

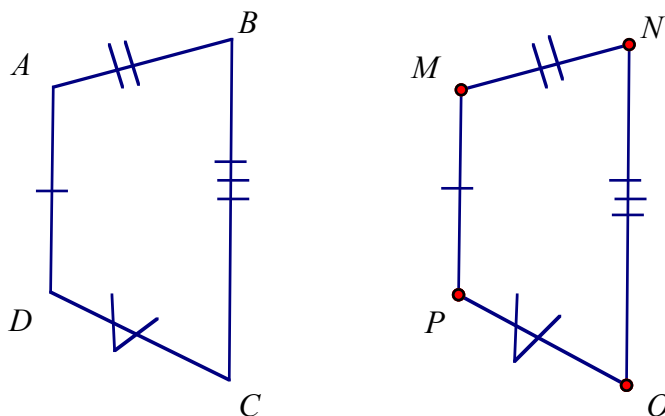
Chapter 7

Congruence Postulates & Theorems - Δ 's

In math, the word *congruent* is used to describe objects that have the same size and shape. When you traced things when you were a little kid, you were using congruence. Neat, don't you think?

Stop signs would be examples of congruent shapes. Since a stop sign has 8 sides with equal lengths and equal angles, they would be congruent octagons.

Congruent polygons – polygons are congruent when all the sides **and** all the angles of one polygon are congruent to all the angles and all the sides of another polygon respectively.



$$\angle A \cong \angle M, \angle B \cong \angle N, \angle C \cong \angle O, \text{ and } \angle D \cong \angle P$$
$$\text{Quad } ABCD \cong \text{Quad } MNOP$$

To show the two quadrilaterals are congruent, I need to show eight congruences – 4 angles and 4 sides respectively.

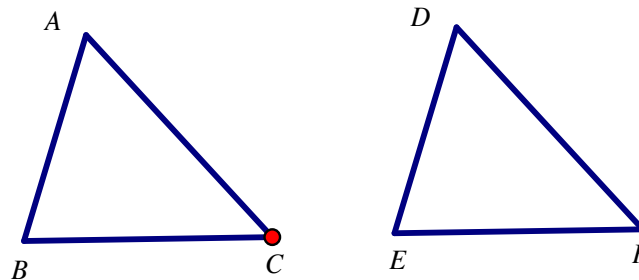
How I name my congruent polygons as congruent is very important. The respective angles and sides must appear in the same order. So, when we write

$$\text{Quad } ABCD \cong \text{Quad } MNOP$$

That means position is important. $\angle C$ corresponds to $\angle O$, notice they are both in the third position.

We are going to look specifically at triangles. To determine if two triangles are congruent, they must have the same size and shape. They must fit on top of each other, they must coincide.

Mathematically, we say all the sides and angles of one triangle must be congruent to the corresponding sides and angles of another triangle respectively.



In other words, we would have to show angles A , B , and C were congruent (\cong) to angles D , E and F respectively, then show \overline{AB} , \overline{BC} , and \overline{AC} were \cong to \overline{DE} , \overline{EF} , and \overline{DF} respectively.

We would have to show those six relationships.

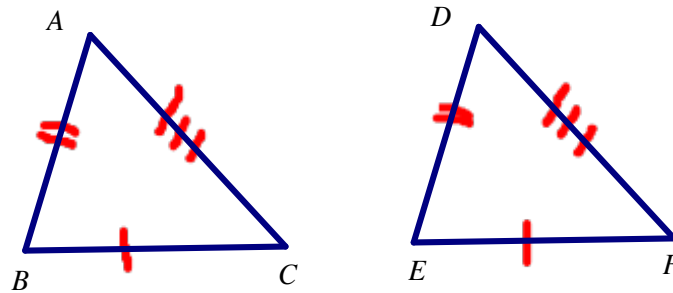
Ah, but there is good news. If I gave everyone reading this three sticks of length 10", 8", and 7", then asked them to glue the ends together to make triangles, something interesting happens. When I collect the triangles, they all fit very nicely on top of each other, they have the same size and shape, they coincide – they are congruent!

Why's that good news? Because rather than showing all the angles and all the sides of one triangle are congruent to all the sides and all the angles of another triangle (6 relationships), I was able to determine congruence just using the 3 sides - a shortcut.

That observation leads us to the side, side, side congruence postulate. Remember, a postulate is something we accept as true – without proof.

SSS Postulate

If three sides of one triangle are congruent, respectively, to three sides of another triangle, then the triangles are congruent.



I could write that as $\triangle ABC \cong \triangle DEF$. Notice how the \cong angles and sides have the same position. Someone else could have written that relationship as $\triangle ACB \cong \triangle DFE$. Again, notice how you name the first triangle effects how you arrange the letters in the second triangle.

There are 6 ways I could have written that congruence statement. What you need to remember is the corresponding parts must line up. Using that notation allows me to determine the congruent parts without looking at a diagram.

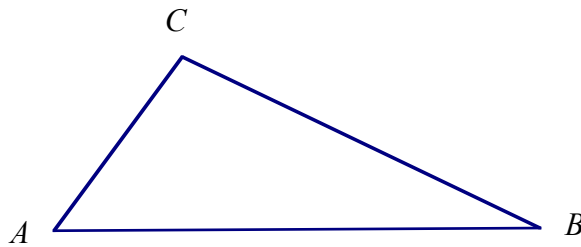
Here are the 6 ways, notice how the angles and sides are aligned.

$$\begin{aligned}\triangle ABC &\cong \triangle DEF \\ \triangle BCA &\cong \triangle EFD \\ \triangle CAB &\cong \triangle FDE \\ \triangle BAC &\cong \triangle EDF \\ \triangle ACB &\cong \triangle DFE \\ \triangle CBA &\cong \triangle FED\end{aligned}$$

So, if I chose $\triangle CAB \cong \triangle FDE$, notice how angles C and F correspond, angles A and D, and angles B and E. Now look at the order of the sides; side CA corresponds to side FD, side AB corresponds to DE, and CB corresponds to FE. Without looking at the picture, I can name these relationships.

Learning the vocabulary and notation is extremely important for success in any field. To be clear, let's examine the words "opposite" and "included" in the context of triangles.

Looking at the following triangle and the phrases in the box, these are important for you to know.

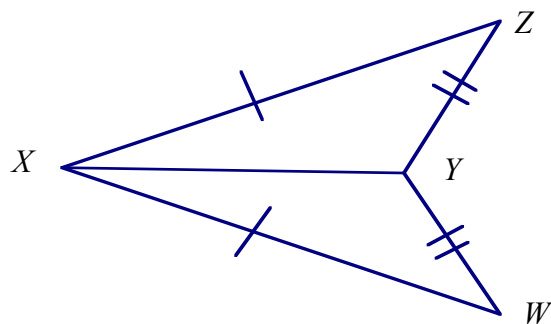


$\angle B$ lies opposite \overline{AC}
 $\angle B$ is included between \overline{AB}
 and \overline{BC}
 \overline{BC} lies opposite $\angle A$
 \overline{BC} is included between $\angle B$
 and $\angle C$

Now, let's use our knowledge and do a proof. When doing proofs, remember to examine the diagram and see if there are relationships we can use. In this next example, for instance, the segment \overline{XY} appears in both triangles.

Example: 1 Given: $\overline{XZ} \cong \overline{XW}$ and $\overline{YZ} \cong \overline{YW}$
 Prove: $\triangle XYZ \cong \triangle XYW$

Marking my congruent sides.



In order to prove triangles congruent, I have to use the definition of congruence or the SSS Congruence postulate. The definition is too much work. The problem I am facing is I only have two sides of one triangle congruent to two sides of another triangle. I need three sides of both triangles. If I looked long enough, I will notice that \overline{XY} is in both triangles. Mark that on

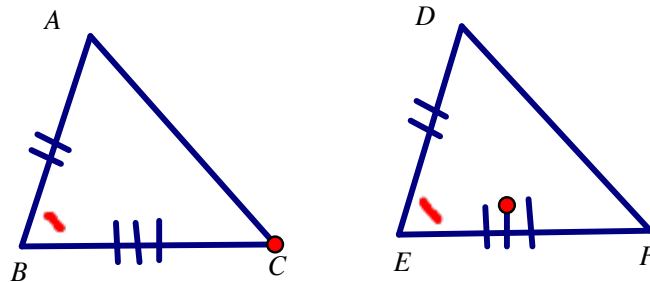
the diagram. Now you can see three sides of one triangle are congruent to three sides of another triangle, respectively.

| | <u>STATEMENTS</u> | <u>REASONS</u> |
|----|---|-------------------|
| 1. | $\overline{XZ} \cong \overline{XW}$ and $\overline{YZ} \cong \overline{YW}$ | Given |
| 2. | $\overline{XY} \cong \overline{XY}$ | Reflexive prop |
| 3. | $\triangle XYZ \cong \triangle XYW$ | SSS \cong Post. |

Using the same type of observations as before, we can come up with two more congruence postulates.

The side, angle, side postulate is abbreviated **SAS Postulate**.

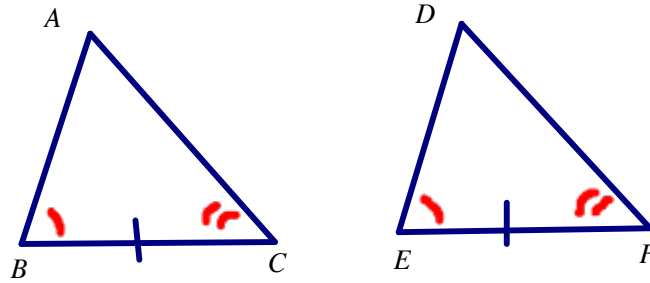
SAS Postulate **If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, respectively, then the two triangles are congruent.**



Using that postulate, we know $\triangle ABC \cong \triangle DEF$. Remember, after you name the first triangle, you must name the second triangle so the letters are positioned with the corresponding angles and segments.

A third postulate is the angle, side, angle postulate.

ASA Postulate **If two angles and the included side of one triangle are congruent to the corresponding parts of another triangle, the triangles are congruent.**



Naming the \cong Δ s, we have $\triangle BCA \cong \triangle EFD$

If you are going to be successful, you need to memorize those three postulates and be able to visualize that information.

SSS, SAS and ASA Congruence Postulates

Combining this information with previous information, we will be able to determine if triangles are congruent. So, study, review and make sure you can visualize these relationships!

Proofs: Congruent Δ 's

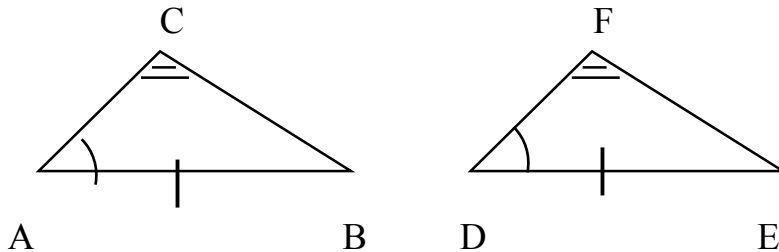
To prove other triangles are congruent, we'll use the **SSS, SAS and ASA** congruence postulates. We also need to remember other theorems that will lead us to more information.

For instance, you should already know by theorem the sum of the measures of the interior angles of a triangle is 180° . A corollary to that theorem is if two angles of one triangle are congruent to two angles of another triangle; the third angles must be congruent. OK, that's stuff we need to remember to successfully prove theorems.

Using that information, let's try to prove this congruence theorem.

AAS Theorem **If two angles and the non included side of one triangle are congruent to the corresponding parts of another triangle, the triangles are congruent.**

First let's draw and label the two triangles.



Given : $\angle A \cong \angle D$, $\angle C \cong \angle F$, $\overline{AB} \cong \overline{DE}$

Prove: $\triangle ABC \cong \triangle DEF$

| Statements | Reasons |
|--|--|
| 1. $\angle A \cong \angle D$ $\angle C \cong \angle F$ $\overline{AB} \cong \overline{DE}$ | Given |
| 2. $\angle B \cong \angle E$ | 2 \angle 's of a Δ congruent 2 \angle 's of another Δ , 3 rd \angle 's congruent |
| 3. $\triangle ABC \cong \triangle DEF$ | ASA |

That was too easy. Now we have 4 ways of proving triangles congruent: **SSS**, **SAS**, **ASA**, and **AAS**. You need to know those. I know what you are thinking; you want to try another one. OK, we'll do it!

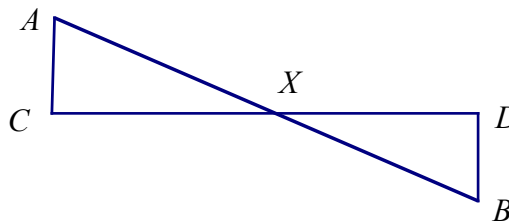
Here's what you need to be able to do. First, label congruences in your picture using previous knowledge. After that, look to see if you can use one of the four methods (SSS, SAS, ASA, AAS) of proving triangles congruent. Finally, write those relationships in the body of proof. You are done. Piece of cake!

Let's look at another diagram and prove the triangles are congruent.

Example 2

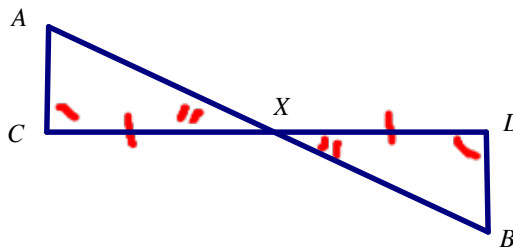
Given: $\overline{AC} \parallel \overline{BD}$,
 \overline{AB} bisects \overline{CD}

Prove: $\triangle ACX \cong \triangle BDY$



Remember two things: 1) to mark up your picture with that information as I did below and 2) Use the geometry you have learned. In this diagram, you have lines that intersect, that leads to vertical angles. And you are given lines are parallel. Parallel lines cut by a transversal leads to a number of relationships – corresponding angles, alternate interior angles, etc.

So, to start, write down what was given. What relationship is formed when something is bisected mark it on the diagram.



| Statements | Reasons |
|--|---|
| 1. $\overline{AC} \parallel \overline{BD}$ \overline{AB} bis. \overline{CD} | Given |
| 2. $\overline{CX} \cong \overline{DX}$ | Def. of bisector |
| 3. $\angle C \cong \angle D$ | \parallel lines cut by tran, alt int \angle 's are \cong |
| 4. $\angle AXC \cong \angle BXD$ | Vert. \angle 's are \cong |
| 5. $\triangle ACX \cong \triangle BDY$ | ASA |

I can't tell you how important it is to fill in the picture by labeling the information given to you and writing other relationships you know from previous theorems, postulates, and definitions.

In order to prove that, we had to know the definition of a bisector and the subsequent mathematical relationship. And, even though vertical angles were not part of the information given, we could see from the diagram vertical angles were formed and we could then use the theorem that all vertical angles are congruent.

4 Ways to Prove Triangles Congruent

SSS, SAS, ASA, AAS

and

Use the Diagram to use other relationships;

Reflexive Property, \angle 's formed by Parallel or Perpendicular Lines,

Vertical \angle 's, Midpoints, and Bisectors

Right Triangles - Congruence

While the congruence postulates and theorems apply for all triangles, we have postulates and theorems that apply specifically for right triangles.

HL Postulate **If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent.**

Since the HL is a postulate, we accept it as true without proof. The other congruence theorems for right triangles might be seen as special cases of the other triangle congruence postulates and theorems we just learned.

LL Theorem **If two legs of one right triangle are congruent to two legs of another right triangle, the triangles are congruent.**

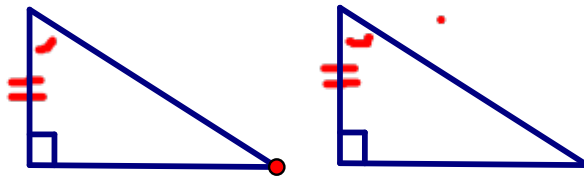
This is a special case of the SAS Congruence Theorem.

HA Theorem **If the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and acute angle of another right triangle, the triangles are congruent.**

This congruence theorem is a special case of the AAS Congruence Theorem. And finally, we have the Leg Angle Congruence Theorem.

LA Theorem **If a leg and an acute angle of one right triangle are congruent to a leg and an acute angle of another right triangle, the triangles are congruent.**

If you drew and labeled the picture of the LA Congruence Theorem, you would see that could be derived from the ASA or AAS congruence theorems.



Right Triangle Congruence Postulates & Theorems

HL, LL, HA, and LA

Proofs: **cpctc**

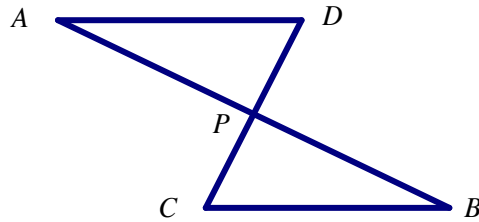
When two triangles are congruent, each part of one triangle is congruent to the corresponding part of the other triangle. That's referred to as ***corresponding parts of congruent triangles are congruent, thus cpctc.***

One way you can determine if two line segments or two angles are congruent is by showing they are the corresponding parts of two congruent triangles.

1. Identify two triangles in which segments or angles are the corresponding parts.
2. Prove the triangles are congruent.
3. State the two parts are congruent, supporting the statement with the reason; "corresponding parts of congruent triangles are congruent"

That reason is normally abbreviated "**cpctc**"

Let's see if we can prove two line segments are congruent.

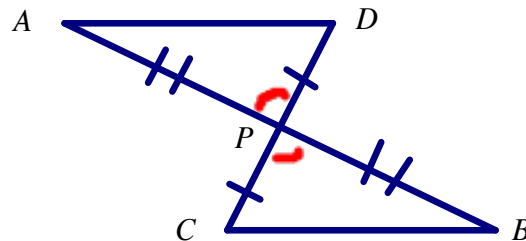


Given: \overline{AB} and \overline{CD} bisect each other

Prove: $\overline{AD} \cong \overline{BC}$

Now the strategy to prove the segments are congruent is to first show the triangles are congruent.

So, let's fill in the picture showing the relationships based upon the information given and other relationships that exist using our previously learned definitions, theorems, and postulates. Look for vertical angles, midpoints, bisectors, etc.



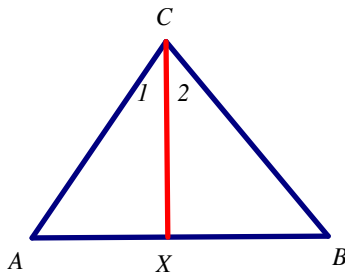
Using the definition of bisector, I can determine that $\overline{AP} \cong \overline{PB}$ and $\overline{CP} \cong \overline{PD}$. I also notice I have a pair of vertical angles. Notice how I marked the drawing to show these relationships. Looking at the triangles, I can show they are congruent by SAS.

Let's go ahead and fill in the body of the proof using the diagram

| Statements | Reasons |
|---|--------------------------|
| 1. \overline{AB} and \overline{CD} bisect each other | Given |
| 2. $\overline{AP} \cong \overline{PB}$ $\overline{CP} \cong \overline{PD}$ | Def. of bisector |
| 3. $\angle APD \cong \angle PBC$ | Vert \angle 's \cong |
| 4. $\triangle APD \cong \triangle PBC$ | SAS |
| 5. $\overline{AD} \cong \overline{BC}$ | cpctc |

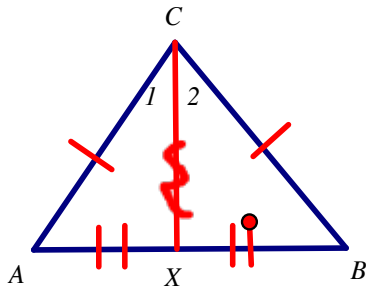
Filling in the body of the proof is easy after you mark the congruences, in your picture.

The strategy to show angles or segments are congruent is to first show the triangles are congruent, then use cpctc. Try this next one on your own.



Given: Diagram
 $\overline{AC} \cong \overline{BC}$, $\overline{AX} \cong \overline{BX}$

Prove: $\angle 1 \cong \angle 2$



Mark the picture with the parts that are congruent based on what's given to you.

Then mark the relationships based upon your knowledge of geometry.

In this case $\overline{CX} \cong \overline{CX}$

From the picture we can see three sides of one triangle are congruent to three sides of another triangle, therefore the triangles are congruent by SSS Congruence Postulate.

$$\triangle AXC \cong \triangle BXC$$

If the triangles are congruent, then the corresponding parts of the triangles are congruent by cpctc. That means $\angle 1 \cong \angle 2$, just what we wanted to prove.

Construct a T-Proof using that argument.

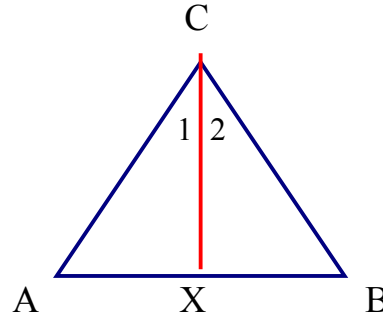
| Statements | Reasons |
|--|-----------|
| 1. $\overline{AC} \cong \overline{BC}$, $\overline{AX} \cong \overline{BX}$ | Given |
| 2. $\overline{CX} \cong \overline{CX}$ | Reflexive |
| 3. $\triangle CXA \cong \triangle CXB$ | SSS |
| 4. $\angle 1 \cong \angle 2$ | cpctc |

We can develop quite a few relationships based upon knowing triangles are congruent. Let's look at a few.

Theorem. If 2 \angle 's of Δ are \cong , the sides opposite those \angle 's are \cong

Given: ΔABC
 $\angle A \cong \angle B$

Prove: $\overline{AC} \cong \overline{BC}$



| Statements | Reasons |
|--|-----------------------|
| 1. Draw \angle bisector CX | Construction |
| 2. $\angle 1 \cong \angle 2$ | Def \angle bisector |
| 3. $\angle A \cong \angle B$ | Given |
| 4. $CX \cong CX$ | Reflexive Prop |
| 5. $\Delta CAX \cong \Delta CBX$ | AAS |
| 6. $\overline{AC} \cong \overline{BC}$ | cpctc |

Hopefully, you were able to follow the proof. It would have been easier to follow if we marked up the diagram!

The converse of that theorem is also true. But we also have a corollary to that theorem that will be helpful to remember.

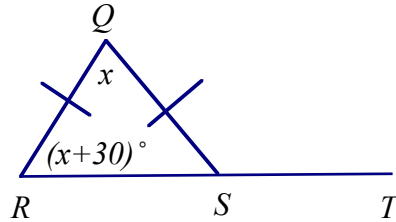
Corollary An equiangular triangle is also equilateral.

Theorem If 2 sides of a triangle are congruent, the angles opposite those sides are congruent. (Base \angle 's of isosceles Δ are \cong)

Corollary An equilateral triangle is also equiangular.

Corollary Each angle of an equilateral triangle has measure 60° .

Example 3 Given the diagram, $\overline{QR} \cong \overline{QS}$, $m\angle Q = x^\circ$, $m\angle R = (x+30)^\circ$
Find the value of x and $m\angle QST$.



Since $\overline{QR} \cong \overline{QS}$, we know the \angle s opposite those sides are \cong .
That means that $\angle R \cong \angle S$.

We also know the sum of the interior \angle s of a $\Delta = 180^\circ$.

$$\begin{aligned} m\angle R + m\angle S + m\angle Q &= 180^\circ \\ (x+30)^\circ + (x+30)^\circ + x &= 180^\circ \\ 3x + 60 &= 180^\circ \\ 3x &= 120^\circ \\ x &= 40^\circ \end{aligned}$$

In order to find the measure of any angle, we first had to find the value of x .

$$\begin{aligned} m\angle Q &= 40^\circ \\ m\angle R &= 70^\circ \end{aligned}$$

To find $m\angle QST$, I know *the exterior angle of a triangle is equal to the sum of the two remote interior angles*. Therefore,
 $m\angle QST = 110^\circ$

If I had forgot that theorem, I could have found the $m\angle QSR$, then used the theorem *if the exterior sides of a triangle lie in a line, the angles are supplementary*.

What you need to remember in math: *the more you know, the easier it gets!*

The more postulates, theorems and corollaries you know makes the proofs shorter.

Remember while you are expected to memorize these, you also need to have a visual so they make more sense to you. And lastly, the easiest way to make solving problems easier or constructing proofs is to mark up your diagram. While information may be given to you explicitly, other information might be given to implicitly. So, use the relationships in the Reflexive Property, Vertical angles, angles formed by parallel or perpendicular lines, midpoints and bisectors.