

## Ch. X5 Logic, An Introduction

Logic allows us to determine if an argument is valid; in other words, to determine if the conclusion follows from the argument. Heck, I thought you won arguments by speaking last or speaking louder.

As always, we'll need to define some terms. You will see some of these have been defined in two of the previous sections, *Sets* and *Venn Diagrams*. And there is some more good news: some of the symbols we used in those sections will also be used in this section.

So let's get started. A **statement** is a declarative sentence that is either true or false, but not both. For example: "The word "cat" begins with the letter c." That's a true statement. An example of a false statement is: " $2 + 3 = 6$ ."

If you do not have a declarative sentence, it's not a statement. A subjective sentence is not considered a statement because, depending upon the person, it could be viewed as either true or false. For instance, if I say: "I'm very good looking." While that's a declarative sentence, it is subjective.

Statements are represented symbolically by lowercase letters, usually  $p, q, r, s \dots$

### Logical Connectors

Two or more statements can be joined or connected to form compound statements. The most common of which are "and", "or", "if, then", and "if and only if" statements.

You'll feel a little more at ease if you understand we have already defined some of these in the *Sets* and *Venn Diagrams* sections. The good news is the definitions don't change.

### Truth Tables

A truth table allows us to look at compound statements and determine when they are true. If  $p$  is the statement, "It is snowing" and  $q$  is the statement, "The sun is shining" then the conjunction of  $p$  and  $q$  is "It is snowing and the sun is shining." A **conjunction** joins two statements by using the word "and".

Symbolically, we write  $p \wedge q$ . Notice the "and" symbol,  $\wedge$ , closely resembles the symbol we use for intersecting two sets,  $\cap$ .

To make a truth table, we examine all the possibilities when joining those two statements. Both  $p$  and  $q$  could be true. Both  $p$  and  $q$  could be false. Statement  $p$  could be true and  $q$  could be false, or statement  $p$  could be false and  $q$  true. Wow! Let's use a truth table to show those relationships. But before we do, remember back to how "and" was defined using sets. In sets, "and" was used to show the intersection of two sets. Both conditions must be met to be in the intersection.

What that means to us is that in order for  $p \wedge q$  to be true, both conditions must be true.

$p$	$q$	$p \wedge q$	<b>AND</b>
T	T	T	
T	F	F	
F	T	F	
F	F	F	

We could have done the same thing with sets by using  $\in$  as True and  $\notin$  as False.

$p$	$q$	$p \cap q$	<b>INTERSECTION</b>
$\in$	$\in$	$\in$	
$\in$	$\notin$	$\notin$	
$\notin$	$\in$	$\notin$	
$\notin$	$\notin$	$\notin$	

By looking at those tables, we can see the tables are equivalent.

Two statements joined by "or" are called a **disjunctive statement**. The symbol for "or",  $\vee$ , resembles the symbol we used for joining two sets, set union,  $\cup$ .

Union was defined when an element belonged to one or both sets. An "or" statement is true under those same conditions, when one or both of the statements are true.

$p$	$q$	$p \vee q$	<b>OR</b>
T	T	T	
T	F	T	
F	T	T	
F	F	F	

We can see an "or" statement is true when at least one of the statements is true. When both statements  $p$  and  $q$  are false, the disjunction is false.

In sets, we also talked about the complement of a set. So if  $B$  was a set, then  $\sim B$  was everything in the universal set, not in  $B$ . We are going to use the same notation in logic. If  $p$  is a statement, then  $\sim p$  negates the statement; they have opposite truth values.

Earlier, we defined  $p$  as “It is snowing.” How do you think we will interpret  $\sim p$ ? That’s right, “It is not snowing.”

Let’s look at a truth table that contains negations and see how that plays out.

$p$	$q$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Noticed we negated the  $p$  and  $q$  statements independently, then we used the definition of  $\vee$  to join the  $\sim p$  to the  $\sim q$  statements.

Now, let’s say I want to negate the statement,  $p \vee q$  how would I do that? Is that the same as  $\sim p \vee \sim q$ ?

A truth table will help me with that. We’ll copy the truth table we did for “or” statements before, then negate the disjunction on the next column.

$p$	$q$	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Notice, to find  $\sim(p \vee q)$ , I first found  $p \vee q$ , then I used the opposite truth value. Piece of cake!

Let’s combine all these truth tables and see what we have.

$p$	$q$	$\sim p$	$\sim q$	$p \wedge q$	$p \vee q$	$\sim p \vee \sim q$	$\sim(p \vee q)$
T	T	F	F	T	T	F	F
T	F	F	T	F	T	T	F
F	T	T	F	F	T	T	F
F	F	T	T	F	F	T	T

Now, if we look at the columns that describe the truth values of  $\sim(p \vee q)$ , we see that it is NOT the same as  $\sim p \vee \sim q$ . That means those two statements are not equivalent. Let's add the negation of  $p \wedge q$  to our table. Remember, you can not make these more difficult – just longer.

$p$	$q$	$\sim p$	$\sim q$	$p \wedge q$	$p \vee q$	$\sim p \vee \sim q$	$\sim(p \vee q)$	$\sim(p \wedge q)$
T	T	F	F	T	T	F	F	F
T	F	F	T	F	T	T	F	T
F	T	T	F	F	T	T	F	T
F	F	T	T	F	F	T	T	T

To get that last column, I merely took the opposite truth values already listed under  $p \vee q$ .

If you looked at this long enough, you might find something interesting. Notice that the truth values for  $\sim(p \wedge q)$  are exactly the same as the values under  $\sim p \vee \sim q$ . We say those two statements are logically equivalent. That means we can interchange them in an argument without changing the meaning.

**Important:** To find the negation of a statement, you change the truth value of the statement. To find the negation of a compound statement, you first combine the statements, and then change their truth values.

Find the truth values for  $\sim p \wedge \sim q$ . Is that the same as  $\sim(p \wedge q)$ ? The answer is no.

Do you see anything interesting if you were to add those two to the previous table?

Hopefully, you saw that  $\sim p \vee \sim q$  has the same truth values as  $\sim(p \wedge q)$ . Therefore, those statements are logically equivalent.

## DeMorgan's Laws

$\sim p \vee \sim q$  and  $\sim(p \wedge q)$  are logically equivalent.

$\sim p \wedge \sim q$  and  $\sim(p \vee q)$  are logically equivalent.

While all this might look impressive, all we have done is combined statements by using definitions we learned for “or” and “and” in the previous sections and assigning truth values.

## Conditionals

Another way to connect two statements is by a conditional. Those are statements in the form of “if  $p$ , then  $q$ .” That’s written,  $p \rightarrow q$ . The “if” part of the statement or implication is called the **hypothesis**, and the “then” part is called the **conclusion**.

Many statements can be written in “if, then” form. For instance, “All second graders are 7 years old.” could be rewritten as: “If a child is a second grader, then the child is 7 years old.”

Now to determine the truth values, I need to think of the conditional statements as a promise. If a promise is kept, the statement is true. If the promise is broken, the statement is false.

This is not hard, but I do need you to stay with me on this. Let’s say I say to you, “If you are good, then I will take you to the movies.”

“You being good” is statement  $p$ , and “going to the movies” is statement  $q$ . There are four possibilities.

$p$	$q$	
T	T	you are good; you go to the movies
T	F	you are good; you don’t go to the movies
F	T	you were not good; you go to the movies
F	F	you were not good; you don’t go to the movies

Did I break my promise in any one of those cases? Yes, in the second case, you were good, but you didn’t go the movies.

This is important, in the third case you were not good, but I still took you to the movies.

Did I break my promise? No, the promise was not broken. The promise was not broken in the fourth case either. So the only statement that would be false is statement two. Here’s the truth table.

$p$	$q$	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

This bugs people. A false implies a false is considered a true statement, but that’s the way we are defining it, in the form of breaking a promise. See, definitions are really important for understanding.

The only time a conditional is false is when a True  $\rightarrow$  False; all other times the conditional statement is true.

Let's make a truth table for some other "if, then" statements.

				Implication	Converse	Inverse	Contrapositive
$p$	$q$	$\sim p$	$\sim q$	$p \rightarrow q$	$q \rightarrow p$	$\sim p \rightarrow \sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Do you see any equivalent statements? In other words, do any have the same truth values? Yes!

OK, we can see that an implication and the contrapositive are equivalent statements. We can also see the converse and the inverse are equivalent.

Let's see what happens if I connect the implication and its converse with the word "and".

From the above table, I have:

$p$	$q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$ or $p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The last statement is called a biconditional and is usually read " $p$  if and only if  $q$ ".

We can continue to combine compound statements by using the truth values associated with conjunctions, and implications. The problems don't get more difficult, just longer.

## Validity

A valid argument is an argument in which the conclusion must be true whenever the hypothesis is true.

An argument is a set of statements in which one of the statements is the conclusion and the rest of the statements make up the hypothesis.

Let's look at the following truth table and determine if the argument  $[(p \rightarrow q) \wedge q] \rightarrow p$  is valid or invalid.

We start with our simple truth values for  $p$ ,  $q$ ,  $p \rightarrow q$ ,  $(p \rightarrow q) \wedge q$ , then we join that with our conclusion.

$p$	$q$	$p \rightarrow q$	$(p \rightarrow q) \wedge q$	$[(p \rightarrow q) \wedge q] \rightarrow p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Notice, the last column has all the statements combined in the hypothesis in brackets. Since I have an “F” in this last column, that argument is invalid.

Let’s look at a slightly different argument:

$p$	$q$	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

In this argument, notice that I have all “T’s”, this argument is valid.

In summary, to determine if an argument is valid;

1. Find the truth values for each statement in the hypothesis of the argument.
2. Combine all the statements in the hypothesis using “and” or “or” as the connector and determine those truth values.
3. Using the truth values from step 2, with the truth values of the conclusion; determine if the implication for each condition is true.

This takes a little patience, it does not get more difficult, it just takes more space.

**Remember:**

For a conjunction to be true, both statements must be “True”.

The only time a disjunction is false is when both statements are “False”.

The only time an implication is false is when you have a “True” implying a “False”.