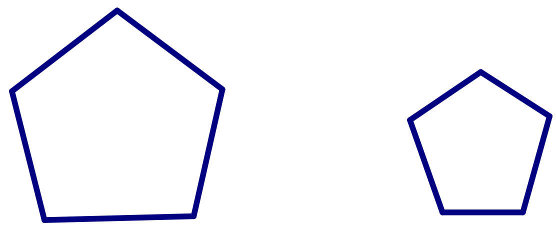
Chapter 10 Similar Polygons

Sec 1. Similar Polygons

When two polygons have the same shape and only differ in size, we say they are similar polygons.



These two pentagons are similar. More formally:

Two polygons are similar if and only if there is a one-to-one correspondence between their vertices such that:
the corresponding angles are congruent, and
the lengths of the corresponding sides are proportional

So just like congruence, similar polygons have the same shape, the corresponding angles are congruent, but their sides are not congruent, they are in proportion.

Since congruent polygons had the same shape and size, we used the congruence symbol $\cong$. And we would write $\Delta ABC \cong \Delta DEF$.

With similar polygons, the size changes so we no longer have the equality, so we take out the = signs in the congruence symbol and if triangles are similar, we would write $\Delta ABC \sim \Delta DEF$ and say triangle ABC is similar to triangle DEF.

Taking this information with the properties of proportion we just learned, we learn more and be better able to apply the geometry we are learning.

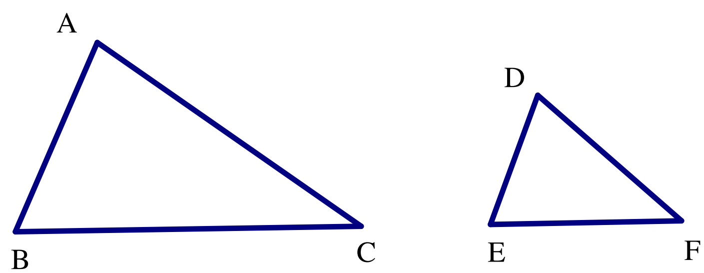
Let's look at a theorem,

Theorem **If two polygons are similar, the ratio of their perimeters equals the ratio of the lengths of any two corresponding sides.**

To prove this theorem, all we have to do is use the last property of proportions we studied in math.

$$\text{if } \frac{a}{b}=\frac{c}{d}, \text{ then } \frac{a}{b}=\frac{a+c}{b+d}$$

As always, the best way to address any new information is by drawing and labeling a picture



The last property of proportion allows me to take the sum of the numerators and place that result over the sum of the denominators. Those sums are the perimeters of the two triangles.

$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF} \to \frac{AB+BC+AC}{DE+EF+DF}$$

$$\frac{Perimeter\,\Delta ABC}{Perimeter\,\Delta DEF}=\frac{AB}{DE}$$

Example 1 If two similar polygons have perimeters of 24 inches and 16 inches respectively and side A of the larger polygon measures 6 inches, what is the length of the corresponding side of the smaller rectangle.

Since the polygons are similar, their sides are in proportion AND from the previous theorem we know the perimeters also have the same ratio as the sides.

$$\frac{24}{16} = \frac{6}{x}$$

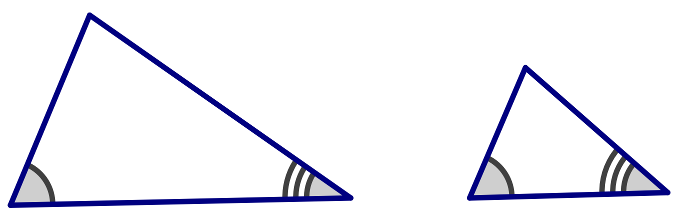
Cross multiplying; $24x = 96$, so $x = 4$

If we reduced 24/16, then set up the equation, that would have made the numbers a lot smaller and probably easier to work with.

Angle – Angle Postulate

Up to this point, the only way we could show triangles similar was by the definition. The next postulate is really going to make our lives easier with respect to triangles.

AA Postulate **If two angles of one triangle are congruent to two angles of another triangle, the triangles are similar.**



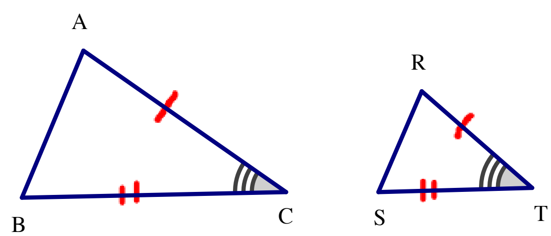
The reason this is a postulate is that we cannot prove it, but as we examine more and more cases, this seems to hold up. Since it does, we accept it as true without proof – a postulate.

Any postulate or theorem that has a name is generally important, so make sure you know what this one says and means. Like parallel lines theorems and postulates, this is a shortcut. Rather than showing all angles are congruent and the corresponding sides are in proportion, all I have to is show two angles congruent.

Using this postulate, let's see what doors that can open to make more observations.

SAS Theorem **If an angle of one triangle is congruent to an angle of another triangle and the lengths of the sides including those angles are proportional, the triangles are similar.**

$$Given: \Delta ABC \text{ and } \Delta RST \text{ with } \angle C \cong \angle T, \frac{AC}{RT} = \frac{BC}{ST}$$

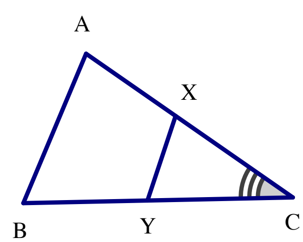


Prove: $\Delta ABC \sim \Delta RST$

To prove that theorem, I will have to use information I know. The first is to draw and label the picture. I know if 2 angles are equal, then the triangles are similar. I only have one angle given. Now what I can do is pick a point X on AC so that

TR = CX, then draw XY || AB.

Let's see what that looks like and how that allows me to play with the geometry I already know.

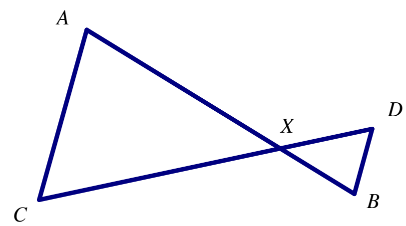


$\angle CXY \cong \angle CAB$	|| lines being cut by t, the corresponding $\angle$s $\cong$.
$\Delta ABC \sim \Delta XYC$	AA Postulate

$\frac{AC}{XC} = \frac{BC}{YC}$	~Δs, sides in proportion
XC = RT	Construction
$\frac{AC}{RT} = \frac{BC}{YC}$	Substitution
YC = ST	Solve for YC/Substitution
ΔXYC ≅ ΔRST	SAS
ΔABC ~ ΔRST	Substitution

Example 2

Given: $\overline{AC} \parallel \overline{BD}$
Prove: ΔAXC ~ ΔBXD



	Statements	Reasons
1.	AC ∥ BD	Given
2.	∠A ≅ ∠B	2 ∥ lines cut by a transversal, alt int. angles are congruent
3.	∠AXC ≅ ∠BXD	Vertical angles are congruent
4.	ΔAXC ~ ΔBXD	Angle- Angle Postulate

Once we know triangles are similar, we then know by definition the sides are in proportion. This will allow us to find lengths of sides of triangles.

So if we wanted to find lengths of sides, we would be able to write the proportions. Here's just one example of a proportion, more can be written.

$$\frac{AX}{BX} = \frac{XC}{XD}$$

Notice, just like in naming congruent triangles, the order of the letters matters. So when you look how the similar triangles were named, that same order will dictate the proportions.

Applications: Similar ’s

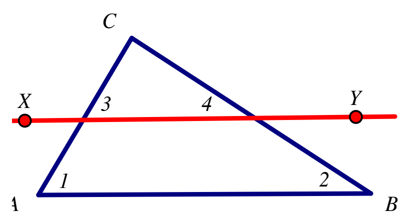
We learned the sides of similar polygons are in proportion. Using that information, we are able to solve problems.

If we continued in our study, we might be able to draw some more conclusions based on our observations.

Theorem **If a line is parallel to one side of a triangle and intersects the other two sides, it divides them proportionally.**

Given: ΔABC, $\overline{XY} \parallel \overline{AB}$

Prove: $\frac{AX}{XC} = \frac{BY}{YC}$



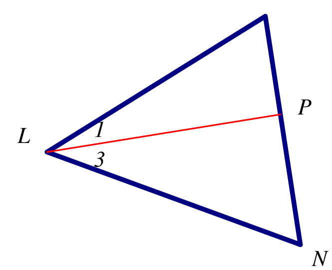
	Statements	Reasons
1.	$\overline{XY} \parallel \overline{AB}$	Given
2.	$\angle 1 \cong \angle 3$, $\angle 2 \cong \angle 4$	|| lines cut by t. corr. angles
3.	$\Delta ABC \sim \Delta XYC$	AA Postulate
4.	$\frac{AC}{XC} = \frac{BC}{YC}$	Similar Δ’s
5.	$AX + XC = AC$ $BY + YC = BC$	Segment Addition Thm
6.	$\frac{AC - XC}{XC} = \frac{BC - YC}{YC}$	Prop of pro.
7.	$AC - XC = AX$ $BC - YC = BC$	Sub. prop =.
8.	$\frac{AX}{XC} = \frac{BY}{YC}$	Substitution

Following as a direct result of that theorem, we have the following corollary.

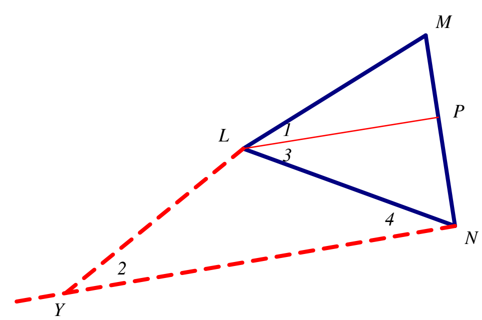
Corollary **If three parallel lines intersect two transversals, they divide them proportionally.**

Moving along, we have another theorem.

Theorem **If a ray bisects an angle of a triangle, it divides the opposite side into segments whose lengths are proportional to the lengths of the other two sides.**



As I initially look at the figure, I'm not sure how to proceed. So, using the idea of changing something I do not recognize into something I know how to do, I adjust my diagram so it fits into my previous knowledge.



Like many other theorems, I will use my knowledge of geometry to make this problem recognizable. We just proved that a line parallel to one side of a triangle, divides the sides proportionally. So, I will create that same condition by constructing $\overline{YN} \parallel \overline{LP}$ and extending $\overline{LM}$ - see red dashed line segments above.

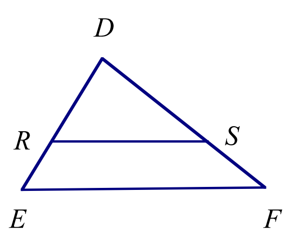
Now we are ready for the proof.

1.	Δ LMN $\overline{LP}$ bisects $\angle$MLN	Given
2.	Extend $\overline{ML}$, $\overline{YN} \parallel \overline{LP}$	Construction
3.	$\frac{MP}{PN} = \frac{LM}{LY}$ for ΔMYN	line $\parallel$, divides Δ pro
4.	$\angle 1 = \angle 2$	Corr $\angle$'s
5.	$\angle 1 = \angle 3$	Def $\angle$bisector
6.	$\angle 3 = \angle 4$	Alt int $\angle$'s
7.	$\angle 2 = \angle 4$	Sub
8.	$\overline{LY} \cong \overline{LN}$	Base $\angle$'s , opp. sides
9.	$\frac{MP}{PN} = \frac{LM}{LN}$	Sub in step 3

In this proof, we drew the picture, then we added on to that picture so we could use our previously learned knowledge of geometry. We constructed a parallel to the angle bisector,. then drew line NY parallel to the angle bisector, we extended line segment ML

Isn't this stuff neat? These theorems and postulates will allow us to find lengths of line segments using similar triangles.

Example 3 Given $\overline{RS} \parallel \overline{EF}$, DR = 4, RE = 5, DS = 5, find SF.

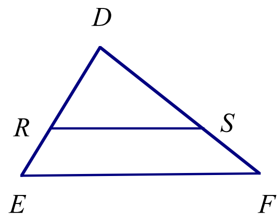


$$\frac{DR}{RE} = \frac{DS}{SF} \qquad\qquad \frac{4}{5} = \frac{5}{SF}$$

Substituting the known values into that proportion, we have
Cross multiplying, 4SF = 25 or SF = 25/4 or 6 ¼

Let's look at the same problem using algebra.

Example 4 Given $\overline{RS} \parallel \overline{EF}$, DR = x+2, RE = 4x–2, DS = 4x–2, SF = 5x–1, find the value of x.



$$\frac{DR}{RE} = \frac{DS}{SF}$$

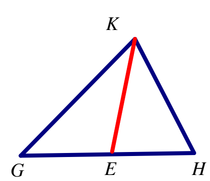
Substituting the algebraic expressions into that proportion, we have by cross multiplying,

$$(x+2)(5x-1) = (4x-2)(4x-2)$$
$$5x^2 + 9x - 2 = 16x^2 - 16x + 4$$
$$0 = 11x^2 - 25x + 6$$
$$0 = (11x - 3)(x - 2)$$

$$\text{so } x = \{2, 3/11\}$$

Notice, this problem was set up the same way as the last example, the only difference was in order to find the value of x, I had to solve a quadratic equation.

Example 5 Given: $\overline{KE}$ bisects $\overline{GH}$, GE = 8, EH = 5, GK = 12, find KH.



By theorem, we know: $\frac{GE}{EH} = \frac{GK}{KH}$

Substituting the given values for those line segments, we have

$$\frac{8}{5} = \frac{12}{KH}$$

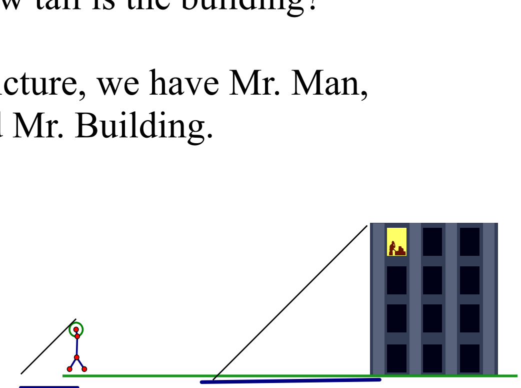
Cross multiplying $8KH = 60$

$KH = 7\frac{1}{2}$

Let's find the height of a building using similar triangles.

A 6 foot tall person casts a shadow 2 feet long. A building's shadow is 30 feet long, how tall is the building?

Drawing s picture, we have Mr. Man, Mr. Sun, and Mr. Building.



Because of the sun's distance from the earth, the top angles will be the same. We also have right angles formed by the ground, therefore by the AA Postulate, the triangles are similar.

Setting up the proportion:

$$\frac{height_{\text{ man}}}{shadow_{\text{ man}}} = \frac{height_{\text{ bldg}}}{shadow_{\text{ bldg}}}$$

$$\frac{6}{2} = \frac{h}{30}$$

$$2h = 90$$

The height of the building is 90 feet.