Ch. Y1 Solutions of Triangles

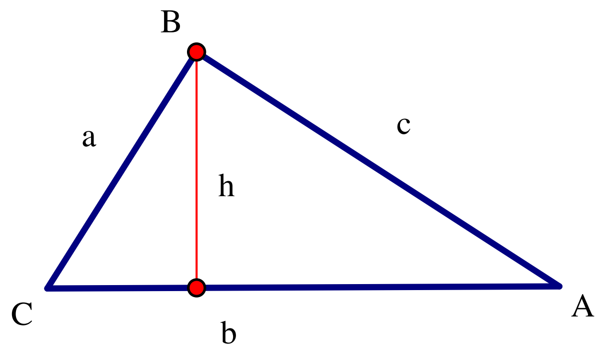
We have studied right triangle trigonometry and learned how to find sides and angles of right triangles. Well, a question that one might ask is what happens when we want to find sides or angles of triangles that are not right triangles. An oblique triangle is a triangle that does not contain a right triangle.

Good news, we can always divide an oblique triangle into right triangles. But, that can be a little cumbersome. So, we will use our knowledge of right triangles and develop some relationships that will work for oblique triangles.

Law of Sines

If two angles and a side of a triangle are known, or if two sides and an angle opposite one of them is known, then the remaining parts can be found.

Given ΔABC, construct an altitude (h), from $\angle$B to $\overline{AC}$



Drawing the altitude forms 2 right triangles. Using sin A = $\frac{opp}{hyp}$, we have

$$\sin C = \frac{h}{a} \quad \text{and the} \quad \sin A = \frac{h}{c}$$

Solving both equations for h, $h = a \sin C$ and $h = c \sin A$

Using substitution $a \sin C = c \sin A$

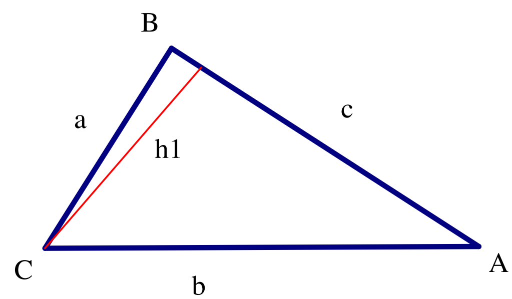
Divide both sides by (sin C)(sin A)

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Using the same ΔABC, construct an altitude h_1 from $\angle C$ to $\overline{AB}$

Go through the same process for the right triangles formed. Note $h \neq h_1$

$$\sin A = \frac{h_1}{b} \qquad \sin B = \frac{h_1}{a}$$



Solving for h_1 $\quad$ $\boldsymbol{h_1 = b \sin A}$ **and** $\boldsymbol{h_1 = a \sin B}$

Using substitution $\quad$ $\mathbf{b \sin A = a \sin B}$

Divide both sides by (sin A)(sin B)

We have already shown that $\frac{b}{\sin B} = \frac{a}{\sin A}$, so using the Transitive Property we have

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

That gives us the **Law of Sines**

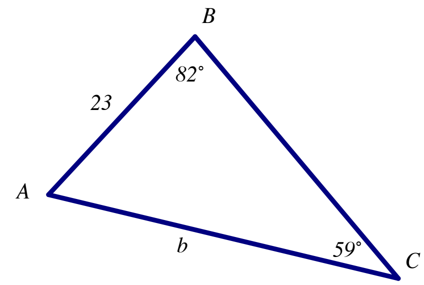
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The ratio of any side of a triangle to the sine of the opposite angle is a constant.

As we said earlier, we use the Law of Sines under a a couple of conditions

A) If two angles and a side of a triangle are known
B) If two sides and an angle opposite one of them are known

Example 1 Given the ΔABC, find *b*.



Using the Law of Sines

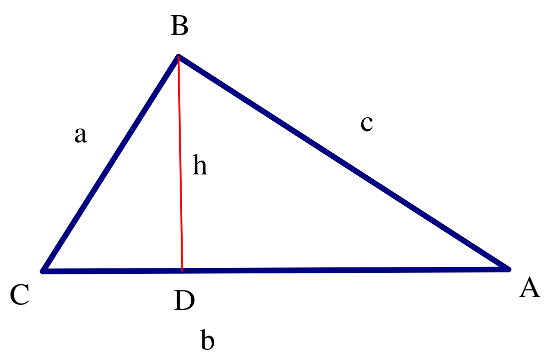
$$\frac{23}{\sin 59^\circ} = \frac{b}{\sin 82^\circ}$$

$$b\sin 59^\circ = 23\sin 82^\circ \quad \text{(look up values for sine)}$$
$$b\,(.86) \approx 23\,(.99)$$
$$b \approx 26.5$$

Law of Cosines

If two sides and the included angle or if three sides of a triangle are given, the Law of Sines can not be applied directly.

Given ΔABC, construct an altitude (h), from $\angle$B to $\overline{AC}$. That altitude forms two right triangles that allows us to use the trig.

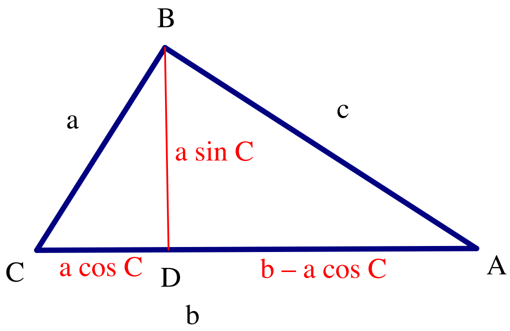


$$\sin C = \frac{h}{a} \quad \text{or} \quad h = a\sin C$$
$$\cos C = \frac{adj}{hyp} \rightarrow \cos C = \frac{CD}{a}$$
$$\therefore \text{ CD} = a\cos C$$

By the diagram AC = b and we just found that CD = a cos C, then

AD = b – *a cos C*

We will use that information to relabel our triangle ΔABC



ΔBDA is a right triangle, we can use the Pythagorean Theorem to write an equation

$$(a \sin C)^2 + \underline{(b - a \cos C)^2} = c^2$$

Squaring the binomial - underlined

$$a^2 \sin^2 C + \underline{b^2 - 2ab \cos C + a^2 \cos^2 C} = c^2$$

Rewriting the equation using the Commutative and associative properties that will result in a trig identity

$$a^2 \sin^2 C + a^2 \cos^2 C + b^2 - 2ab \cos C = c^2$$

Factoring a^2 out of the first two terms

$$a^2 (\sin^2 C + \cos^2 C) + b^2 - 2ab \cos C = c^2$$

Substituting 1 for $\sin^2 C + \cos^2 C$

$$a^2 (1) + b^2 - 2ab \cos C = c^2$$

$$a^2 + b^2 - 2ab \cos C = c^2$$

OR

Law of Cosines

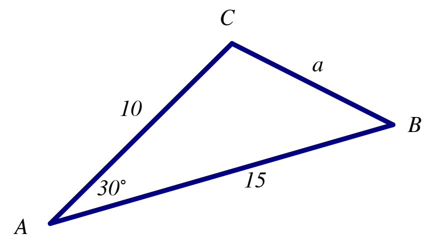
$$\mathbf{c^2 = a^2 + b^2 - 2ab \cos C}$$

In any triangle the square of a side equals the sum of the squares of the other two sides decreased by twice the product of those sides and the cosine of the included angle.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

Example 2 Given ΔABC, find the value of *a*.



Using the Law of Cosines

$a^2 = b^2 + c^2 - 2bc \cos A$

$a^2 = 10^2 + 15^2 - 2(10)(15)\dfrac{\sqrt{3}}{2}$

$a^2 \approx 100 + 225 - 300(.86)$

$a^2 \approx 325 - 258$

$a^2 \approx 67$

$a \approx 8.2$

Example 3 Given three sides of a Δ, $a = 15$, $b = 17$, and $c = 19$, find $m\angle C$, the side opposite side *c*.

$c^2 = a^2 + b^2 - 2ab \cos C$

$19^2 = 15^2 + 17^2 - 2(15)(17) \cos C$

$361 = 225 + 289 - 510 \cos C$

$361 = 514 - 510 \cos C$

$-153 = -510 \cos C$

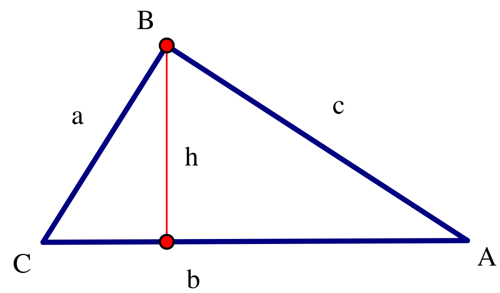
$-153/-510 = \cos C$

$0.3 \approx \cos C$

$73^\circ \approx \angle C$

Area of a Triangle

Let's use ΔABC, and the area formula for a triangle; $A = \frac{1}{2}bh$



$$\sin C = \frac{h}{a} \quad \rightarrow \quad h = a \sin C$$

$$\text{Area}_{\Delta} = \frac{1}{2}(AC)h$$

$$\text{Area}_{\Delta} = \frac{1}{2}(b)a\sin C$$

$$\text{Area}_{\Delta} = \frac{1}{2}ab\sin C$$