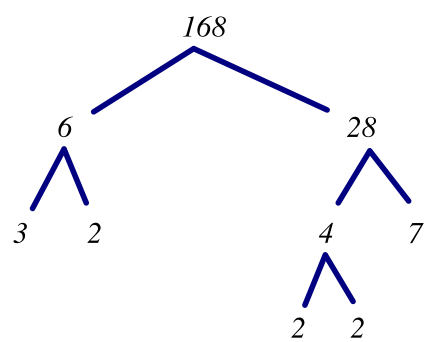
By convention, not rule, we usually write the product by writing the smallest numbers first. That is, to help people read them, we would normally see 12 written as 2 x 2 x 3.

Prime Factorization

Prime factorization is the process of re-writing numbers as product of primes. There are different ways of finding prime factors. One method is using a factor tree, use your knowledge of multiplication facts or rules of divisibility to find factors and continue to rewrite the numbers until you **only** have primes.

Ex. 2. **Write 168 as a product of primes.**

Write 168 at the top middle of the page and rewrite as a product of two factors.



The factors are 3 x 2 x 2 x 2 x 7 or $3 \times 2^3 \times 7$. By convention, we would write that as

$$2^3 \times 3 \times 7$$

Another method is to divide the original number by the smallest prime numbers until the last quotient is prime.

Ex. 3. **Write 168 as a product of primes.**

Divide 168 by 2, then continue dividing by prime numbers.

$$\begin{array}{r} 2\overline{)168} \\ 2\overline{)84} \\ 2\overline{)42} \\ 3\overline{)21} \\ 7 \end{array}$$

The prime factors are 2^3 x 3 x 7, just like before.

Number of Factors in a Number

Besides writing numbers as products of primes, another question might be how many factors does a number have? For examples how many factors does 12 have? You might be able to answer that because of your knowledge of the multiplication facts. 1x12, 2x6, and 3x4. So the factors of 12 are 1, 2, 3, 4, 6 and 12. Twelve has 6 factors. That was easy because of our familiarity with the number 12.

How about 84? How many factors are there in 84? Again, I could find factors of 84 using the multiplication facts or rules of divisibility.

1x84, 2x42, 3x28, 4x21, 6x14, 7x12. So the factors of 84 are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, and 84. There are 12 factors.

Rather than looking at these factors by listing as we have done, let's use prime factorization and use exponents.

$12 = 2x2x3$ or 2^2x3^1. 12 has 6 factors.

$40 = 2x2x2x5$ or 2^3x5^1. 40 has 8 factors.

$84 = 2x2x3x7$ or $2^2x3^1x7^1$ 84 has 12 factors

We don't normally write the exponent when it is 1, but to make our observation clearer I did.

Now, stay with me on this because this pattern does not just jump out at you. One way to determine the number of factors in a number is to write the product of primes in exponential notation as we just did. That is, $12 = 2^2x3^1$. We said 12 had 6 factors. If I played long enough, I might see a pattern set up that would tell me the number of factors a number had – not the factors themselves. If I add one to each exponent, then find the product of the exponents gives the number of factors. In 12, the exponents are 2 and 1. Adding one to each exponent, I have 3 and 2 and the product is 6. That's the number of factors.

In the number 84, rewritten as $2^2x3^1x7^{1,}$ the exponents are 2, 1, and 1 respectively. If I add one to each exponent and then find their product, I have 3, 2, and 2. The product of those numbers is 12. That is how many different factors are in 84.

Theorem **The number of factors in the number *p* that can be written as a product of primes, a^mxb^n, is given by the formula (m+1)(n+1)**

Ex. 4. **How many factors are in 200?**

$200 = 5x2x5x2x2 = 2^3x5^2$. Add one to each exponent, then multiply. The exponents are 3 and 2. Add one to each exponent and find the product. 4x3 = 12. There are 12 factors. We just don't what the factors are.

Determine if a Number is Prime

How can we determine if a number is prime? We know from the Fundamental Theorem of Arithmetic, that every composite number can be written as a product of primes. Suppose p is the *least prime* factor of a number n, then we know that p times some other prime number q is equal to n. Since p is the least prime, that would suggest the px$p \le n$ or $p^2 \le n$.

Looking at that from a slight different angle, we have

Theorem **If n is an integer greater than one and not divisible by any prime number p, where $p^2 \le n$, then n is prime.**

Ex. 1. **Is 397 prime or composite?**

Look for the possible prime numbers p so that $p^2 \le 397$. Taking the square root of a perfect square larger than 397, 400 is close. I only have to check the primes numbers less than 20. The primes are 2, 3, 5, 7, 11, 13, 17 and

19 are the only primes where $p^2 \leq 397$, none of them are factors of 397, so 397 is prime.

Ex. 2 **Is 43 a prime number?**

Checking all the possible prime numbers so when I square them they are less than or equal to 43, I have 2, 3, and 5. None of those numbers divide into 43, so 43 is prime.

Why didn't I continue checking the other primes, 7, 11, 13, etc.?

Ex. 2. **Is 637 prime or composite?**

The prime numbers we want to check are 2, 3, 5, 7,11, 13, 17, and 23.

$23^2 \leq 637$. 637 is not even, so it is not divisible by 2, the sum of the digits is 16, so it is not divisible by 3, it does not end in 5 or 0, so it is not divisible by 5. The next number prime we check is 7, 7 is a factor. Therefore, 637 is composite.

Greatest Common Factor

Common factor: A number that is a factor of two or more nonzero numbers.

Ex. 1. Find common factors of 18 and 24.

Factors of 18: **1**, **2**, **3**, **6**, 9, 18
Factors of 24**:** **1**, **2**, **3**, 4, **6**, 8, 12, 24

Greatest Common Factor (GCF): Factors shared by two or more numbers are called *common factors*. The largest of the common factors is called the *greatest common factor*.

In the last example, the greatest common factor, GCF, of 18 and 24 is 6. That is the largest factor in both of those numbers.

There are a number of ways of finding the GCF.

Method 1

To find the GCF, list all the factors of each number. Look at the common factors and the largest one is the GCF.

Ex. 2. Find the GCF of 24 and 36.

Factors of 24 1, 2, 3, 4, 6, 8, 12, 24
Factors of 36 1, 2, 3, 4, 6, 9, 12, 18, 36

The GCF is the greatest factor that is in both lists; 12.

Method 2

To find the GCF, write the prime factorization of each number and identify which factors are in each number.

Ex. 3. Find the GCF of 24 and 36.
$36 = 2^2 3^2$
$24 = 2^3 3^1$

Each number has two 2's and one 3, therefore the GCF is $2^2 x 3^1$ or 12.

1, **2**, **3**, **4**, **6, and 12** are all common factors of 36 and 24, but 12 is the greatest common factor.

It might be easier for students to see if the common prime factors are circled. An easier way to find the GCF is to use the smallest exponent of each common factor.

Ex. 4. Find the GCF of $3^4x5^4x7^2$ and 3^2x5^6.

The common factors are 3 and 5. Use the smallest exponent on each of these numbers to find the GCF. The smallest exponent on the 3's is 2 and the smallest exponent on the 5's is 4. Therefore, the GCF is 3^2x5^4.

Generally, the GCF problems are not given to you written as a product of primes as in example 4, you have to perform that process first, then use the smallest exponents.

Least Common Multiples

Least common multiple is a lowest multiple (number) of two or more numbers. That means all the factors of each number have to be contained in a common multiple.

Methods of finding the LCM

There are three methods for finding the Least Common Multiple (LCM) between two numbers:

Method I: Make a list.

Write the multiples of each numbers until there is a common number.

Ex. 1. Find the LCM of 12 and 16.

Multiples of 12: 12, 24, 36, 48, 60, 72, …
Multiples of 16: 16, 32, 48,
48 is the smallest multiple of both numbers, therefore 48 is the LCM.

Method II: Prime factorization.

Write the prime factorization of both numbers. The LCM has to contain ALL the factors of BOTH numbers. Write the prime factors, use the highest exponent because we want to find the greatest factor in EACH number.

Ex. 2. Find the LCM of 72 and 60.

Prime factors of 72: 2x2x2x3x3=72
Prime factors of 60: 2x2x3x5=60

The Prime factors of 60 and 72 are made up of 2's, 3's, and a 5. How many of each prime factors are used to make up the LCM?

72 has three factors of 2 (2^3) and 60 only has two factors of 2 (2^2). Since the highest exponent is 3 use 2^3.

72 has two factors of 3 (3^2) and 60 has one factor of 3 (3^1). Since the highest exponent is 3 use 3^2.

72 has no factors of 5 (5^0) and 60 has one factor of 5 (5^1). Since the highest exponent is 1 use 5^1.

The LCM = $2^3 x 3^2 x 5^1 = 360$.

Method III: Reduce the fraction.

Another way of using Method II is to write the factors as a fraction, then reduce and cross multiply. The product is the LCM.

Ex. 3. **Find the LCM of 60 and 72.**

The factors of 60 are **2x2x3x5**
The factors of 72 are **2x2x2x3x3**

$$\frac{2x2x3x5}{2x2x2x3x3} = \frac{5}{2x3}$$

Ex.4. **Find the LCM of 18 and 24.**

Make a fraction using 18 and 24. Order does not matter.
Reduce that fraction.

$$\frac{18}{24} = \frac{3}{4}$$

Cross multiply. Either 18x4 or 3x24. Both product equal 72.
The **LCM** is 72.

Note: When adding or subtracting fractions, the **LCM** is referred to as the Least Common Denominator or **LCD**. Method III works great!