Chapter 15 Exponential Equations

Sec. 1 Solving Exponential Equations

Most of us are familiar and comfortable with problems such as $5^2 = 25$ and $2^3 = 8$.

In algebra, we are sometimes asked to solve equations like $x^2 = 25$. Without too much fanfare, most of us would answer by saying $x = 5$ or $x = -5$. If you didn't know that, you could have solved that by factoring, then using the Zero Product Property or you could have solved by the definition as we did:

$$\textit{if } x^2 = b, \text{ then } x = \pm\sqrt{b}$$

You'll notice the variables in the equations we have been solving for are not in the exponent. What would happen if the variable was in the exponent, a problem like

$$\mathbf{3^x = 81.}$$

One way to solve it is by trying to plug a number in for x by trial and error that would make the equation true. Using intelligent guessing, trial and error is OK, but it's time consuming.

If we worked with enough of these, we might see a way of solving this same equation by rewriting 81 as a power of 3.

For instance; $81 = 3^x$

By substituting, we have $= 3^4$. Now, since the bases are equal, then the exponents must be equal. In other words, $x = 4$.

Mathematically, we write:

Theorem For $b > 0$, $b \neq 1$, $b^x = b^y$ if and only if $x = y$

Don't you just love how we write things mathematically? What's this ***b*** has to be greater than zero and not equal to one business?

Let's try ***b*** being negative, not greater than zero, and see what happens. $(-2)^2 = 2^2$, the exponents are equal, are the bases then equal?

How about when the base equals one: $1^5 = 1^{12}$ in this case, the bases are equal, do the exponents have to be?

Now you know why $\boldsymbol{b} > 0$ and $\boldsymbol{b} \neq 1$.

Example 1 Find the value of x; $2^5 = \mathbf{2^{2x-1}}$

Since the bases are equal, then the exponents must be equal.
Therefore,

$5 = 2x - 1$ Solving

$6 = 2x$

$3 = x$

Substituting 3 makes the original equation true

Example 2 **Solve;** $2^{6x^2} = 4^{5x+2}$

Notice the bases are not the same, we therefore can not set the exponents equal. That's too bad, things were working out so well, But alas! I've always wanted to use that expression.

Is it possible to make the bases the same? Can I write the number 4 as a power of 2? You wouldn't have asked if there was not a way you say.

$2^{6x^2} = 4^{5x+2}$ Given

$2^{6x^2} = (2^2)^{5x+2}$ Substitution

$2^{6x^2} = 2^{10x+4}$ Power Rule, Exp.

$6x^2 = 10x + 4$ $b^x = b^y$ Theorem

$6x^2 - 10x - 4 = 0$

$$2(3x^2 - 5x - 2) = 0$$
$$2(3x + 1)(x - 2) = 0$$
$$x = 1/3 \text{ or } x = 2$$

Yes, say it, you love math!!!

You might be thinking that the problem was more difficult than the first one; $3^x = 81$. But in reality, it was not. It was longer because we solved a quadratic equation instead of a linear equation– not harder.

Algorithm for Solving Exponential Equations

1. **Express each side of the equation as a power in the SAME base.**
2. **Simplify the exponents**
3. **Set the exponents equal**
4. **Solve the resulting equation**

Example 3 Solve for n. $9^{n-1} = (1/3)^{4n-1}$

I have to write each side of the equation using the SAME base. I can write 9 as 9^1 or as 3^2.

Remember, that is equal to 3^{-1}

Therefore, I can write both sides having base 3

$9^{n-1} = (1/3)^{4n-1}$	Given
$(3^2)^{n-1} = (3^{-1})^{4n-1}$	Make the bases the same
$3^{2n-2} = 3^{-4n+1}$	Simplify exponents
$2n - 2 = -4n + 1$	$b^x = b^y$ Theorem
$6n = 3$	
$n = ½$	

Example 4 Solve for x. $9^{3x} = 27^{x-2}$

Since the bases are not the same, I can not set the exponents equal. So, can I make the bases equal?

$$9 = 3^2 \quad \text{and} \quad 27 = 3^3$$

Making those substitutions into the original equation, we have:

$(3^2)^{3x} = (3^3)^{x-2}$	- Substitution
$3^{6x} = 3^{3x-6}$	- Exp. (power rule)
$6x = 3x - 6$	- $b^x = b^y$ Theorem
$3x = -6$	
$x = -2$	

Sec. 2 Graphing Exponentials of the form $y = b^x$, $b > 1$

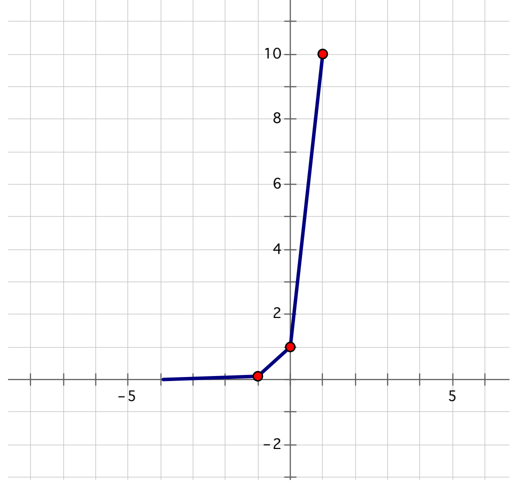
If I were to ask you to graph an exponential equation in two variables such as $y = 10^x$, my guess is you'd construct an x-y chart, plug in convenient values of x and find the corresponding values of y.

Example 1 Graph $y = 10^x$

x	-3	-2	-1	0	1	2	3
y	1/1000	1/100	1/10	1	10	100	1000

As you can see from the chart, the values of y get large very quickly. So quickly, it's almost impossible to actually plot the points. Who want to go over 3 and up 1000 to plot (3, 1000)?

So, to graph this function, I will plot a few key points to get an idea and extend the graph and

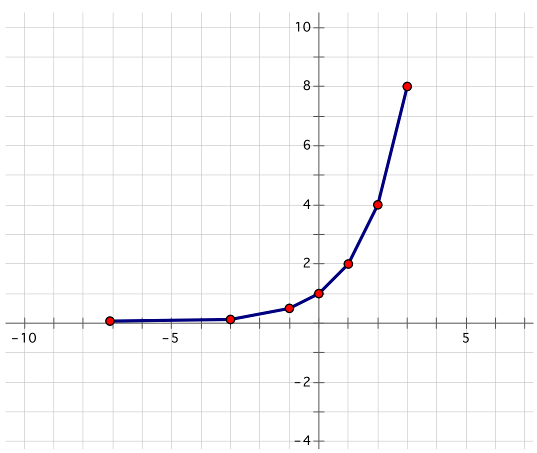


If I were to graph enough of these equations, we would begin to see an exponential equation of the form, $y = b^x$, all look pretty much the same when $b \geq 1$.

All the graphs would go through the point (0, 1), they would slide down to the left getting closer and closer to the x-axis but never touching it. The values of y are always positive no matter what values of x are chosen! If $x = 5$, then $y = 10^5$ or 100,000. If $x = –5$, the $y = 10^{–5}$ which is 1/10,000.

Example 2 Graph $y = 2^x$

Let x equal –3, –2, –1, 0, 1, 2, 3 and find the corresponding values of y. then plot those points



Since these are exponentials, just like in a geometric sequence, these numbers get very, very large – quickly.

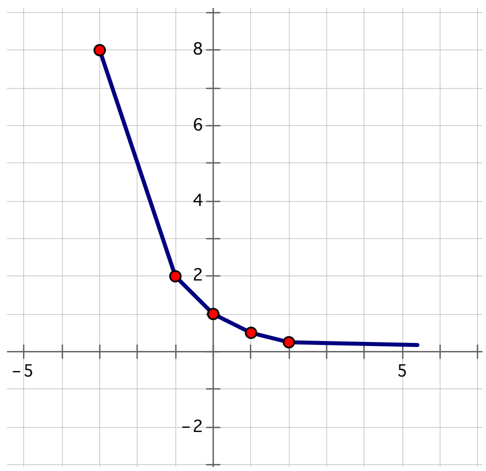
Graph the following.

1. $y = 3^x$ 2. $y = 4^x$ 3. $y = 6^x$

Sec 3 Graphing Exponentials of the form $y = b^x$, $0 < b < 1$

Now, I mentioned that $b \geq 1$, the question becomes, what happens if $y = b^x$, $0 < b < 1$? In other words, what happens if b is a fraction between zero and one?

Well, we could graph $y = (1/2)^x$ and see what occurs. Since any number to the zero power, except 0, equals 1, the graph should go through the point (0, 1) just like it did before. How else is the graph different? Well, as x gets larger, the values of y get smaller. It appears the graph gets closer and closer to the x-axis, but never touches it.



The biggest difference in the graphs is that one graph slides to the left, the other graph slides to the right.

Graph the following

1. $y = (1/3)^x$ 2. $y = (1/5)^x$ 3. $y = (1/10)^x$

Sec. 4 Extending the Laws of Exponents to Rational Numbers

So far, we have learned the laws of exponents dealing with Integers, positive and negative Whole Numbers. However, when we graph exponential equations, we typically connect the ordered pairs and make a nice curve as we did in the last two examples. That suggests that x can take on other values besides the set of integers.

We have seen that that integers can be used as exponents, when the base is not equal to zero. That is

$5^1 = 5$ $(-3)^2 = 9$ $7^0 = 1$ $4^{-2} = 1/16$

When you first studied exponents, you learned an exponent tells you how many times to use the base as a factor. That is, $5^3 = 5x5x5$. From there, we saw that when we multiplied numbers with the same base, rather than writing that all out, we could add the exponents and shorten our work. We also noticed that when we divided numbers with the same base, we could subtract the exponents. But that introduced some complications. When we divided a number by itself, that resulted in having an exponent of zero – that did not make sense. So, we had a choice, drop the new rule or adapt it to fit the rest of the math we previously learned.

We noticed any time we divided a number by itself, by the Multiplicative Inverse, the answer was one and by using the rule of exponents, we had an exponent of zero. Using Substitution, it was easy to see that any number to the zero power, except zero, would always be equal to one as well. The reason we have except zero is because we can't divide by zero in the first place. So we made a third rule.

The rule for dividing exponents led to another rule that involved negative numbers. A negative exponent does not make sense by itself, we can not use a base a negative number of times. We did see some relationships develop when we simplified exponentials using the subtraction rule and when we did it out the long way. That lead us to the fourth rule, $a^{-n} = 1/a^n$.
We will continue with this line of reasoning for rational exponents –

Let's look at $5^{½}$. Using the definition of an exponent, this does not make sense either. But, if we played with it, we might notice we can make it fit the rest of the mathematics we learned just as we did for negative and zero exponents.

The following should be true using the laws of exponents we already know.

$$(5^{½})^2 = 5^{½\,2} = 5^1 = 5$$

This clearly suggests that if I square $5^{1/2}$, I get 5. Or the square root of 5 is $5^{1/2}$.

$$(4^{1/3})^3 = 4^{(1/3)\,3} = 4^1 = 4$$

This suggests that the third root of 4 can be written as $4^{1/3}$.

These observations will lead to an extension of the laws of exponents so the exponents can be rational expressions.

If ***p*** is an integer, ***r*** is a positive integer, and ***b*** is a positive real number, then

$$b^{\frac{p}{r}} = (\sqrt[r]{b})^p$$

and

$$\boldsymbol{(b^p)^{1/r} = (b^{1/r})^p}$$

Let's see what all that means:

Example 1 Write in exponential form $\sqrt{5y}$

The index, when not written, is understood to be two.
So , in the fraction, the denominator is 2.

$$(5y)^{1/2}$$

Example 2 Write in $\sqrt[3]{27x^2y5}$ exponential form

$$= 27^{1/3}\,x^{2/3}\,y^{5/3}$$
$$= 3\,x^{2/3}\,y^{5/3}$$

Example 3 Write $(7y)^{1/2}$ in radical form.

$$= \sqrt{7y}$$

Example 4 Write $7^{1/3}\ x^{2/3}\ y^{1/3}\ z^{7/3}$ in radical form.

The index is 3. So all I have to do is label the index and place the appropriate exponents with each factor

$$\sqrt[3]{7^1 x^2 y^1 z^7}$$

Now, we typically do not write an exponent when it is 1. So the answer would look like

$$\sqrt[3]{7x^2 \cdot y \cdot z^7}$$