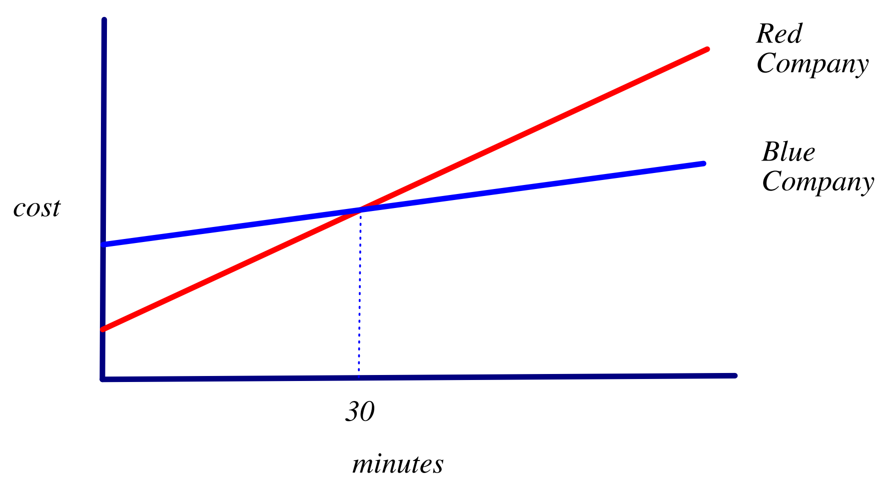
Chapter 6 Systems of Equations

Sec. 1 Systems of Equations

How many times have you watched a commercial on television touting a product or services as not only the best, but the cheapest? Let's say you were watching TV tonight and you saw the following commercials back to back. The first commercial representing Red Phone Company declares they are the least expensive telephone service. Immediately following that commercial, a different Blue Phone Company makes the same claim, they have the cheapest rates, what can you conclude from these two commercials?

Most people might conclude on first blush one of the phone companies is being less than truthful. But, if we were to look at this problem from a mathematical standpoint (logic), we might see that both of these phone companies could be, in fact, telling the truth.



Let's look at two telephone companies and see how this might occur from a graphical standpoint. The graphs reflect the amount the phone companies charge by the minute. Notice their graphs intersect at the 30 minute mark. That means, if you speak for exactly 30 minutes, the cost is the same for each company.

Looking at the graphs, if a person normally speaks less than 30 minutes, Red Company's rates would be cheaper. If you speak more than 30 minutes, Blue Company rates are cheaper. So they both made true statements.

Red Company charges a monthly fee of $10.00 plus ten cents per minute. Mathematically, we'd write:

$$C_R = \$10 + .10m$$

Blue Company charges $11.50 per month plus five cents per minute. Writing that mathematically, we have

$$C_B = \$11.50 + .05m$$

Two different phone companies, two different monthly charges, two different charges per minute. To determine which is the better deal, we plug in each formula the number of minutes the phone is being used.

In other words, both companies were telling the truth in their respective commercials depending upon the number of minutes the phone was being used. The break even point is 30 minutes, both companies would charge $13.00.

So, which phone company offers the better deal? That answer depends upon your usage. Those type decisions have to be made all the time in real life. Do you buy a car or lease one?. Do you rent a car or use a taxi? Do you buy a house or rent?

Systems of equations can be solved by trial and error, graphically, substitution, or by linear combination. Trial and error may take a little time, graphing both equations would give us an idea where the costs are the same, the other two methods will tell us exactly where the break even point is.

Sec. 2 Graphing Systems of Equations

When we graph 2 equations on the same coordinate axes, the point of intersection is an ordered pair that satisfies both equations.

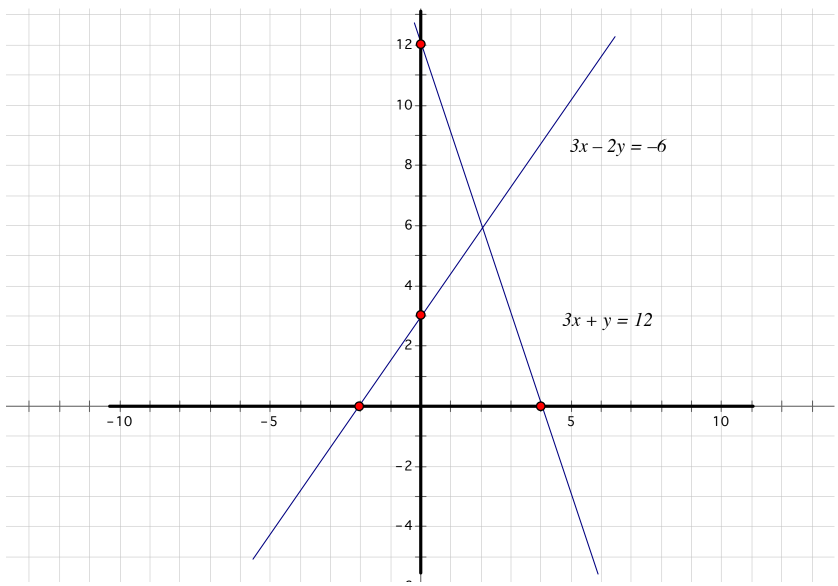
If the lines do not intersect, then they have no points (ordered pairs) in common. Mathematically, we say there is not a solution, and ordered pair, that makes both equations true.

The reason we solve these equations by graphing is to see, if in fact, there will be a point of intersection. If there is, then by graphing them, we can see a point, described by an ordered pair that is approximately the answer.

Example 1 Solve this system of equations by graphing

$3x + y = 12$

$3x - 2y = -6$



Looking at those 2 graphs, it looks like they intersect as (2, 6).

To see if they did, we would substitute (2, 6) into both equations to see if they were true.

Doing that in the 1st equation.

$3x + y = 12$
$3(2) + 6 = 12$ That checks.

Doing the same for the other equation.

$3x - 2y = -6$
$3(2) - 2(6) = -6$ That also checks.

Therefore the point (2, 6) is on both lines. It satisfies both equations. It is a solution.

If you are real neat with your graphing, you can find ordered pairs that approximate your answer.

But sometimes, the answers are ordered pairs that are made up of fractions. That's why we will learn more precise ways of solving systems of equations.

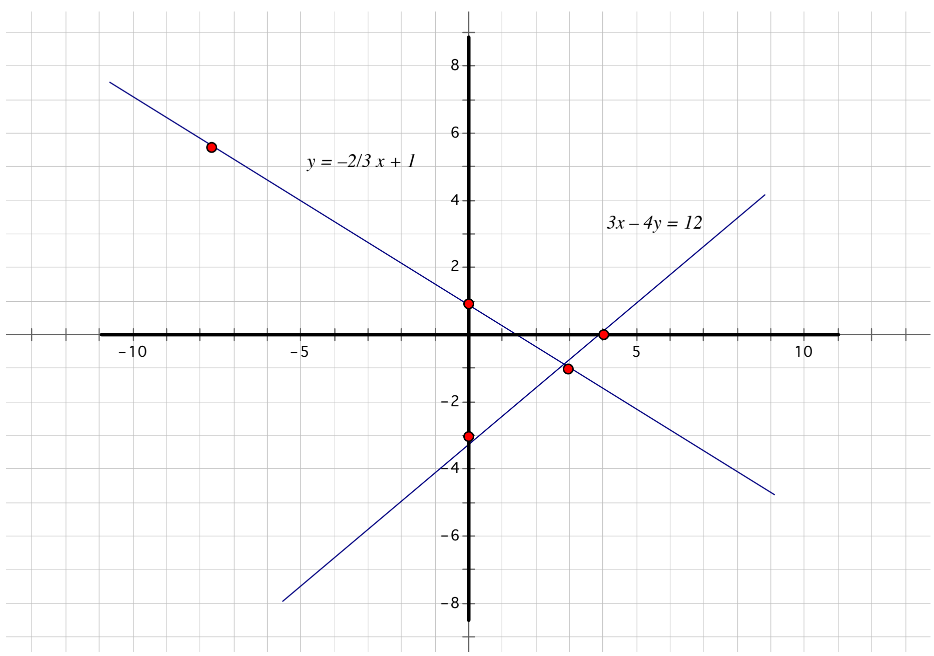
But, knowing how to graph these is important so you have an idea of whether or not there is a point of intersection, an ordered that satisfies both equations.

Let's try another, you say.

Ok.

Example 2 Solve: by graphing $y = -2/3\,x + 1$
$3x - 4y = 12$

Graphing



The point of intersection appears to be around (3, –1).
If we substituted that ordered pair into both of those equations, we'd find it does not work. It does not satisfy both equations.

But, we know there is a point, an ordered pair, that satisfies both equations. And we know it's around (3, –1).

Let's look at a more precise way of finding that point of intersection – Can't wait, can you?

Sec. 3 Solving Systems of Equations – Linear Combination

When you solve two equations at the same time, we call that solving the equations "simultaneously," or solving a "system" of equations. What we are doing is looking for a solution (answer) that satisfies both equations. The geometric interpretation is we are finding the point where two or more lines intersect. A point that is on both lines.

2 Methods of Solving Systems of Equations

1. Linear Combination / Elimination
2. Substitution

In either method, what you are really doing is taking a problem that you don't know how to solve and changing that into a problem you do know how to solve.

You might recall, when we had equations that were not in the $ax + b = c$ format, we used the Properties of Proportion to transform it into that format.

So, let's look at an example that will make sense to you. If I told you 4 quarters = 1 dollar and 5 nickels = 1 quarter, I think you'd have no problem with those statements. Let's write them mathematically.

$$4\text{ Q} = 1\text{ D}$$
$$5\text{ N} = 1\text{ Q}$$

If I added those together using the Addition Property of Equality, I would have

$$4\text{ Q} + 5\text{N} = 1\text{ D} + 1\text{Q}$$

In other words, if I added equals to equals, the results would be equal.

Let's use that idea, Addition Property of Equality, and try a solving for a system of equations.

Example 1 Find the ordered pair that satisfies both equations.

$$\begin{aligned} x + y &= 1 \\ x - y &= 5 \end{aligned}$$

What I have is two equations with two unknowns (variables). I don't know how to solve for those variables. So, using the strategy of changing a problem I don't know how to solve into one I do, I'm going to try to change this system of 2 equations with two variables into one equation with one variable.

How do I do that? Through a series on transformations using the Properties of Real Numbers.

$$\begin{array}{l} x + y = 1 \\ \underline{x - y = 5} \\ 2x \quad\; = 6 \end{array}$$

Notice, the coefficients of the y are the same, but opposite in sign. If I added these two equations together by the Addition Property of Equality, the y's add out. By adding the equations together, as can be seen above, the y's fall out.

$$2x = 6.$$

Solving the resulting equation, I have $x = 3$.

If $x = 3$, substituting that value back into one of my original equations, we find $y = -2$

That means that these two graphs intersect at $(3, -2)$

The strategy for solving equations by LINEAR COMBINATION is to add the equations together in such a way as to eliminate a variable. Eliminating a variable changes the system of equations into one equation with one variable – which I know how to solve. Then you solve the resulting equation and voila – You're done. Piece of cake.

Example 2 Solve and check.

$$x + y = 14$$
$$x - y = 22$$

Notice the coefficients of the y's are the same in both equations, but opposite in sign.

If I add these two equations together, the y's will fall out resulting in one equation and one unknown.

$$2x = 36$$
$$x = 18$$

If $x = 18$, then in the first equation $x + y = 14$, substitute $x = 18$, That results in $y = -4$. So the ordered pair that satisfies both of these equations is $(18, -4)$.

What if the coefficients were the same but not opposite in sign? If I added those equations together, nothing would fall out resulting in one equation with two variables. Since I don't know how to solve that, I would run into some difficulty. How could I solve such a problem so I end up with one equation with one unknown?

Subtracting the equations would result in one of the variables disappearing. That wasn't that hard, was it? Or I could have multiplied one of the equations by by (-1) to make the coefficients the same but opposite in sign, then add them

What would happen if the coefficients were not the same?

If they were not the same, it wouldn't matter if I added or subtracted the equations, I would end up with one equation with two variables. A problem I don't know how to solve.

What I could do is make the coefficients the same in one variable, but opposite in sign, then add the equations together. That would make the system look like the problems we were just doing.

Let's write an algorithm that will help us accomplish that strategy.

Algorithm Linear Combination/Elimination

1. If necessary, multiply either or both equations by number(s) whichn will make the coefficients of one of the variables the same but opposite in sign
2. Add the two equations together
3. Solve the resulting equation
4. Substitute that value into one of the given equations to find the value the other variable
5. Write the solution as an ordered pair.

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Example 3 Solve

$$3x + 10y = 2$$
$$x - 2y = 6$$

Notice the coefficients are not the same.

First, I have to determine what variable I want to get rid of. If I multiply the bottom equation by (–3) and add the equations, that will get rid of the x's.

OR

I could multiply the bottom equation by 5 and add the equations and that will get rid of the y's.

It DOES NOT MATTER WHICH YOU CHOOSE. Choose which ever will make the computation easier.

I'll multiply the bottom equation by (–3).

$3x + 10y = 2$		$3x + 10y = 2$	
$x - 2y = 6$	mult by –3	$\underline{-3x + 6y = -18}$	
	Adding	$16y = -16$	
	Solve	$y = -1$	

Now we substitute that value back into either of the original equations. The second equation looks simpler, so I'll substitute it in there.

$x - 2y = 6$ Substitute in $y = -1$

$x - 2(-1) = 6$ Simplify

$x + 2 = 6$

$x = 4$

The answer is the ordered pair (4, –1). That's the point where these two lines intersect. Again, the strategy is to add the equations together to eliminate a variable. Yes, I can hear you. You love math, math is your life!

Example 4 Solve

$$\begin{aligned} 2x + 3y &= -4 \\ 5x - 2y &= 9 \end{aligned}$$

First, I have to determine which variable is easier to get rid of. Oh boy, in the last equation, all I had to do was multiply one equation by a number to get the coefficients the same. This time I'll have to multiply both equations. I can multiply the top by –5 and the bottom by 2, that will make the coefficients of the x's the same

OR

I could multiply the top by 2 and the bottom by 3, that will get rid of the y's. I'll choose to get rid of the y's.

I'll multiply the top equation by 2 and the bottom equation by 3.

$2x + 3y = -4$	mult by 2	$4x + 6y = -8$
$5x - 2y = 9$	mult by 3	$\underline{15x - 6y = 27}$
	Adding	$19x = 19$
	Solve	$x = 1$

Now plug that value for x back into either of the original equations. Using the top equation:

$2(1) + 3y = -4$

$3y = -6$

$y = -2$ The answer is the ordered pair (1, –2).

Solve by linear combination and check.

1. $x + y = 1$
   $x - y = 5$

2. $x + y = 14$
   $x - y = 22$

3. $a - b = -1$
   $-a + 2b = 4$

4. $-m + 3n = 5$
   $m - 2n = 25$

5. $2x + y = -5$
   $2x + 3y = 10$

6. $x + 2y = 1$
   $2x + y = 12$

7. $x + 2y = 1$
   $2x - y = -23$

8. $r - s = 5$
   $2r + 3s = -20$

9. $3x - 2y = -7$
   $2x - 5y = 10$

10. $2x - 5y = 1$
    $4x + 2y = 14$

11. $3m + 2n = 1$
    $2m - 3n = 18$

$4a - 3b = 3$
$3a - 2b = 4$

13. $6x + 3y = 0$
    $4x - y = 4$

14. $a - 2b = 0$
    $a + 2b = 0$

15. $5x + 2y - 4 = 0$
    $3x - 2y - 4 = 0$

16. $x + 2y + 7 = 0$
    $3x + 4y + 21 = 0$

17. $5x = 4 - 2y$
    $4y = 8 - 10x$

18. $4x = -5y + 5$
    $7y = -6x + 7$

19. $4a + b - 1 = 0$
    $a - 2b - 16 = 0$

20. $3m = 2n - 19$
    $4n = 3 - m$

Sec. 4 Solving Systems of Equations - Substitution

The other method used to solve systems of equations is Substitution. Again, the strategy is to transform two equations and two unknowns into one equation and one unknown so we can write an equation in $\boldsymbol{ax + b = c}$ format.

To accomplish that, I can solve one of the equations for one variable and substitute that expression into the other equation. That will result in one equation in one variable.

Algorithm for Substitution

1. Solve one of the equations for one of the variables.
2. Substitute that expression in the OTHER equation.
3. Solve the resulting equation.
4. Plug that value into either of the original equations to find the other variable.

Let's redo **Example 3** using this method.

Example 1 Solve

$$3x + 10y = 2$$
$$x - 2y = 6$$

First, determine which equation has the easiest variable to solve for. I'm choosing x in the second equation because its coefficient is 1.

$3x + 10y = 2$
$x - 2y = 6,$ solving for x, $x = 6 + 2y$

Everywhere you see an x in the OTHER equation, substitute 6 + 2y. The other equation is:

$3x + 10y = 2$ Substitute $6 + 2y$
$3(6 + 2y) + 10y = 2$ Solve
$18 + 6y + 10y = 2$
$18 + 16y = 2$
$16y = -16$

$y = -1$

Substitute $y = -1$ into either of the original equations, you find $x = 4$. $(4, -1)$.

By substituting, we end up with one equation with one unknown.

As we can see, mathematics is more than just a body of knowledge, it is a way of thinking. As a student of mathematics, you continually are making decisions. In this chapter, you would decide what method to choose to solve systems, then what variable to either solve for or get rid of.

Generally I'll use the Elimination/Linear Combination Method to solve systems of equations. I only consider Substitution when one of the coefficients of one of the variables is 1, as in above example. When the coefficients are not 1, I'll use linear combination.

Try this one both ways:

$$2x + y = 11$$
$$3x - 2y = -1$$

Solution $(3, 5)$

Solve the following systems of equations eliminating by substitution, and check:

1. $y = 2x - 1$
   $x + 4y = 23$

2. $y = 4x + 2$
   $y = 6\frac{1}{2} - 5x$

3. $2x + 3y = -1$
   $x = 3y - 23$

4. $x = -6y - 1$
   $3y - 2x = 12$

5. $x = 1 - 4y$
   $2x + y = -6\frac{3}{4}$

6. $3x + 4y = 2$
   $y = 4x - 9$

7. $x = 34 - 5y$
   $7x - 6y = 33$

8. $2x + 3y = 5$
   $y = x$

9. $x = 3y$
   $x + 3y = 102$

10. $4x - y = 0$
    $3x + 2y = 33$

11. $2x - y = 5$
    $5x - 2y = 14$

12. $5s - 5t = 50$
    $5t - 7s = -56$

13. $4x + 10y = 19$
    $2y = 3x$

14. $5x + 7y = 50$
    $3x + y = 14$

Sec. 5 Word Problems – 2 variables

Word problems, you know you love them. A while ago we solved word problems in one variable. The trick you might remember was to read the word problem to determine what type it was, then you read it again to find out what you are looking for, then you read it again to build relationships and you read it again to make an equation. Finally, you solved the equation.

The point I want to make is you have to read a word problem 5 or 6 times just to get the information you need to solve it. If you read the problem only once or twice, then you are not giving yourself the opportunity to be successful. Enough said.

Now, when you encounter a word problem, you are going to be faced with a choice, a decision. The choice will be to solve using one variable or to solve using more than one variable.

To solve using more than one variable, you will need to find the same number of equations as variables – a system of equations.

Let's look at an example.

Example 1

The sum of two consecutive even numbers is 62. Find the numbers.

First, we'll do the problem in one variable like we have done before. We are looking two numbers.

I will call them # 1 and # 2. The smallest we'll call x. The second was is a consecutive even number, how do I get that? I add 2 to the first number.

#1- x
#2- $x + 2$

Now read the question again to find an equation. The sum of the numbers is 62. That means #1 + #2 = 62. Plugging those numbers in, I have,

$x + (x + 2) = 62$

Now, solve for x.

$2x + 2 = 62$

$2x = 60$

$x = 30$

If $x = 30$, that means the first number is 30, the second number is 32.

Doing that same problem using more than one variable, I again identify what am I looking for. I will use #1 and #2 again. Now, I'll call the first number x and the second number y.

#1- x
#2- y

Now this is very important. Since I have 2 variables I must now find two equations. If I had three variables, I would need to three equations.

What do I know about these numbers? Their sum is 62.

Mathematically we write: $x + y = 62$

That's one equation, are there any other relationships? The numbers are consecutive even integers. How can I express that mathematically? The second number minus the first must be 2. Therefore we have

$$y - x = 2$$

Putting those two equations together, we have this system.

$$\begin{array}{lcr} x + y = 62 & \text{or} & x + y = 62 \\ y - x = 2 & & \underline{-x + y = 2} \end{array}$$

Now I solve that system of equations by linear combination or substitution. Easy as pie.

$$2y = 64$$
$$y = 32$$

If $y = 32$, then $x = 30$

The nice thing about solving equations in more than one variable is I call my unknowns x, y, z, etc., then I look for relationships and solve the resulting systems of equations.

I must have at least the same number of equations as variables. Otherwise I can't solve the system.

Write two equations for each problem and solve:

1. Find two numbers whose sum is 114 and difference is 58.

2. The difference of two numbers is 17 and their sum is 33. Find the numbers.

3. One number is four times another numbers, the sum of the numbers is 140. Find the numbers.

4. One airplane costs four times as much a a certain car. Two such planes cost $6000 more than six of the cars. Find the cost of each.

5. A part of $5000 was invested at 4.5% and a part at 5.5%. the 4.5% investment yields $75 more each year than the 5.5% investment. How much is invested at each rate?

6. Five pounds of tea and 8 pounds of coffee cost $11.36, while 10 pounds of tea and three pounds of coffee cost $10.76. What is the price of each per pound?

7. How many pounds of 75-cent candy and how many pounds of $1.25 candy must be mixed to make a mixture of 90 pounds to sell at 96-cents per pound?

8. A man rowed up a river 10 miles in 5 hours and back in 2,5 hours. Find the rate of the current and his rate of rowing in still water.

9. A chemical company has in storage a 15% solution and a 25% solution of disinfectant. How many gallons of each should be used to make 50 gallons of a 22% solution?

10. What number must be added to both the numerator and denominator of 11/12 to equal the fraction 2/3?

11. Sarah bought 90 stamps for $12.75. Some were 15¢ stamps and the rest were 10¢ stamps. How many of each kind did she buy?

12. Sherman bought 23 stamps for $5.85. Some were 31¢ stamps and the rest were 15¢ stamps. How many of each kind did he buy?

13. In cashing a check for $135, Hank asked for the whole amount in $5 and $10 bills. He received a total of 18 bills. How many of each kind of bill did he receive?

14. In cashing a check for $220, Chris asked for the whole amount in $10 and $20 bills.She received a total of 13 bills. How many of each kind of bill did she receive?

15. Estela has twice as many nickels as half-dollars. Their value is $2.40. How many nickels and how many half-dollars does she have?

16. Aponi has three times as many dimes as quarters. In all she has $3.30. How many coins of each type does she have?

17. A book club charges a membership fee and a fixed cost per book. After buying 6 books, Susan owes $17. After buying 10 books, she owes $25. What is the membership fee and the cost of each book?

18. To join a health club, Jason pays monthly dues and an initiation fee. After 3 months, he had paid $160. After 5 months, he had paid $210. What are the monthly dues and initiation fee?

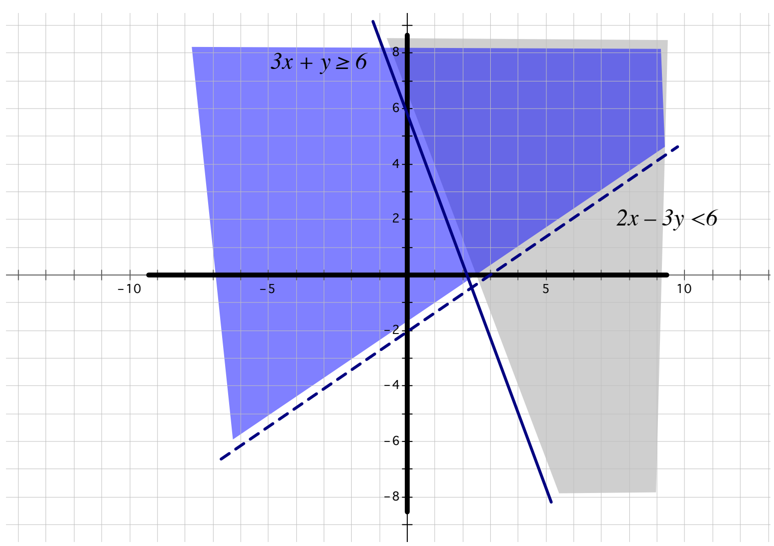
Sec. 6 Systems of Inequalities

When solving a system of equalities, we were looking for the point, ordered pair, that satisfied all the equations. When we solve a system of inequalities, we are looking for all the points, ordered pairs, that satisfy each inequality. There's an infinite number of those points, so the best way to identify them is by graphing.

You might recall from graphing inequalities, we identified the boundary line, then shaded the points that satisfied the inequality.

To find the solution set for a system of inequalities, we graph each inequality, then look to see when all (both) conditions are met (true). That occurs where the graphs of the inequalities overlap.

Example 1 Graph the solution set $3x + y \geq 6$
$2x - 3y < 6$



The graphs overlap in the deeper purple region, so that's the solution. The set of ordered pairs that will make both statements true.

In a coordinate plane, graph the solution set of each system.

1. $y < x$
   $y > -x$

2. $y > x + 1$
   $y < x + 2$

3. $3x - 2y < 6$
   $2x + 3y > 6$

4. $x + y > -6$
   $y < -3$

5. $0 < x < 2$

6. $-1 < y - x < 5$

Linear Programming

Many applications in business involve optimization. Optimization is the process of finding maximum or minimum values for a specific quantity. When the quantities being optimized are represented by linear equations, the process is called linear programming.

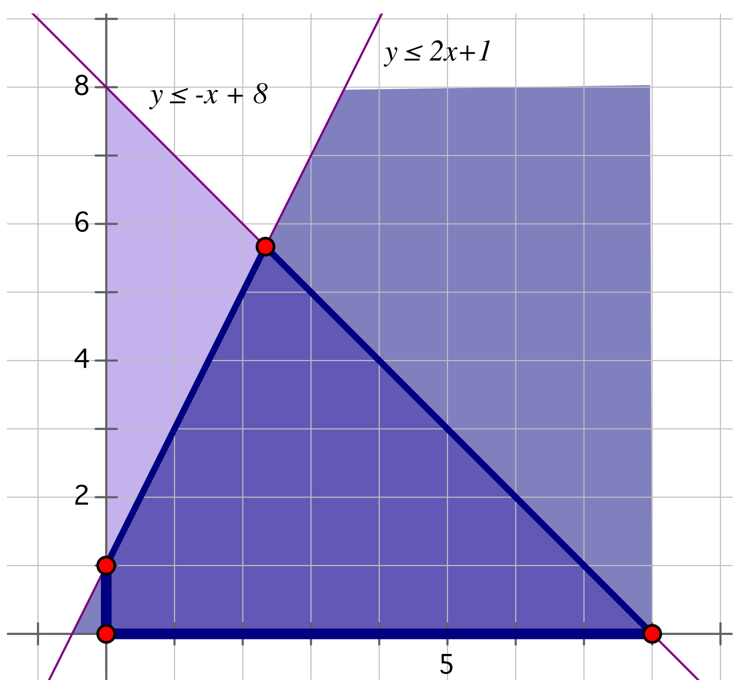
To optimize a function, we will graph systems of linear inequalities just like we did in the last section. We will find the solution for that system by determining where the graphs overlap. So, if you could do solve linear equations algebraically and inequalities graphically, linear programming will be a piece of cake.

But as always, we will introduce some new phrases. An objective function is a linear function that will be optimized. Then we will have a system of inequalities called constraints. A constraint is noting more than limitations placed on a given situation. A very simple example might be you can not park more than 4 cars in your garage. That would be a limitation for storage, a constraint, described by the inequality, $x \le 4$.

Procedure for Optimizing a Linear Function

1. Graph the system of constraints (limitations) which are described by the inequalities to form feasible (polygonal) region.
2. Find the coordinates of the vertices (corners) of that region by solving a system of equations
3. Evaluate the objective function for each corner described by an ordered pair

Example 1 Find the maximum and minimum values of the objective function $f(x, y) = 7x - 2y$ subject to the following constraints.
$x + y \le 8 \qquad y \le 2x + 1 \qquad x \ge 0 \qquad y \ge 0$



Notice I only graphed in the first quadrant because both x and y are positive! Of the 4 corners, I can determine three of the points by inspection.

(0, 0), (0, 1) (8, 0). To find the 4th corner, I will need to find the intersection of the two lines; $y = -x + 8$ and $y = 2x + 1$. Setting the equations equal, we have:

$$2x + 1 = -x + 8$$
$$3x = 7$$
$$x = 7/3$$

Substituting 7/3 into one of the equations, we have; $y = -7/3 + 8$. So $y = 17/3$. The 4th ordered pair (corner) is (7/3, 17/3)

So, following the procedure above, I graphed the inequalities and found the corners. My last step to substitute those ordered pairs into the objective equation which was given as **$f(x, y) = 7x - 2y$**

Evaluating f for those 4 corners,
$f(0, 0) = 7(0) - 2(0) = 0$
$f(0, 1) = 7(0) - 2(1) = -2$
$f(8, 0) = 7(8) - 2(0) = 56$
$f(7/3, 17/3) = 7(7/3) - 2(17/3) = 49/3 - 34/3 = 15/3 = 5$

We can see the largest value for f is 56 – the maximum occurs at (8, 0) the smallest value for f is –2 occurs at (0, 1)

Example 2

A manufacturer makes two kinds of radios, FM and AM-FM. The company has the equipment to manufacture any number of FM sets up to and including 600 per month. Or any number of AM-FM sets up to 525 per month. It takes 30 man-hours of labor to produce an FM radio, and 40 hours to produce an AM-FM. The firm has up to 24,000 man-hours available for radio production each month. If the profit gained on each FM radio is $16 and $24 for each AM-FM radio, find the number of each kind of radio the firm should manufacture to gain the greatest profit per month.

Going through our procedure, let's graph the inequalities. To begin, we know they can't not make a negative number of radios, so we know we will only be graphing in the 1st quadrant.

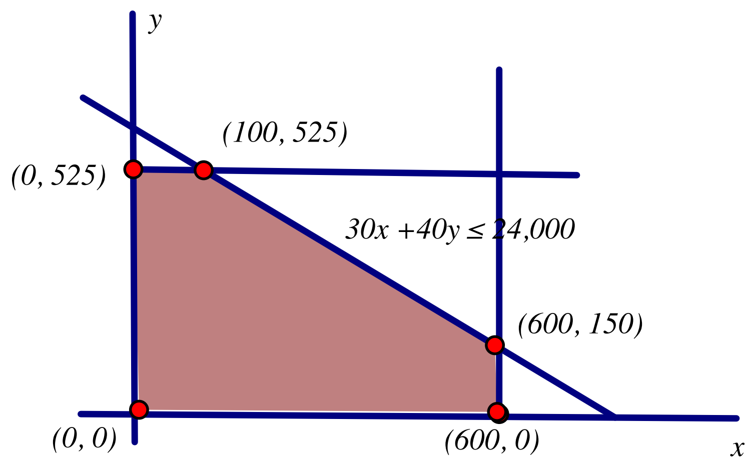
# of FM radios – x
# of AM/FM radios – y

They can't make more than 600 FM radios so $0 \leq x \leq 600$. The can't make more than 525 AM/FM radios $0 \leq y \leq 525$

It takes 30 hours to make FM radios; so time is 30x hours
It takes 40 hours to make AM/FM radios, so time is 40y hours
They have 24,000 hours available, so $30x + 40y \leq 24{,}000$

And the profit statement, the objective function, to maximize the profit is

$$\mathbf{P(x, y) = 16x + 24y}$$



P(0, 0) = 0
P(0, 525) = 12,600
P(100, 525) = 14,200
P(600, 150) = 13, 200
P(600, 0) = 9,600

The maximum value occurs at P(100, 525). That means to make the most profit, $14,200, the company would make 100 FM radios and 525 AM/FM radios.