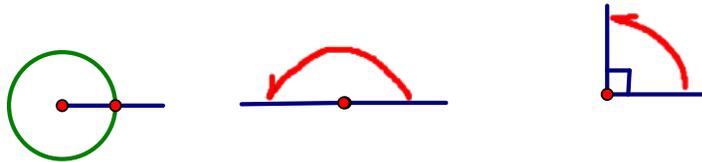


## Chapter 3

### Angles

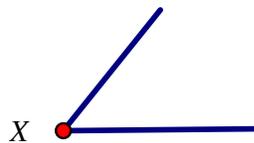
What's the secret for doing well in geometry? Knowing all the angles. As we did in the last chapter, we will introduce new terms and new notations, the building blocks for our success. Again, we will need take our time to familiarize ourselves with these and become comfortable using them.

An angle can be seen as a rotation of a ray about a fixed point. In other words, if I were mark a point on a paper, then rotate a pencil around that point, I would be forming angles. One complete rotation measures  $360^\circ$ . Half a rotation would then measure  $180^\circ$ . A quarter rotation would measure  $90^\circ$ .



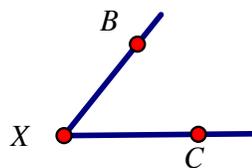
Let's use a more formal definition. An angle is the union of two rays with a common end point.

The common endpoint is called the vertex. Angles can be named by the vertex -  $\angle X$ .



That angle is called angle X, written mathematically as  $\angle X$ .

The best way to describe an angle is with three points. One point on each ray and the vertex always in the middle.



That angle could be NAMED in three ways:  $\angle X$ ,  $\angle BXC$ , or  $\angle CXB$ .

## Classifying Angles

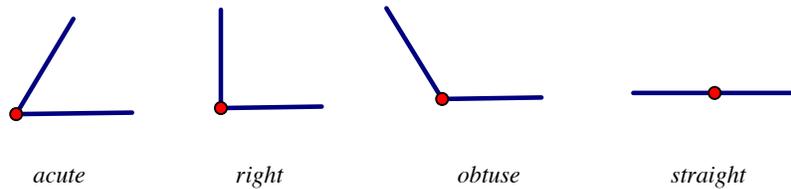
We classify angles by size.

Acute angles are angles less than  $90^\circ$ . In other words, not quite a quarter rotation.

Right angles are angles whose measure is  $90^\circ$ .

Obtuse angles are greater than  $90^\circ$ , but less than  $180^\circ$ . That's more than a quarter rotation, but less than a half turn.

Straight angles measure  $180^\circ$ .



Congruent angles are angles that have the same measure.

Before we continue with our study of angles, let me make a comment. You will find many similarities in math if you stop, think, reflect, visualize, and organize your thoughts. Let's look at the next few items and come back to this discussion, a discourse that will make learning mathematics easier.

### Postulate

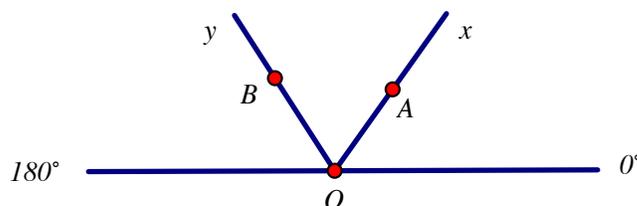
For every angle there is a unique number between 0 and 180 called the measure of the angle.

The next postulate parallels the Ruler Postulate introduced earlier.

### Protractor Postulate

The set of rays which have a common endpoint  $O$  in the edge of a half-plane, and which lie in the half-plane, can be paired with the numbers between 0 and 180 in such a way that:

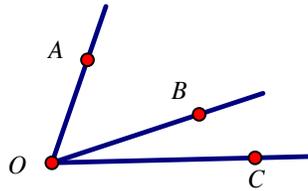
1. One of the rays in the edge is paired with 0 and the other is paired with 180;
2. If  $\overline{OA}$  is paired with  $x$  and  $\overline{OB}$  is paired with  $y$ , then  $m\angle AOB = |x - y|$



The next postulate, Angle Addition Postulate, parallels the Segment Addition Postulate introduced earlier. That is to say, an angle is equal to the sum of its parts.

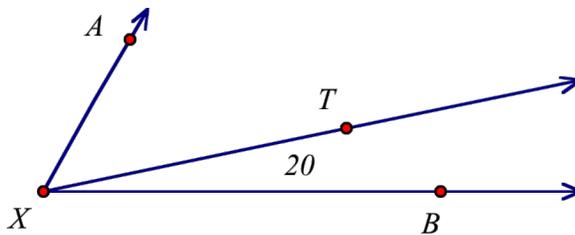
### Angle Addition Postulate

If B lies in the interior of  $\angle AOC$ , then  $m\angle AOC = m\angle AOB + m\angle BOC$



**Example 1** Given T lies in the interior of  $\angle AXB$  and the  $m\angle AXB = 60^\circ$  and  $m\angle TXB = 20^\circ$ , find  $\angle AXT$

As always, draw and label the picture.



Now, if the whole angle,  $m\angle AXB = 60^\circ$ , and  $m\angle TXB = 20^\circ$  then by the Angle Addition Postulate

$$m\angle AXB = m\angle AXT + m\angle TXB$$

Substituting we have

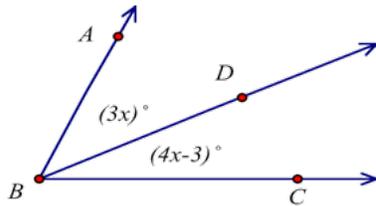
$$60^\circ = m\angle AXT + 20^\circ$$

$$40^\circ = m\angle AXT$$

Knowing that the entire angle measured  $60^\circ$  and part of it was  $20^\circ$ , I'm guessing that many students could have looked at the picture and determined that was  $40^\circ$  in their head. So why do we do the problem out? The answer is simple, we want to make sure you understand the concept so if you run into variations or problems that are described algebraically, you can still solve them.

**Example 2** Given that D lies on the interior of  $\angle ABC$  and the measure of  $m\angle ABD = (3x)^\circ$ ,  $m\angle ABC = 46^\circ$ , and the  $m\angle DBC = (4x - 3)^\circ$ , Find the value of x and  $m\angle DBC$ .

Always start by drawing the picture and labeling.



By the Angle Addition Postulate,  $\angle ABC = \angle ABD + \angle DBC$

$$46^\circ = (3x) + (4x - 3)$$

$$46^\circ = 7x - 3$$

$$49^\circ = 7x$$

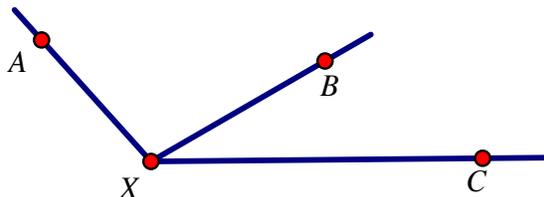
$$7^\circ = x$$

The Symmetric Property allows me to say  $x = 7$ . That's only part of the answer, we were also asked to find the  $m\angle DBC$ . Since  $x = 7$ , plug in 7 into  $4x - 3$ ;  $4(7) - 3 = 25^\circ$ .  $m\angle DBC = 25^\circ$

If you know your theorems, postulates and definitions, you just can not make math difficult. The more you know, the easier math gets!

## Angle Pairs

Adjacent angles are two angles that have a common vertex, a common side, and no common interior points.



$\angle AXB$  and  $\angle BXC$  are adjacent angles. They have a common vertex – X, they have a common side XB and no common interior points.

**Complementary angles - 2 angles while sum is  $90^\circ$ .** For instance, if  $\angle P = 40^\circ$  and  $\angle Q = 50^\circ$ , then  $\angle P$  and  $\angle Q$  are complementary angles. If  $\angle A = 30^\circ$ , then the complement of  $\angle A$  measures  $60^\circ$ .

**Supplementary angles - 2 angles whose sum is  $180^\circ$ .** If  $\angle M = 100^\circ$  and  $\angle S = 80^\circ$ , then  $\angle M$  and  $\angle S$  are supplementary angles.

**Example 1** Find the value of x, if  $\angle A$  and  $\angle B$  are complementary  $\angle$ 's and  $\angle A = (3x)^\circ$  and  $\angle B = (2x + 10)^\circ$ .

I don't need a picture to solve this problem since I know complementary angles are two angles whose sum is  $90^\circ$ .

$$\angle A + \angle B = 90^\circ$$

$$3x + (2x + 10) = 90^\circ$$

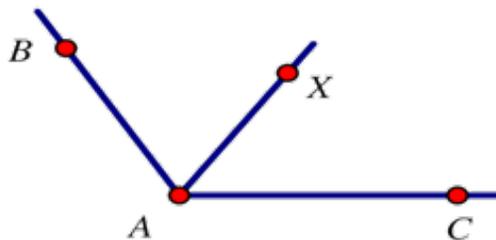
$$5x + 10 = 90^\circ$$

$$5x = 80$$

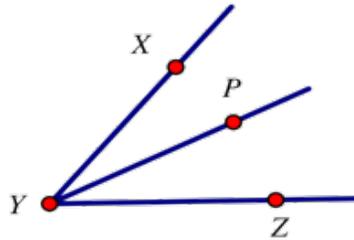
$$x = 16$$

If I were asked to find the measure of each angle, I would substitute 16 into those expressions.

**Angle bisector;**  $\overline{AX}$  is said to be the bisector of  $\angle BAC$  if X lies on the interior of  $\angle BAC$  and  $m\angle BAX = m\angle XAC$ .



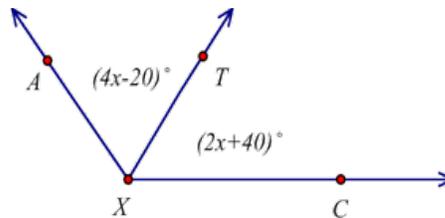
**Example 2** Given  $YP$  is an angle bisector and  $m\angle XYZ = 80^\circ$ , find the  $m\angle PYZ$ .



We know by the definition of angle bisector that the 2 angles have the same measure. We also know the whole angle  $= 80^\circ$ . Combining that with the Angle Addition Postulate, the  $m\angle PYZ = 40^\circ$

Let's look at another example involving an angle bisector, this time using algebraic expressions.

**Example 3**  $\overline{XT}$  bisects  $\angle AXC$ ,  $m\angle AXT = (4x - 20)^\circ$  and  $m\angle TXC = (2x + 40)^\circ$   
Find the value of  $x$  and  $m\angle TXC$



$\overline{XT}$  bisects  $\angle AXC$  which means it cuts the angle into two equal angles.

$$m\angle AXT = m\angle TXC$$

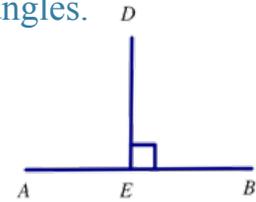
$$4x - 20 = 2x + 40$$

$$2x = 60$$

$$x = 30$$

To find the , we substitute 30 into the expression  $2x + 40$ . Therefore,  $m\angle TXC = 100^\circ$ .

Perpendicular lines are two lines that form right angles.

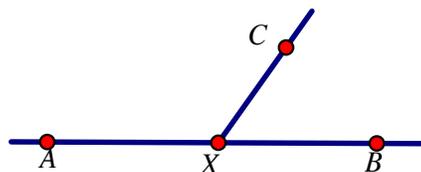


As in the study of any subject, success depends upon the acquisition of language, terminology and notation. In this chapter we introduced new terms and notation that is part of the language of mathematics. Not learning that language will result in a lot of difficulty later.

In the previous chapter, we studied points, lines and planes. Did you notice any similarities in the two chapters? In the last chapter we came across the [Segment Addition Postulate](#), in this chapter we learned about the [Angle Addition Postulate](#). In the last chapter we learned about a [midpoint](#) of a segment, in this chapter we learned about an [angle bisector](#). The point I'm trying to make is that those ideas are the same, they are identified and labeled differently because they are being used in a different context.

**Theorem** If the exterior sides of two adjacent angles lie in a line, then the angles are supplementary. Sometimes called a linear pair.

This theorem follows directly from the Protractor Postulate and is an important theorem that you will use over and over again that you will need to commit to memory.

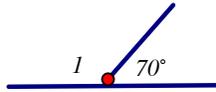


$\angle AXC$  and  $\angle CXB$  supplementary.

This theorem is extra important because it provides us with a relationship – an equation.

$$m\angle AXC + m\angle CXB = 180^\circ$$

**Example 1** Find the  $m\angle 1$ .



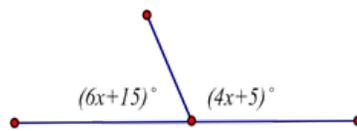
Since the exterior sides of these two adjacent angles lie in a line, they are supplementary angles.

$$m\angle 1 + 70^\circ = 180^\circ$$

$$m\angle 1 = 110^\circ$$

The good news is we just can't make these problems more difficult, but we can add a little algebra.

**Example 2** Given the figure, find the value of  $x$ .



Since the exterior sides of these 2 adjacent angles lie in a line,

$$(6x+15)^\circ + (4x+5)^\circ = 180^\circ$$

$$10x + 20 = 180$$

$$10x = 160$$

$$x = 16$$

The next theorems are just as straight forward. See if you can draw the pictures and talk your way through the theorems to convince other you are correct.

Theorem [An angle has exactly one bisector.](#)

Theorem [All right angles are congruent](#)

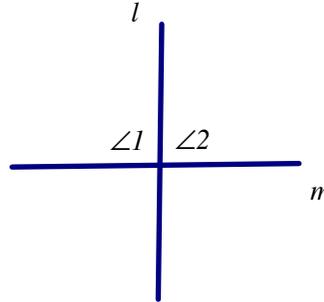
Theorem [If two lines are perpendicular, they form congruent adjacent angles.](#)

Let's prove this next theorem.

**Theorem** If two lines form congruent adjacent angles, the lines are perpendicular.

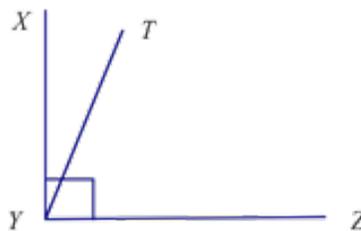
Given:  $\angle 1$  and  $\angle 2$  are  $\cong$

Prove:  $l \perp m$



	<u>STATEMENTS</u>	<u>REASONS</u>
1.	$\angle 1 \cong \angle 2$	Given
2.	$m\angle 1 + m\angle 2 = 180^\circ$	Ext sides lie in a line
3.	$2m\angle 1 = 180^\circ$	Substitution
4.	$m\angle 1 = 90^\circ$	Div Prop =
5.	$m\angle 2 = 90^\circ$	Substitution
6.	$l \perp m$	2 lines form right $\angle$ s

**Example 3** Given  $\overline{XY} \perp \overline{TZ}$  and  $m\angle XYT = 30^\circ$ , find  $m\angle TYZ$

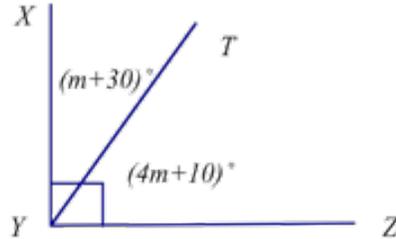


Using the Angle Addition Postulate and definition of perpendicular lines, we know that

$$\begin{aligned}
 m\angle XYT + m\angle TYZ &= 90^\circ \\
 30^\circ + m\angle TYZ &= 90^\circ \\
 m\angle TYZ &= 60^\circ
 \end{aligned}$$

Let's do that same problem, this time with algebra.

**Example 4** Given  $\angle XYT = (m+30)^\circ$ ,  $\angle TYZ = (4m+10)^\circ$ , find  $m\angle TYZ$ .



The only difference between this example and the previous one is these angle measures are represented algebraically. So we start off with the same equation.

$$\begin{aligned}m\angle XYT + m\angle TYZ &= 90^\circ \\(m+30) + (4m+10) &= 90 \\5m + 40 &= 90 \\5m &= 50 \\m &= 10\end{aligned}$$

Now, we were not asked to find  $m$ , we were asked to find  $m\angle TYZ$ .

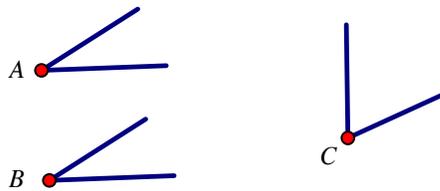
$$\begin{aligned}m\angle TYZ &= 4m + 10 \\&= 4(10) + 10 \\&= 50^\circ\end{aligned}$$

**Theorem** If the exterior sides of two adjacent acute angles lie in perpendicular lines, the angles are complementary.

This theorem parallels the previous theorem, if the exterior sides of two adjacent angles lie in a line, they are supplementary.

If you can visualize these theorems, it will help you remember them. Associating the visualizations of these theorems with other theorems will also help you recall information over time. The theorem whose exterior sides lie in a line – supplementary, if the exterior sides lie in perpendicular lines, the angles are complementary.

Theorem If two angles are complementary to the same angle, then they are congruent to each other.



Given:  $\angle A$  and  $\angle C$  are complementary.  
 $\angle B$  and  $\angle C$  are complementary.

Prove:  $\angle A \cong \angle B$

By definition of complementary angles, that means that  $\angle A + \angle C = 90^\circ$  and  $\angle B + \angle C = 90^\circ$ .

Since their sums both equal 90, then they are equal to each other

$$\angle A + \angle C = \angle B + \angle C .$$

$$\angle A \cong \angle B$$

Theorem If two angles are complementary to two congruent angles, then they are congruent to each other.

With a little thought, we can extend the ideas expressed in those two theorems to include supplementary angles.

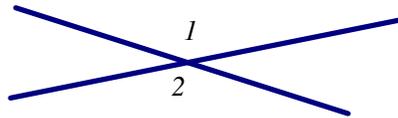
Theorem If two angles are supplementary to the same angle, then they are congruent to each other.

Theorem If two angles are supplementary to two congruent angles, then they are congruent to each other.

Exactly the same reasoning would be required to demonstrate the last two theorems.

## Vertical Angles

Vertical angles are formed by intersecting lines, they are non-adjacent angles. The mathematical definition of vertical angles is: two angles whose sides form pairs of opposite rays.

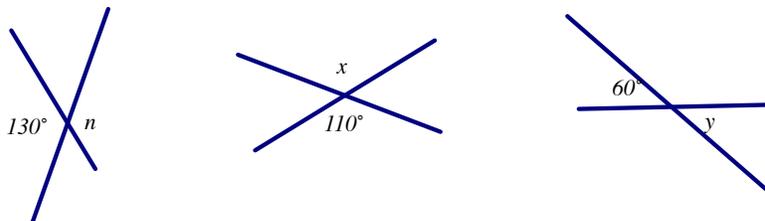


$\angle 1$  and  $\angle 2$  are a pair of vertical angles.

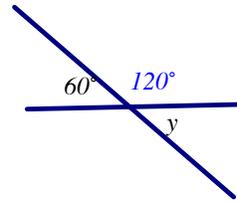
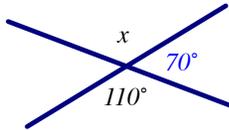
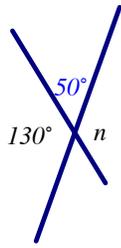
An observation we might make if we were to look at a number of vertical angles is they seem to be equal. We might wonder if they would always be equal.

Well, I've got some good news for you. We are going to prove vertical angles are congruent. Proving something is true is different than showing examples of what we think to be true.

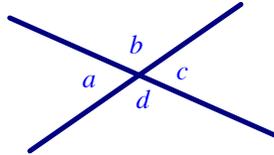
If we are going to be successful in geometry, then we have to have a body of knowledge to draw from to be able to think critically. What that means is we need to be able to recall definitions, postulates, and theorems that we have studied. Without that information, we are not going anywhere. So every chance you have, read those to reinforce your memory. And while you are reading them, you should be able to visualize what you are reading.



Can you find the values of  $n$ ,  $x$ , and  $y$ ? How were you able to make those calculations? In all those drawings, we had two adjacent angles whose exterior sides lie in a line. That means they are supplementary, their sums are  $180^\circ$ . Filling in those angles would then help you find the measures of  $\angle n$ ,  $\angle x$ , and  $\angle y$ .



**Example 1** Name two pairs of vertical angles.



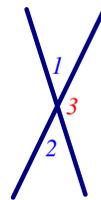
$\angle a$  and  $\angle c$  is one pair, the other pair is  $\angle b$  and  $\angle d$

**Theorem - Vertical angles are congruent**

To prove this theorem, we write the statement, draw and label the picture describing the theorem, write down what is given, write down what we are supposed to prove, and finally prove the theorem.

Given:  $\angle 1$  and  $\angle 2$  are vertical angles

Prove:  $\angle 1 \cong \angle 2$



If I just labeled  $\angle 1$  and  $\angle 2$  in the diagram, I would be stuck. Notice, and this is important, by labeling  $\angle 3$  in the picture, I can now use a previous theorem – If the exterior sides of 2 adjacent angles lie in a line, the angles are supplementary. That would mean  $\angle 1$  and  $\angle 3$  are supplementary and  $\angle 2$  and  $\angle 3$  are supplementary because their exterior sides lie in a line. If I didn't know my definitions and theorems, there is no way I could do the following proof.

After drawing the picture and labeling it, I will start by writing down what's given as Step 1. My second and third steps follows from the picture about supplementary angles, and my last step is what I wanted to prove.

<u>Statements</u>	<u>Reasons</u>
1. $\angle 1$ and $\angle 2$ are vert $\angle$ s	Given
2. $\angle 1$ and $\angle 3$ are supp $\angle$ s	Ext sides, 2 adj $\angle$ 's in a line
3. $\angle 2$ and $\angle 3$ are supp's	Same as #2
4. $\angle 1 = \angle 2$	Two $\angle$ 's supp to same $\angle$

A proof is nothing more than an argument whose conclusion follows from the argument. Proofs can be done differently, all we care about is the conclusion follows from the argument - that it is valid - and you can follow it with understanding.

If I believe the audience can follow what I am doing in the proof, I leave out some steps. For instance, if 2 angles are congruent, then by definition, they are equal. I interchange congruence and equality - I leave out the step. In the proof above,  $\angle 3$  was not part of the original problem, so I just wrote it in there because I needed it. In the proof below, I used the Reflexive Property as a reason to use  $\angle 3$ .

Take note, the first proof was completed in 4 steps, the second proof for the same theorem was completed in 8 steps.

Let's look at another way someone might prove vertical angles are congruent. I might suggest that as you begin to prove theorems, you write the statement, draw and label the picture, put more information into the picture based upon your knowledge of geometry, write down what is given, and what it is you are going to prove.

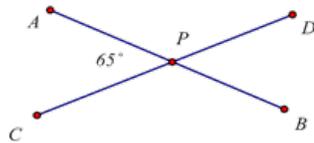
Now you are ready to go, make your T-chart. Your first statement could be to write down what is given, the last step will always be what you wanted to prove.

<u>Statements</u>	<u>Reasons</u>
1. $\angle 1$ and $\angle 2$ are vert's	Given
2. $\angle 1 = \angle 2$	Def of Congruence
3. $\angle 3 = \angle 3$	Reflexive Prop
4. $\angle 1$ and 3 are sup $\angle$ s	Ext sides, 2 adj $\angle$ 's in a line
$\angle 2$ and $\angle 3$ are sup $\angle$ s	
5. $\angle 1 + \angle 3 = 180^\circ$ $\angle 2 + \angle 3 = 180^\circ$	Def of supp $\angle$ s
6. $\angle 1 + \angle 3 = \angle 2 + \angle 3$	Substitution
7. $\angle 1 = \angle 2$	Subtraction Prop of Equality
8. $\angle 1 \cong \angle 2$	Def of congruence

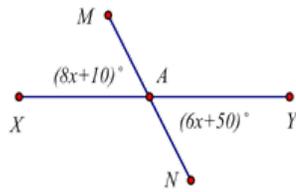
This proof is clearly longer than the first way we proved it, but the conclusion still follows from the argument.

### Example 2

Find the  $m\angle BPD$  .



Since  $\angle BPD$  and  $\angle APC$  are vertical angles, they are congruent, so  $m\angle BPD = 65^\circ$

**Example 3**Find the measure of  $m\angle MAX$ 

$\angle MAX$  and  $\angle NAY$  are vertical angles. We know vertical angles are congruent – they have the same measure.

$$\begin{aligned}8x + 10 &= 6x + 50 \\2x &= 40 \\x &= 20\end{aligned}$$

$$\begin{aligned}m\angle MAX &= (8x+10)^\circ \text{ was given to us, substituting } x = 20, \\m\angle MAX &= 170^\circ\end{aligned}$$

There is information in this section that provide relationships we need to know to be successful – Angle Addition Postulate, complementary angles, supplementary angles, angle bisector, exterior sides of an angle lie in a line, exterior sides of an angle lie in perpendicular lines and vertical angles.

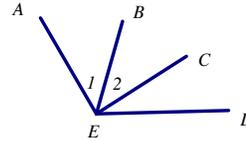
Theorem Through a point outside a line, exactly one parallel can be drawn to that line.

Theorem Through a point outside a line, exactly one perpendicular can be drawn to that line.

By memorizing and being able to visualize previous definitions, postulates, and theorems, being able to do proofs will be a great deal easier.

## Review Questions

1. The vertex of  $\angle RST$  is point
2. In the plane figure shown,  $\angle 1$  and  $\angle 2$  are \_\_\_\_\_ angles.



3. How many angles are shown in the figure?
4. In the plane figure shown,  $m\angle AEC + m\angle CED$  equals
5. If EC bisects  $\angle DEB$  and the  $m\angle DEC = 28^\circ$ , then  $m\angle CEB$  equals
6. If  $m\angle 1 = 30^\circ$  and the  $m\angle 2 = 60^\circ$ , then  $\angle 1$  and  $\angle 2$  are
7. If  $m\angle 1 = 3x$  and the  $m\angle 2 = 7x$ , and  $\angle 1$  is a supplement of  $\angle 2$ , then  $x =$
8. If the exterior sides of two adjacent angles lie in perpendicular lines, the angles are
9. If  $\angle 1$  is complementary to  $\angle 3$ , and  $\angle 2$  is complementary to  $\angle 3$ , then
10.  $\angle T$  and  $\angle A$  are vertical angles. If  $m\angle T = 2x + 8$  and  $m\angle A = x + 22$ , then  $x =$
11. Name the 5 components of a proof.