Ch. 11 Solving Quadratic & Higher Degree Inequalities

We solve quadratic and higher degree inequalities very much like we solve quadratic and higher degree equations.

One method we often use to solve quadratic and higher degree equations is by factoring using the Zero Product Property or the Quadratic Formula if the polynomial was not easily factored To accomplish that, we used the following algorithm:

> **Place everything on one side, zero on the other side of the equal sign.**
> **Factor completely**
> **Set each factor equal to zero**
> **Solve the resulting equations**

Those zeros will become critical points. They will be points on a number line that divide the number line not intervals. In addition to the zeros, we might encounter rational expressions. You will remember, we cannot have values of a variable that will result in a denominator being zero. So, besides the zeros being critical points, values of the variable that make a denominator zero will also be critical points.

Let's look at how we solved an equation and how that relates to solving inequalities in examples 1 and 2 below.

Example 1 Solve $x^2 + x = 12$

Placing everything on one side, zero on the other side, we have

$$x^2 + x - 12 = 0$$

$$(x + 4)(x - 3) = 0$$

$$x + 4 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = -4 \quad \text{or} \quad x = 3$$

So we have our 2 solutions.

Now, how would changing this equality into an inequality change the way we find the values of x that make this open sentence true? The answer is the steps are pretty much the same, but we will need to look at the problem using logic. That is, asking when a product could be positive or negative.

Example 2 Solve $x^2 + x > 12$

$$x^2 + x - 12 > 0$$
$$(x + 4)(x - 3) > 0$$

Now here is where the logic comes into play. In Example 1, we set those two factors equal to zero, solved the resulting equations and we were done. We did that because we knew that when two numbers are multiplied together and their product was zero, one of the numbers had to be zero. When solving inequalities, we need to ask ourselves this question; When is

$$(x + 4)(x - 3) > 0?$$

In other words, under what circumstances are two factors multiplied so their product is positive? We know that a product of two numbers is positive if and only if both of the given numbers are positive <u>or</u> both are negative.

So let's set this up and see what that means algebraically. $(x + 4)(x - 3) > 0$

Case I Both Positive Numbers	OR	Case II Both Negative Numbers
$x + 4 > 0$ and $x - 3 > 0$		$x + 4 < 0$ and $x - 3 < 0$
$x > -4$ and $x > 3$		$x < -4$ and $x < 3$
Solving this, we see		Solving this, we see
$\rightarrow x > 3$		$\rightarrow x < -4$

So under Case I, both numbers being positive, any number greater than 3 will make the open sentence true. Under Case II, both numbers being negative, any number less than –4 will make the open sentence true.

So the solution set, all the values of x that make the open sentence true, is any number greater than three or any number less than a negative four.

$\{x/x > 3\}$ or $\{x/ x< -4\} \rightarrow \{x/ x > 3\} \cup \{x/ x< -4\}$

So clearly, while the first couple of steps of solving the inequality look like the first few steps of solving an equality, we have to bring in our knowledge of math, signed numbers, to see what numbers work.

Example 3 Find the solution set $t^2 + 3t - 18 < 0$

$$(t + 6)(t - 3) < 0$$

When is a product of two numbers negative? When one number is positive and the other is negative. That gives us 2 cases to consider.

Case I $(+)(-)$

$t + 6 > 0$ and $t - 3 < 0$

$t > -6$ and $t < 3$

$\rightarrow \quad -6 < t < 3$

OR

Case II $(-)(+)$

$t + 6 < 0$ and $t - 3 > 0$

$t < -6$ and $t > 3$

Never! $\rightarrow \varnothing$

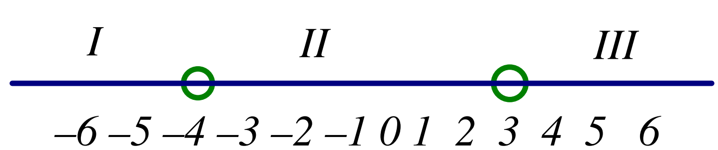
$$\rightarrow \quad \{t / -6 < t < 3\}$$

An easier way to solve inequalities like these is to look at critical points – the zeros – where graphs cross the x-axis. Depending upon where you are in your study, you know linear equations, equations of degree one, cross the x-axis once. There is only one solution. Quadratic equations (degree two) can cross the x-axis up to two times, cubic equations (degree 3) can cross up to three times, etc.

In Example 2, we began solving the equation the way we normally would,

$$x^2 + x > 12$$
$$x^2 + x - 12 > 0$$
$$(x + 4)(x - 3) > 0 \quad \text{then broke into 2 cases.}$$

This time we will plot the zeros, the critical points, $x = -4$ and $x = 3$ on the number line. Those two points break the number line into three segments (interval I, II, III). Numbers less than – 4, numbers between – 4 and 3, and numbers greater than 3. I can choose a convenient number from each interval to determine if numbers in that interval will make our open sentence true. If it does, that entire interval works. If it does not make the open sentence true, then no number in that interval works.



Pick a convenient number less than –4 in interval I and substitute that into

$x^2 + x > 12$. Pick (– 5), Substituting, we have $(-5)^2 + (-5) > 12$? That is true. So the interval from negative infinity to – 4 works.

Let's try zero from interval II. $(0)^2 + 0 > 12$? Is not true, so the interval between – 4 and 3 does not work.

And for the third interval, let's try +5. $(5)^2 + 5 > 12$? That is true. So the interval from 5 to positive infinity also works.

So, we have the same solution as before, $\{x/x > 3\}$ U $\{x/x < -4\}$.

The algorithm for solving quadratic and higher degree inequalities

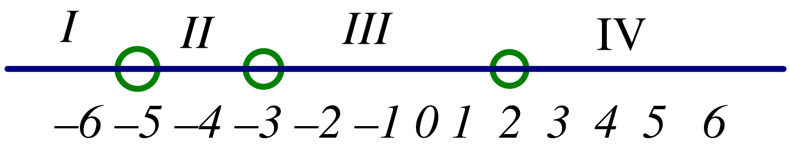
1. **Find the critical points; solutions to equation and where denominator equals zero**
2. **Graph the critical points on the number line**
3. **Check the intervals to determine if the equation is true**

The way I like to think of quadratic or higher degree inequalities is to think of their graphs in two variables, that is $y = x^2 + x - 12$ as a boundary. That's a graph of a parabola that will intersect the x-axis at –4 and 3. If that was an inequality, I would either shade above the graph or below the graph - just like we did with linear inequalities. In other words, if I picked one interval, such as we did with – 5 and it worked, then that interval would be below the graph as would the interval where x was greater than three (draw the picture). The interval in the middle would be above the line. When we graph inequalities, we are shading above or below the line.

If I thought about this for two minutes or so and drew some pictures of parabolas and higher degree equations, I could pretty much determine the intervals by inspection – just by looking. That's the real nice thing about math, the more you know, the easier it gets!

Example 4 Find the solution set. $(x-2)(x+3)(x+5) > 0$

I could do this problem by using different cases, but it would be time consuming. So, an easier method would be to look at the four intervals created by those three critical points.



Interval x < – 5, choose –10 and substitute; (–12)(–7)(–5) is negative; does not work.*
Interval –5< x <–3, choose –4 and substitute; (–6)(–1)(1) is positive, works
Interval –3< x <2, choose 0 and substitute; (–2)(3)(5) is negative, does not work.
Interval x > 2, choose 3 and substitute: (1)(6)(8) is positive, works.

So the solution is {x/ –5< x <–3 U x > 2}. The reason I have an * after the first interval is that I know this graph looks like a snake and the graph of an inequality in two variables will either be above the line or below the line. So, once I know an interval either works or does not work, all I need to do is alternate the intervals to know what works or does not work. Look at our solution in terms of shading above or below the curve.

Caution – if any of the factors are raised to a power other than one, check each interval – alternating the interval will not work!

More on Inequalities

A similar method of solution can be employed for rational inequalities such as

$$\frac{2x+7}{x+2} > 0$$

Because fractional equations are solved by multiplying both sides of the equation by the common denominator, one might be tempted to multiply both sides of the inequality by (x + 2). The problem with that is if we multiply or divide an inequality by a negative number, that changes the order of the inequality. We don't know if (x + 2) is positive or negative, so we will solve this inequality by NOT multiplying or dividing.

Just like quadratic and higher degree inequalities, we will put everything on one side and zero on the other side of the inequality. Let's try one.

Example 5 Find the solution set

$$2+\frac{3}{x+2}>0$$

$$\frac{2x+7}{x+2}>0$$

The critical values are – 2 because that makes the denominator zero and –7/2, plot those on the number line.

$$2+\frac{3}{x+2}>0$$

I simplified the expression above by treating it as a mixed number. I multiplied the whole number by the denominator, added the numerator and placed that result over the original denominator.

Back to the problem. $$\frac{2x+7}{x+2}>0$$

You could check all three intervals created by those two critical points or you might try looking at one interval with the most convenient numbers – zero is always a convenient number! So I'm going to check the interval on the far right,

x > –2 and choose 0.

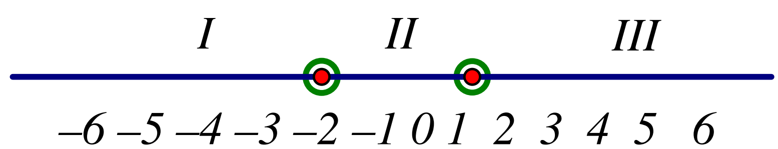
Substituting 0 into the inequality, I see that works. That suggests, by skipping intervals, that x < – 7/2 also works if I choose a convenient number like x = –10 to check. you can see the interval that contains x = –3 does not work.

My solution is {x < –7/2 U x > –2}.

If you did not want to chance skipping intervals, you could have checked a number from each interval. That is, you might have chosen x = –10, x = – 3 and x = 0 to find out which of the three intervals worked.

Example 6 Find the solution set $(x + 2)^2(3x - 4) < 0$

If I expand those binomials, we would have an equation of degree three, which might typically result in having three critical values and four intervals. First thing I want to point out is that the binomial is raised to the second power. That results in two critical values and three intervals. You cannot skip intervals in problems like these. You must choose a number from each interval.



Interval $x < -2$, substitute (–10), (–8)2(–34) is negative, interval I works.

Interval $-2 < x < 4/3$, substitute (0), (2)2(–4) is negative, interval II works.

Interval, $x > 4/3$, substitute (10), (12)2(26) is positive, interval III does not work.

Therefore, the solution is interval I and interval II; $x < 4/3$

If I could graph quadratic and cubic equations, I could look at these graphs as they cross the x-axis at the identified critical points, then use the above the graph and below the graph strategy we used with linear inequalities to determine which intervals worked.