Ch. 11 Solving Quadratic Systems

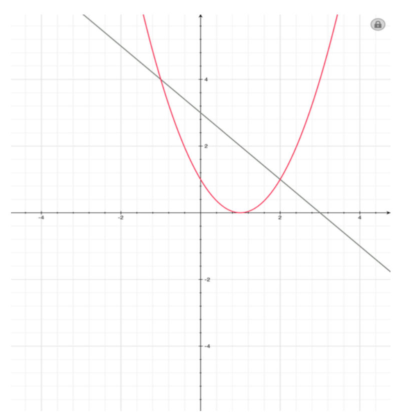
Sec 1. Graphing

We saw earlier when graphing systems of linear equations how graphing gave us a visual of what was occurring. We could see where lines intersected and approximate the solution. Graphing is also a great way to determine real number solutions of systems of equations in 2 variables in which one or both equations are quadratic.

The directions to solving quadratics might be, *find the solution set of a system of quadratics by graphing*. What should be clear is we can only find real number solutions by graphing. If we were directed to find solutions over the complex numbers, then, as we will see, the graphs don't necessarily intersect. So, while graphing will give us an idea of how many solutions a system will have and where those points of intersection are located, graphing is little help when solving over the complex numbers.

Example 1: Find the approximate solution sets by graphing

$y = x^2 - 2x + 1$
$x + y = 3$



Graph both equations on the same coordinate axes and determine points of intersect. The points of intersection are points (ordered pairs) that satisfy both equations.

The solution, the points, appear to be {(–1, 4), (2, 1)}

When solving systems of 2 equations containing quadratics, we could have as many as no real solutions or as many as four. There would be no real solutions if the graphs did not intersect.

Solve the following systems by graphing and estimating the closest order pair to the nearest unit.

1. $x^2 + y^2 = 16$
   $x + y = 5$

2. $y = x^2 - 3$
   $y = 2x + 1$

2. $x^2 + y^2 = 25$
   $y = x^2 - 3$

4. $x^2 + y^2 < 9$
   $y \le x + 1$

Sec 2. Linear Quadratic Systems – Substitution

The most effective way to solve a system of equations that contain a linear and quadratic equation is by substitution. This substitution method is exactly the same substitution method learned for systems of linear equations.

Procedure for Solving Linear – Quadratic Systems by Substitution

1. Solve for one of the variables in the linear equation
2. Substitute that expression into the quadratic equation
3. Solve the resulting quadratic equation in one variable
4. Substitute those values back into the linear equation
5. Write the possible solutions as ordered pairs.
6. Check EACH ordered pairs in both equations

Be aware that systems of equations that involve quadratic equations many have complex roots as well as real roots. Using the graphing method only yields real roots.

Example 2. Solve the following system by substitution

$y = -x^2$
$y = 1$

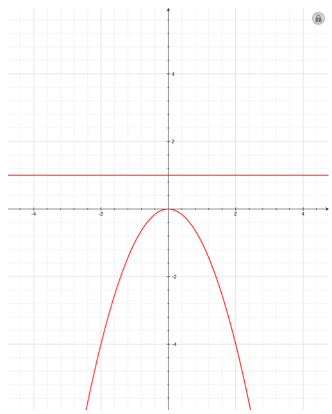
Substituting $y = 1$ into the first equation, we have $1 = -x^2$
$-1 = x^2$

so $x = i$ or $x = -i$

Substituting those values, we have the following possible solutions

$$(i, 1) , (-i, 1)$$

Both of these are true, therefore the solution set is $\{(i, 1) , (-i, 1)\}$



You can see by graphing, there are no real solutions – the graphs do not intersect.

Example 3. Solve the following system by substitution

$$9x^2 + 16y^2 = 2$$
$$3x + 4y = 0$$

Using our procedure, I will solve for one of the variables in the linear equation and substitute that expression into the quadratic.

$$3x + 4y = 0$$
$$3x = -4y \;\rightarrow\; x = -4y/3$$

Substituting that expression into the quadratic

$9x^2 + 16y^2 = 2$

$9(-4y/3)^2 + 16y^2 = 2$

$9(16y^2/9) + 16y^2 = 2$

$16y^2 + 16y^2 = 2$

$32y^2 = 2$

$y^2 = 1/16$

$y = \pm 1/4$

$x = -4y/3$

$x = -4(1/4)/3 = -1/3$

and

$x = -4(-1/4)/3 = 1/3$

{ (–1/3, 1/4), (1/3, –1/4)}

Before I go on, let me make a point. As always, you can follow an algorithm to help you find answers. good. But, it is always better to think before starting.

$$9x^2 + 16y^2 = 2$$
$$3x + 4y = 0$$

If I looked at these two equations, I may have noticed a relationship in one or both of the variables.

$$9x^2 + 16y^2 = 2$$
$$3x + 4y = 0 \quad \rightarrow \quad 3x = -4y$$

So we still have $3x = -4y$, but if I square both sides, I get $9x^2 = 16y^2$

Now substitute $16y^2$ for $9x^2$, we get $16y^2 + 16y^2 = 2$, that could gave saved us a lot of work, don't you think? Yes, thinking before doing is good.

Sec 3. Quadratic – Quadratic Systems

You can solve quadratic-quadratic systems in two variables by graphing, substitution or by the elimination method. To choose which method is most appropriate and then determine which variable to solve for or eliminate means you should "think" in advance. That will cut your work down considerably. Remember, there could be no solutions or as many as four in quadratic-quadratic systems. We sure don't wan to be needlessly bogged down in arithmetic.

Example 1. Find the solution set of the system

$$x^2 + 2y^2 = 17$$
$$2x^2 - 3y^2 = 6$$

To solve this by substitution, there is a quadratic term whose coefficient is one. That would be the easiest equation to solve for because it would eliminate fractional equations. So, that's what I will do, solve for x^2 in the first equation.

$$x^2 = 17 - 2y^2$$

Substituting that expression into the second equation, we have:

$$\begin{aligned} 2(17 - 2y^2) - 3y^2 &= 6 \\ 34 - 4y^2 - 3y^2 &= 6 \\ 34 - 7y^2 &= 6 \\ -7y^2 &= -28 \\ y^2 &= 4 \end{aligned}$$

So we have $y = 2$ or $y = -2$

Substituting those values into the quadratic for BOTH those values of y

For $y = 2$	For $y = -2$
$x^2 = 17 - 2y^2$	$x^2 = 17 - 2y^2$
$x^2 = 17 - 2(2)^2$	$x^2 = 17 - 2(-2)^2$
$x^2 = 17 - 8$	$x^2 = 17 - 8$
$x^2 = 9$	$x^2 = 9$
so $x = 3$, $x = -3$	so $x = 3$, $x = -3$
$(3, 2), (-3, 2)$	$(3, -2), (-3, -2)$

$\longrightarrow \{(3, 2), (-3, 2), (3, -2), (-3, -2)\}$

Remember to check these ordered pairs in BOTH equations!

Linear combination can also be used to solve quadratic systems of equations.

Example 2. Find the solution set of the system

$$\begin{aligned} 3x^2 + 4y^2 &= 16 \\ x^2 - y^2 &= 3 \end{aligned}$$

To use linear combination, we made one of the coefficients the same but opposite in sign on one of the variables, then added the equations together to eliminate a variable.

$$\begin{aligned} 3x^2 + 4y^2 &= 16 \\ x^2 - y^2 &= 3 \end{aligned} \quad \xrightarrow{\times 4} \quad \begin{aligned} 3x^2 + 4y^2 &= 16 \\ \underline{4x^2 - 4y^2} &\underline{= 12} \\ 7x^2 \qquad &= 28 \end{aligned} \qquad \text{Now add}$$

$$x^2 = 4$$

$x^2 = 4$, so $x = 2$ or $x = -2$, now substitute those values for x into one of the equations as we have been doing.

The solution is $\{(2, 1), (2, -1), (-2, 1), (-2, -1)\}$

Check your solutions!

Find the solution set of the following over the complex numbers.

1. $y = x^2 + 2x - 3$
   $y = 2x^2 - x - 1$

2. $4x^2 + y^2 = 25$
   $x^2 - y^2 = -5$

3. Find two positive numbers so that the sum of tier squares is 74 and the difference of their squares is 24.

4. Walking diagonally across the rectangular field is 65 feet. If the area of the field is 1500 square feet, what are the dimensions of the field?