CONGRUENCE

In math, we use the word ***congruent*** to describe objects that have the same shape and size. When you were a little kid, you probably traced objects like squares and triangles, what you were doing was forming congruent shapes.

So, very informally, we can say two geometric figures are congruent $(\cong)$ if we can superimpose them so they coincide with each other. The segments and angles that coincide are referred to as the corresponding parts.

Let's get to some formality, if two line segments have the same length, we say they are congruent. By the same token, if two angles have the same measure, we say the angles are congruent.

And what would math be without a little notation? Let's say that two line segments $\overline{AB}$ and $\overline{XY}$ have the same length, that would mean $\overline{AB}$ is congruent to $\overline{XY}$, written $\overline{AB} \cong \overline{XY}$.

If we have two angles, A and B, that have the same measure, then those two angles would be congruent, written $\angle A \cong \angle B$.

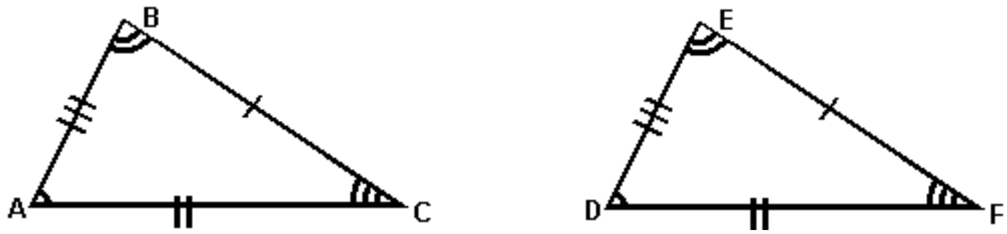
Congruent Polygons - two polygons are congruent if all the corresponding sides and angles of the polygons are congruent, respectively

Triangle Congruence

If we wanted to show two triangles were congruent using the definition, we would have to show all three sides and all three angles of one triangle are congruent to the corresponding three sides and angles of another triangle.

That's showing six separate congruences, three angles and three segments.

Let's look at an example.



Notice that these angles and sides all correspond:

$$\angle A \cong \angle D, \angle B \cong \angle E, \angle C \cong \angle F, \overline{AC} \cong \overline{DF}, \overline{AB} \cong \overline{DE} \text{ and } \overline{BC} \cong \overline{EF}$$

We'd write the triangles are congruent using mathematical notation; $\Delta ABC \cong \Delta DEF$

It's very important when you label the triangles congruent, you label the corresponding vertices in the same order. In other words, in the triangles above, we would **not** say $\Delta ABC \cong \Delta EFD$. Be careful to make sure you label the second triangle so the angles are in the same order (corresponding) as then first triangle.

There are 6 different ways to name a triangle, with respect to congruence, it does not matter how you name the first triangle. But, you must name the second triangle using the corresponding parts.

1. ΔABC ≅ ΔDEF
2. ΔBCA ≅ ΔEFD
3. ΔCAB ≅ ΔFDE
4. ΔBAC ≅ ΔEDF
5. ΔACB ≅ ΔDFE
6. ΔCBA ≅ ΔFED

By using this notation, I can determine without the aid of a diagram which angles are congruent and which segments are congruent.

Side Side Side Congruence

Rather than showing all three angles and all three sides congruent to show triangles are congruent, we might notice something special by looking at a few examples.

Let's say I was to give everyone in the class three sticks; one 10 inches long, another 8 inches, and the third 5 inches. I then asked everyone to glue the ends of the sticks together to form a triangle.

Guess what happens when I collect all the triangles formed by gluing the sticks together? That's right, they all fit very nicely on top of each other, they have the same size and shape – they coincide. They are congruent.

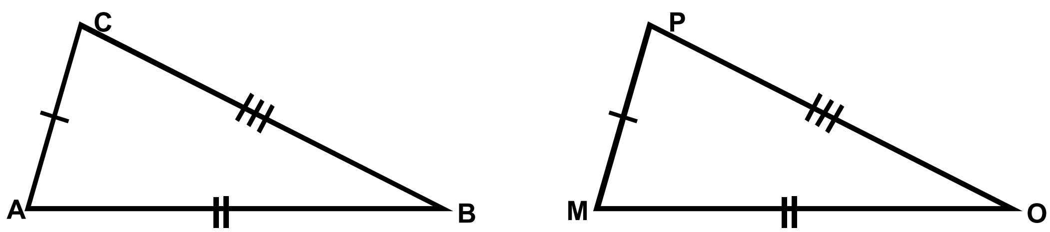
What we notice is a triangle's size and shape can be completely determined by its three sides. That's cool, but what's even cooler is its application to congruence of triangles.

If these triangles coincide, they must be congruent and we didn't even discuss angle measurements required in the definition of congruent polygons.

In other words, we found a shortcut to determine if triangles are congruent.

Side, Side, Side Congruence Postulate (SSS)

> If three sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles must be congruent.



Notice that the lines segments that are congruent have the same number of hatch marks and the order of the vertices on both triangles go in the same order as the hatch marks. That's very important!

$$\Delta ABC \cong \Delta MOP$$

Now remember, I can name this congruence 6 different ways. But how I name the first figure determines how I name the second.

Let's define some of the words we're going to use in this section.

A ***postulate*** or ***axiom*** is something we believe without proof, a basic assumption. In math, we believe it because it keeps happening and we can't find a circumstance when it does not happen.

A ***theorem*** is a statement that has to be proved. And a ***corollary*** is a statement that follows directly from a theorem – it also has to be proved.

Now that we have some of the terminology out of the way, let's look at some other triangles to see if we can determine congruence with<u>out</u> showing all six relationships, three congruent sides and three congruent angles.

Side Angle Side Congruence

What would happen if I gave everyone in the class two sticks, one 5 inches, the other 7 inches and asked the ends be glued together with a specific angle measurement?

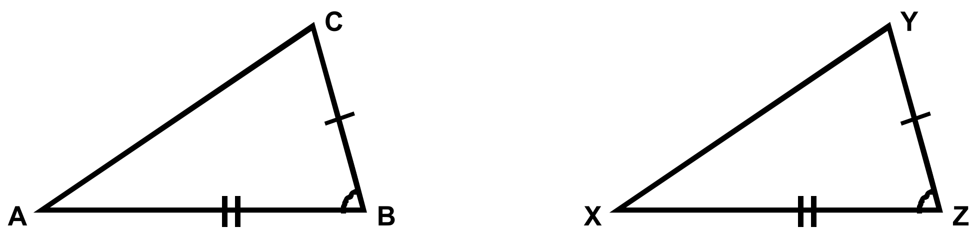
The angle formed by joining the two sides is called the included angle. If I then asked everyone to tie a string to the ends of those sticks, a triangle would be formed. Would the triangles be congruent? You're probably thinking I would not have asked the question unless that was true.

If I then collected those triangles, I might notice something special, the triangles are all the same size and shape - they coincide. That would lead me to believe the triangles are congruent.

Yes, another shortcut!

Side, Angle, Side Congruence Postulate (SAS)

> If two sides and the included angle of one triangle are congruent to the corresponding parts of another triangle, then the triangles are congruent.



$$\Delta ABC \cong \Delta XZY$$

Remember, this is very important, the included angle is the angle between the two sides.

We have now seen that triangles can be determined to be congruent by **SSS** or **SAS**. Those postulates shorten our work, rather than showing six congruences, I was able to cut my work in half. Do you think we can go through similar processes to determine if other triangles are congruent?

Of course, but being a little more sophisticated, rather than passing out sticks, we just draw them.

Angle Side Angle Congruence

Can we find a shortcut to determine if two triangles are congruent knowing the measurements of two angles and a side?

Well, we have two possibilities. One possibility has the side included (between) the two angles. If we were all to draw a line 4 inches long (the length really does not matter), then measure two different angles off each end, we'd end up with congruent triangles.

Angle, Side, Angle Congruence Postulate (ASA)

> If two angles and the included side of one triangle are congruent to the corresponding parts of another triangle, then the triangles are congruent.

$$\Delta CAB \cong \Delta XZY$$

Angle Angle Side

What happens if the side is not included?

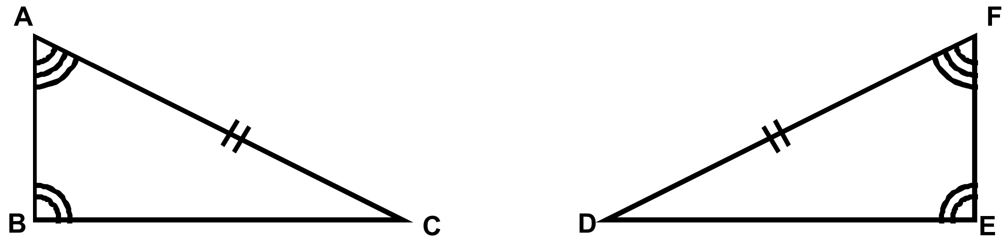
With further investigation, kids call that playing, we might see another relationship that follows directly from the ASA Postulate.

It turns out if we know two angles of one triangle are congruent to the two angles of another triangle, the third angles must also be congruent. Remember, the sum of the interior angles of a triangle is 180°.

That would lead us back to the Angle, Side, Angle Congruence Theorem. But rather than having to go back to that theorem, we can take a short cut called the Angle, Angle, Side

Angle, Angle, Side Congruence Theorem

> If two angles and the non-included side of one triangle are congruent to the corresponding parts of another triangle, the triangles are congruent.



$$\Delta ABC \cong \Delta FED$$

We have two angles and the non-included side. However, by quickly realizing that angle C must be equal to angle D, we can see that those two triangles could have been shown to be congruent by **ASA Postulate**.

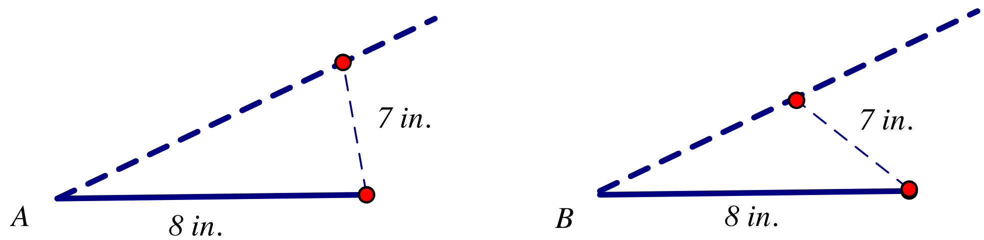
By using the congruence postulates and theorems, we cut our work of showing triangles congruent in half. Rather than showing all three sides and all three angles are congruent by the definition, we can just look at half that information. Don't you just love math?

You are wondering, can finding congruent triangles be difficult? The answer is no. I can try to camouflage the relationships by flipping them, rotating them, or joining them. But, if you know the SSS Postulate, the SAS Postulate, the ASA Postulate, and the AAS Theorem, then you can determine if triangles are congruent.

Remember the order of those letters is important. The SAS Postulate is two sides and the included (between) angle, that's why the A is in the middle. The ASA Postulate has the S between the two A's because its an included side.

Notice we do not have an Angle, Side, Side theorem. The reason for that is simple, given an angle and two sides does not guarantee a unique triangle as the previous three postulates guaranteed.

Let's look, let's say I have a triangle with two sides measuring 8 inches and 6 inches and I want an angle measurement of 30 degrees. One person could draw the triangle on the left, another using the same measurements could draw the triangle on the right. Notice, they are not congruent. That's why there is no ASS theorem.



Angle A is congruent to Angle B

Notice those two triangles do not have the same size and shape, they are not congruent.

Showing Triangles Congruent

In the following problems, show the triangles are congruent. You can not go by the picture alone. Either information has to be given to you explicitly, two lines or angles are equal, or that information has to be derived from the geometry you have already learned.

You have 4 ways of showing triangles are congruent:

SSS SAS ASA AAS

If the problems assigned were direct applications of those postulates and theorem, then your work would be pretty much done. Unfortunately, that's not the case. In order to use SSS, SAS, ASA and AAS, we usually have to use our previously learned geometry to show angles or segments are congruent.

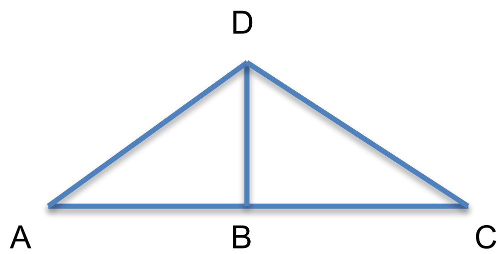
The following are relationships you should keep in mind so you can show congruence in the diagrams you are given.

Angles formed by parallel lines Vertical angles Perpendicular lines

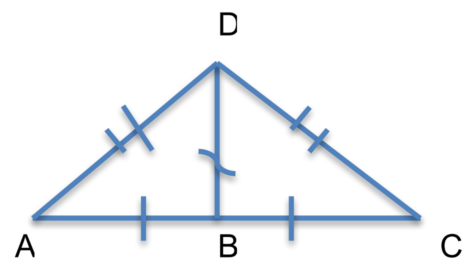
Reflexive Property Midpoints Bisectors

When you identify these on your diagram, it's usually easy to determine if the triangles are congruent and the method used to show congruency.

Example 1 Name the congruent triangles and the method used to show congruence. Given B is the midpoint of $\overline{AC}$ and $\overline{AD} \cong \overline{CD}$



First place tick marks on all already learned, if B is a midpoint, then $\overline{AB} \cong \overline{BC}$. And using the Reflexive Property, $\overline{BD} \cong \overline{BD}$. The explicit information given $\overline{AD} \cong \overline{CD}$. Now, using the geometry we have



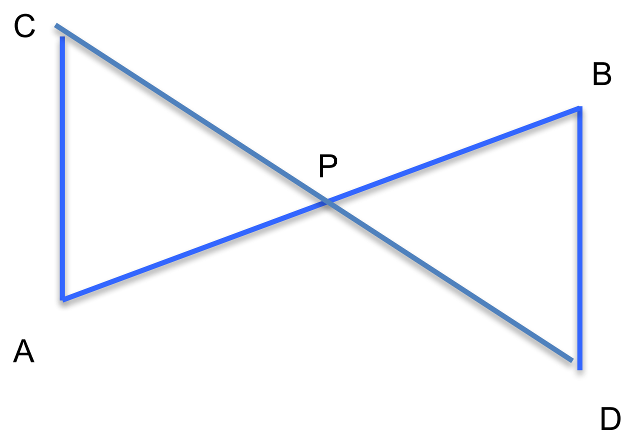
We can now see we have three sides of on triangle congruent to three sides of another triangle, therefore the triangles are congruent by SSS.

Δ ABD $\cong$ Δ CBD by SSS

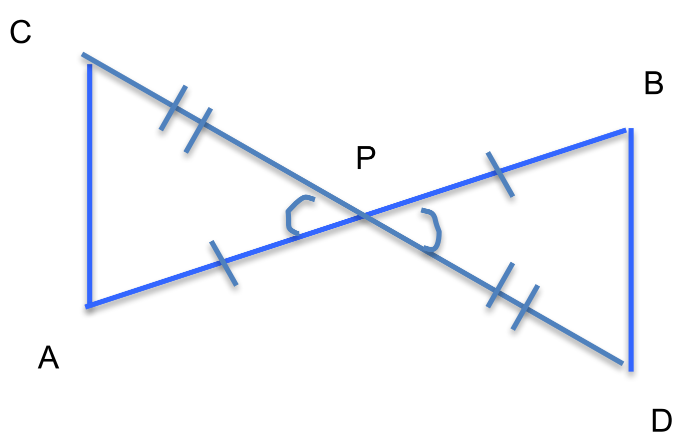
It is important that you mark up your diagram, it will help you see the congruent parts so you can determine if you are going to use SSS, SAS, ASA, or AAS.

And to get there, you have to remember the math you learned; look for angles formed by parallel or perpendicular lines, vertical angles, midpoints, bisectors, and the reflexive property, etc.

Example 2 Name the congruent triangles and the method used to show congruence.
Given: CD and AB bisect each other

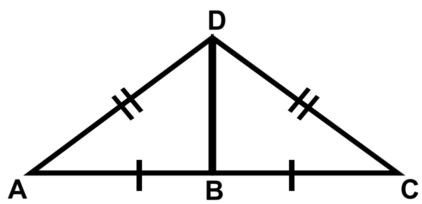


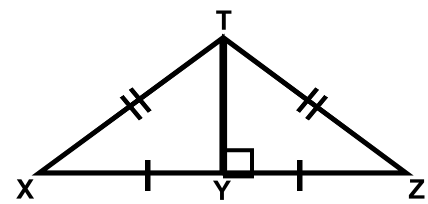
Marking the diagram with ticks with congruences, we are given bisectors and we can see vertical angles.

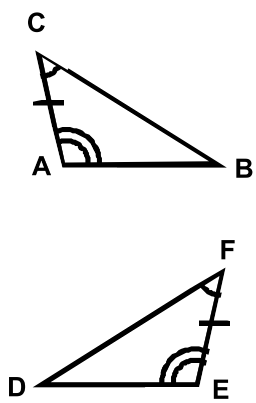


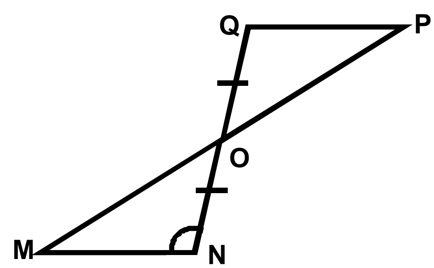
So ΔAPC ≅ ΔBPD by SAS

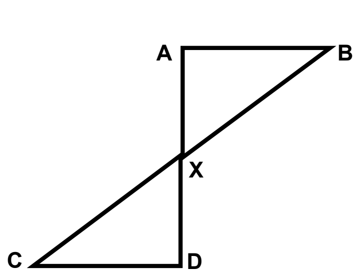
In the following, name the congruent triangles and the method used to show congruence.



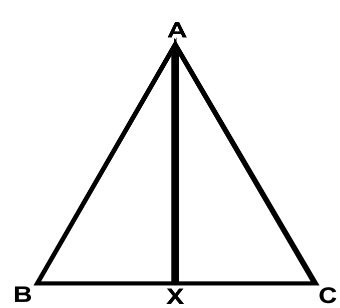




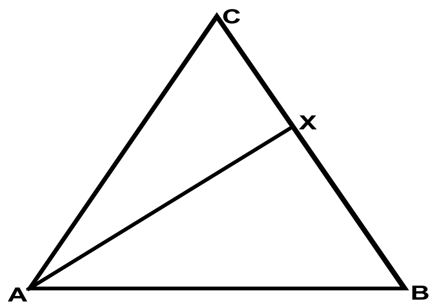




$\overline{AB} \parallel \overline{CD}$, $\overline{BC}$ $\underline{b \perp}$ $\overline{AD}$

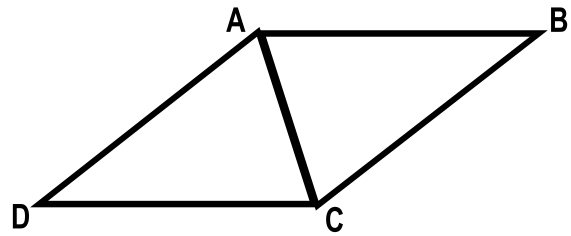


$\overline{AX}$ *is* $\perp$ $\underline{b}$ $\overline{BC}$

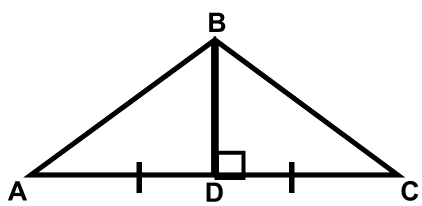


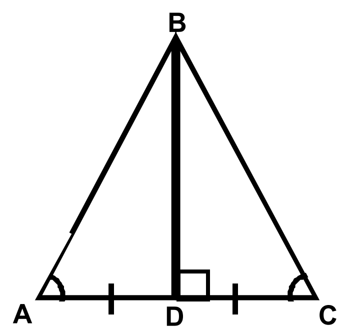
$\overline{AX}$ $\underset{=}{b}$ $\angle CAB$

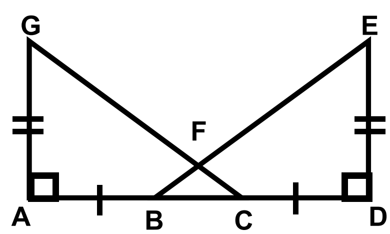
ΔABC is equilateral

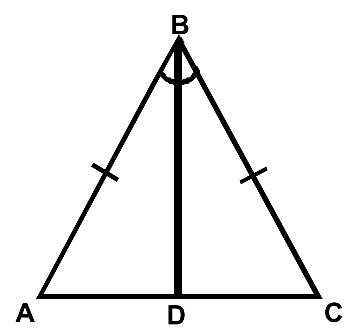


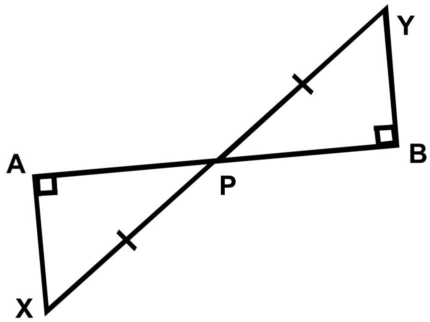
□ $ABCD \rightarrow$ parallelogram







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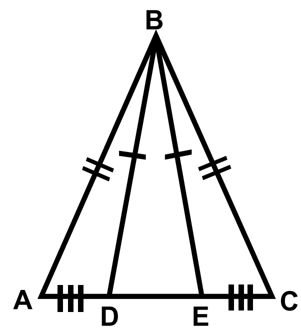
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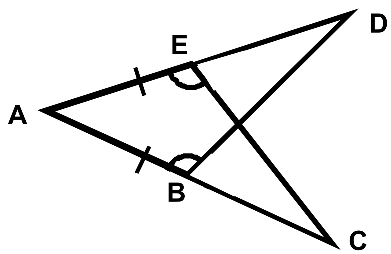
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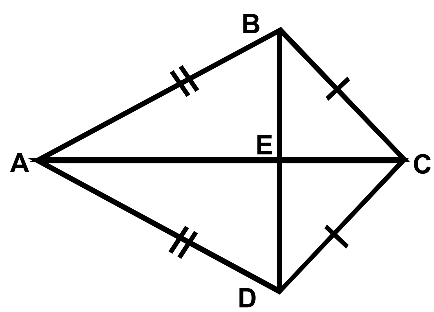
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Right Triangle Congruence

The theorems and postulates we just studied also work for right triangles and are often just corollaries to those. As an example, we will look at the Leg Leg Theorem for right triangles.

LL Theorem

> If two legs of one right triangle are congruent to two legs of another right triangle, the triangles are congruent.

If you drew two right triangles and marked the congruent legs, you would quickly see the right angles are the included angles of the triangles. That would quickly see that the triangles would be congruent by SAS.

The next postulate, not a theorem, would have to be demonstrated as we did earlier with the SSS, SAS, and ASA postulates. In other words, we can not prove it, we accept it as true.

HL Postulate

> If the hypotenuse and leg of one right triangle are congruent to a hypotenuse and a leg of another right triangle, then the triangles are congruent.

The other right triangle theorems are corollaries, direct results of the SSS, SAS and ASA postulates.

You should be able to draw the pictures associated with the following theorems and explain how they relate to SSS, SAS, and ASA postulates as we did for the LL Theorem,

HA Theorem

> If the hypotenuse and acute angle of one right triangle are congruent to the corresponding parts of another right triangle, the triangles are congruent.

LA Theorem

> If a leg and acute angle of one right triangle are congruent to the corresponding parts of another right triangle, then the triangles are congruent.

The HL Theorem is important because it is not a direct result of one of the postulates which means you have to remember it, but it does come in handy.

So now we have eight ways to prove triangles are congruent.