Chapter 11

Circle Theorems

A circle is a set of points in a plane that are a given distance from a given point, called the center. The center is often used to name the circle.

This circle shown is described as \( \bigcirc T \).

As always, when we introduce a new topic we have to define the things we wish to talk about.

A radius of a circle is a segment that joins the center of a circle to a point on the circle. \( \overline{TD} \)

A chord is a segment whose endpoints lie on the circle; \( \overline{AG} \). A diameter is a chord that contains the center; \( \overline{CE} \). A secant is a line that contains a chord;

A tangent to a circle is a line that lies in the plane of the circle and intersects the circle in exactly one point.

A central angle of a circle is an angle whose vertex is the center of the circle.

Angle 1 is a central angle of \( \bigcirc A \)
A **minor arc** of a circle is the union of two points on the circle and all the points of the circle that lie in the interior of the central angle whose sides contain the two points.

The measure of a minor arc is defined to be the measure of its central angle.

What that means is that if the measure of the minor arc is 50°, then the measure of \( \angle 1 \) is 50°.

A **semicircle** is the union of the endpoints of a diameter and all the points of the circle lying on one side of the diameter.

An **inscribed angle** is an angle whose vertex lies on the circle and whose sides contain chords of a circle.

Before we start, let’s look at two theorems.

**Theorem**  In the same or congruent circles, if two central angles are congruent, their arcs are congruent.

**Theorem**  In the same or congruent circles, if two minor arcs are congruent, the central angles are congruent.

If I played long enough with arcs and chords, I would find that congruent arcs have congruent chords and congruent chords have congruent arcs. Those would be easily proven using the congruence theorems for triangles.
**Theorem**  In the same or congruent circles, congruent chords have congruent arcs.

**Theorem**  In the same or congruent circles, congruent arcs have congruent chords.

**Theorem**  A diameter that is perpendicular to a chord bisects the chord and its two arcs.

To prove this theorem, we draw the picture, draw lines so triangles are formed, prove the triangles are congruent by HL Congruence Postulate, the rest falls into place nicely.

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**Theorem:**  A diameter that is \( \perp \) to a chord bisects the chord and its 2 arcs

Given: Circle O  
\( AB \perp XY \)

Prove:  
\( XK \cong YK \)  
\( \widehat{XB} = \widehat{YB} \)

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<td>Give</td>
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| 6. \( XK \cong YK \)  
\( \angle 1 \cong \angle 2 \) | cpctc |
| 7. \( \widehat{XB} \cong \widehat{YB} \) | 2 central \( \angle \)'s are \( \cong \), the arcs are \( \cong \) |

The only information we have about the measure of angles formed with a circle is the definition of a central angle. The measure of the central is defined to be the measure of the arc that it intercepts. So
if we want to find the measure of other angles, it would seem we would have to somehow the angle, such as an inscribed angle, can be related to a central angle.

These theorems and postulates will allow us to find more information about the measures of angles and chords when dealing with circles. The next theorem is an example of how all this information fits together and results in more deductions.

**Arc Addition Postulate**  

The arc addition postulate is parallel to the segment addition postulate and the angle addition postulate. That is, an arc is equal to the sum of its parts.

\[ \text{arc } AC = \text{arc } AB + \text{arc } BC \]

**Theorem**  

The measure of an inscribed angle is equal to half the measure of the intercepted arc.

\[ \angle 1 = \frac{1}{2} \overarc{AB} \]

To show this relationship about inscribed angles, I’m going to draw one of the rays through the center of the circle as shown below.

By drawing the ray through the center, I can then construct a triangle where one of the angles is a central angle – as shown below. \( \triangle AOB \) is isosceles because two of the sides are radii, that means that \( \angle A = \angle B \); base angles of an isosceles triangle are congruent.
∠AOC is a central angle, but it is also an exterior angle of a triangle, which means that it is equal to the sum of the two remote interior angles. With that background, let’s look at the proof of this theorem.

**Theorem**  The measure of an inscribed ∠ is half the measure of the intercepted arc.

Given: ∠ABC inscribed

Prove: ∠ABC = $\frac{1}{2} \overarc{AC}$

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<td>2. $\overline{OB} \cong \overline{OA}$</td>
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<td>3. ∠A \cong ∠B</td>
<td>2 sides of Δ are \cong, ∠’s opposite arc \cong</td>
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<td>4. ∠AOC = ∠A + ∠B</td>
<td>Ext ∠ = 2 remote int ∠’s</td>
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<td>5. ∠AOC = ∠B + ∠B</td>
<td>Sub</td>
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<td>6. ∠AOC = 2∠B</td>
<td>D-Prop</td>
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<td>7. $\frac{1}{2} \overarc{AOC} = \overarc{B}$</td>
<td>DPE</td>
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<tr>
<td>8. ∠AOC \cong \overarc{AC}</td>
<td>Central ∠, arc</td>
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<tr>
<td>9. $\frac{1}{2} \overarc{AC} = \overarc{B}$</td>
<td>Sub</td>
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That proof was a special case because I drew my ray through the center of the circle. What would have happened if I did not?
Again, always go back to what you know. I could construct a ray from B going through the center of the circle – Ray BO.

Now I can use the information I have just proven, plus I know the Angle Addition Postulate; that $\angle ABC = \angle ABX + \angle XBC$.

That information will allow us to find the measure of an arc formed by an inscribed angle or the measure of an inscribed angle as shown below.

![Diagram of circles and angles](image)

- **Arc SH = 80°**
- **$\angle X = 85°$**

**Corollary** If two inscribed angles intercept congruent arcs, the angles are congruent.

![Diagram of circle and angles](image)

- $\angle 1$ and $\angle 2$ both intercept arc AB, $\angle 1 = \angle 2$

**Corollary** If a quadrilateral is inscribed in a circle, the opposite angles are supplementary.

**Corollary** An angle inscribed in a semicircle is a right angle.

![Diagram of semicircle and angle](image)

- The intercepted arc of a semicircle is 180°
Theorem When a secant ray and a tangent ray are drawn from a point on a circle, the measure of the angle formed is equal to half the measure of the intercepted arc.

\[ \angle 3 = \frac{1}{2} h^\circ \]

Theorem When two secants intersect in the interior of a circle, the measure of the angle is equal to half the sum of the measures of the arcs intercepted by that angle and its vertical angle.

\[ \angle 4 = \frac{1}{2} (h^\circ + k^\circ) \]

To prove this theorem, I will continue to use the strategies I have used before. First, I will draw the picture with the information given to me. Second, I will add lines so I can relate this to problems I have solved before, and third I will label the diagram.

In this proof, I will be able to construct a triangle by connecting two points, that triangle will have an exterior angle formed (the angle I want to find), and there will be two inscribed angles formed. The relationship between the exterior angles and the two remote interior angles is the basis for our proof.

Let’s start, first the picture and the line that connects the points X and W. Label the angles.
Theorem: When 2 secants intersect in a circle, the \( \angle \) formed is equal to \( \frac{1}{2} \) the sum of the arcs formed by the vertical \( \angle \).

Given: \( XY \) & \( ZW \) intersect

Prove: \( \angle 1 = \frac{1}{2}(XZ + YW) \)

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<tr>
<td>2. ( \angle 1 = \angle 2 + \angle 3 )</td>
<td>Ext ( \angle ) of ( \triangle ) = sum of 2 remote int ( \angle )’s</td>
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<tr>
<td>3. ( \angle 2 = \frac{1}{2} \overarc{ZX} )</td>
<td>Inscribed ( \angle ) = ( \frac{1}{2} ) intercepted arc</td>
</tr>
<tr>
<td>( \angle 3 = \frac{1}{2} \overarc{YW} )</td>
<td></td>
</tr>
<tr>
<td>4. ( \angle 1 = \frac{1}{2} \overarc{ZX} + \frac{1}{2} \overarc{YW} )</td>
<td>Substitution in step 2</td>
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<tr>
<td>5. ( \angle 1 = \frac{1}{2}( \overarc{ZX} + \overarc{YW} ) )</td>
<td>Distributive Prop</td>
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Notice the importance of the triangle theorems in these proofs.

To find the measure of \( \angle 1 \), take half the sum of the intercepted arcs

\[
\angle 1 = \frac{1}{2} (120 + 40) \\
\angle 1 = \frac{1}{2} (160) \\
\angle 1 = 80^\circ
\]
Theorem  When two secant rays, a secant ray and a tangent ray, or two tangent rays are drawn to a circle from an exterior point, the measure of the angle formed is equal to half the difference of the measures of the intercepted arcs.

\[
\angle 5 = \frac{1}{2} (k - h)
\]

To prove that theorem, I would draw the picture, draw a line and start labeling. Let's look. Notice by drawing BC, I have formed two inscribed angles and a triangle.

\[
\angle 2 = \angle 1 + \angle 5 \quad \text{Ext angle of } \Delta
\]

\[
\angle 1 = \frac{1}{2} h \quad \text{and} \quad \angle 2 = \frac{1}{2} k \quad \text{Inscribed angles}
\]

Finding \( \angle 5 \) is just a matter of substituting those values for angles 1 and 2 into the first equation.

\[
\angle T = \frac{1}{2} (90 - 40)
\]

\[
\angle T = 25^\circ
\]
Theorem  A tangent to a circle is perpendicular to the radius drawn to the point of tangency. (The converse of that theorem is true also.)

Theorem  In the same or congruent circles, if two chords are congruent, they are equally distant from the center. (Converse is true)

To prove that theorem, again you would draw the picture, try to make triangles, prove the triangles are congruent or similar, then use cpctc or a proportion.

Theorem  When two chords intersect within a circle, the product of the lengths of the segments of one chord is equal to the product of the lengths of the other chord.

![Diagram of two intersecting chords in a circle]

Theorem  When 2 chords intersect, the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other.

Given: Chords $XY$ & $ZW$

Prove: $a \cdot b = c \cdot d$

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<td>2. $\angle X \cong \angle W$ $\angle Z \cong \angle Y$</td>
<td>Inscribed $\angle$’s intercept same arc</td>
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<td>3. $\angle XKZ$ and $\angle YKW$ are Vert $\angle$</td>
<td>Def. Vertical $\angle$</td>
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<td>4. $\triangle XKZ \sim \triangle WKY$</td>
<td>AA Postulate</td>
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<td>5. $\frac{a}{d} = \frac{c}{b}$</td>
<td>$\sim \triangle$’s proportion</td>
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<td>6. $ab = cd$</td>
<td>Prop of Proportion</td>
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Using the previous theorem, find the value of $x$.

Using the previous theorem, we know the products of the segments are equal. That means that $12 \cdot x = 6 \cdot 6$ or $12x = 36$.

$x = 3$

**Theorem**  
If two secants are drawn to a circle from an exterior point, the product of the lengths of one secant and its external segment is equal to the product of the other secant and its external segment.
Theorem  If 2 secants are drawn to a circle from an exterior pt, the product of the lengths of one secant segment and its external segment is equal to the product of the other secant and its external segment.

Given:  secants $\overline{PX}$ and $\overline{PY}$

Prove:  $PX \cdot PY = PZ \cdot PW$

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<td>Inscribed $\angle$, same arcs</td>
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<td>3. $\angle P \cong \angle P$</td>
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<td>4. $\triangle XPW \sim \triangle ZPY$</td>
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<td>5. $\frac{PX}{PZ} = \frac{PW}{PY}$</td>
<td>$\sim \triangle$’s, sides in proportion</td>
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<td>6. $PX \cdot PY = PZ \cdot PW$</td>
<td>Prop of Proportion</td>
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Theorem  If a tangent and a secant are drawn to a circle from an exterior point, the square of the length of the tangent segment is equal to the product of the lengths of the secant segment and its external segment

$$t^2 = m \cdot u$$