Ch. 8   Polynomial Factoring

Sec. 1

Factoring is the process of changing a polynomial expression that is essentially a sum into an expression that is essentially a product.

Factoring polynomials is not much different than finding the prime factors of composite numbers done in earlier grades. What has to happen in both situations is students have to look at the problem at hand, then using their background knowledge, identify patterns, then select a method to rewrite the composite number as a product.

Let's look at a couple of examples. If I asked a student to find the prime factors of 117, the first thing many students would do is check to see if 117 is divisible by numbers they are very familiar, such as 2, 5, and 10. Since 117 is not even and does not end in 0 or 5, they students will deduce that they can't use those numbers as factors.

Now, they will have to look for other ways of finding factors. Hopefully, rather than using trial and error and dividing, they would use the rules of divisibility. Since the sum of the digits of 117 is 9, we know that 117 is divisible by 3 and by 9.

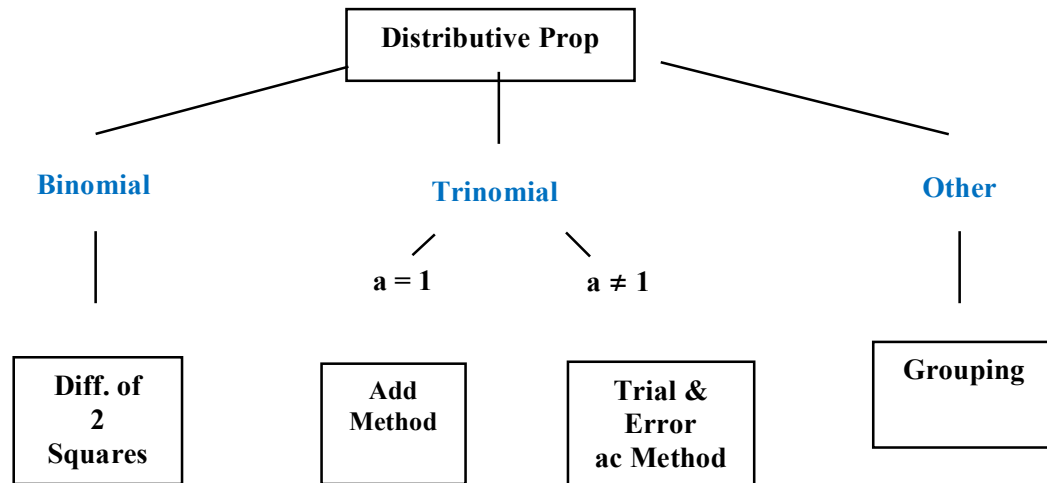So 117 = 9 x 13, factoring the 9, we have 117 = 3 x 3 x 13

Polynomial factoring uses the same type of reasoning. That is, you look at a polynomial and determine if there are common factors, much like a student would have done by checking if a number was divisible by 2, 5, or 10. In algebra, we would factor out those common factors using the Distributive Property.

Once that was accomplished, a student would look for a method to factor the rest of the polynomial – using special products. With composite numbers, students determined which rule of divisibility might apply.

Just like we used our knowledge of the rules of divisibility to factor numbers, we checked our most common rules, 2, 5 and 10, then checked the

sum of the digit rules, then the combination rules. We will follow a procedure to factor polynomials.

We will use the schematic below to factor polynomials.

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                        ┌─────────────────────┐
                        │  Distributive Prop  │
                        └─────────────────────┘
              ╱                    │                    ╲
       Binomial              Trinomial                  Other
          │                 ╱         ╲                   │
          │            a = 1           a ≠ 1              │
   ┌───────────┐    ┌──────────┐   ┌───────────┐   ┌───────────┐
   │  Diff. of │    │   Add    │   │  Trial &  │   │ Grouping  │
   │     2     │    │  Method  │   │   Error   │   │           │
   │  Squares  │    │          │   │ ac Method │   │           │
   └───────────┘    └──────────┘   └───────────┘   └───────────┘
```

Factoring is used to simplify algebraic expressions and solve higher degree equations.

The diagram suggests that you always factor by first trying to use the **Distributive Property.** After that, you then determine if you have a binomial, trinomial or other.

If you have a binomial, we'll try to use the method called the **Difference of 2 Squares**.

If you have a trinomial and the coefficient if the quadratic term (number in front of the squared term) is one, we'll use the **Addition Method**. If the coefficient of the quadratic term is NOT one, then we will use **Trial & Error or the ac Method** to factor.

If I don't have a binomial or trinomial, then I will use **Grouping**.

Sec 2   Distributive Property

Let's look at factoring using the **Distributive Property**.

$$a(b + c) = ab + ac$$

To factor, we look for numbers of letters that appear in each term of a polynomial.

Example 1   Factor   4x + 12

Is there a number or letter that appears on both terms? Hopefully you realized there was a factor of 4 in both terms. Taking the 4 out, we have

4x + 12 = 4(x + 3)

Example 2   Factor 4x + 12xy

I have a 4x in both terms, factoring that we have
4x + 12xy = 4x(1 + 3y)

We use the Distributive Property all the time when multiplying mentally. Multiplying 6 by 13 would be done by 6x10 + 6(3) = 60 + 18 = 78.

Using the Distributive Property to 12 x 24 = 24(10 + 2) = 240 + 48 = 288

Recognizing those patterns makes computation much easier.

Sec. 3   Difference of 2 Squares

Now let's see what happens if I have a binomial. The diagram suggests I use the **Difference of Two Squares.**

$$a^2 - b^2 = (a - b)(a + b)$$

I use the Difference of Two Squares if I have a binomial, both terms being perfect squares, and a " – " sign separates them.

3

To factor a binomial using the Difference of Two Squares:

1. Factor using the Distributive Property – if possible
2. Take the square root of each term, write them twice in parentheses, as shown below, put a "+" sign between one and a " – " between the other.

Example 3   Factor $x^2 – 9$

The square root of x^2 is x, the square root of 9 is 3.
$x^2 – 9 = (x + 3)(x – 3)$

Example 4   Factor $49x^2 – 81y^2$

The square roots of $49x^2$ and $81y^2$ are 7x and 9y. Therefore, we have

$49x^2 – 81y^2 = (7x + 9y)(7x – 9y)$

Factor completely using the Distributive Property or Difference of Two Squares.
Remember, always try to use the Distributive Property FIRST!

We can use the Difference of 2 Squares in computation as well. To multiply 22 by 18, I recognize 22 is (20 + 2) and 18 is (20 – 2); in my head I have $(20+2)(20–2) = 20^2 – 2^2 = 400 – 4 = 396$.

When you use the Difference of 2 Squares for computation, we are looking for a nice midpoint, then adding and subtracting the same number. Multiply 54x46. What's a nice midpoint? If you said 50, that's great. So re-writing the multiplication we have $(50+4)(50–4) = 2500 – 16 = 2486$. Piece of cake with a little practice.

A	B	C
1) $3x + 6$	$5y^2 – 15y$	$2x^2 – 4x$
2) $5xy – 10xy^2$	$6xy^2z^3 – 12xy^2$	$8x^2 – 4x^3y^2$

3) $x^2 - 25$ $y^2 - 16$ $z^2 - 100$

4) $3x^2 - 12$ $3x^2 - 75$ $2x^3 - 32x$

5) $3x^2y - 27xy$ $5y^2 - 5$ $x^2 - 1$

6) $x^{16} - 1$ $x^2 - 81$ $x^2 + 25$

$5x^3y - 20xy$ $y^2 + 16$ $2x^2 - 100$

So far, we have factored using the Distributive Property and the Difference of Two Squares. Now, we'll look at TRINOMALS and use the Addition Method or Trail & Error.

Sec. 4   **Addition Method**

The general form of a quadratic equation is

$$a\mathbf{x^2 + bx + c = 0}; \quad a = 1$$

The "*a*" represents the coefficient of the quadratic term. When $a = 1$, we'll use the **Addition** **Method** of factoring.

To factor using the Addition Method:
1. Factor using the Distributive Property - if possible
2. Off to the side, write all the factors of the constant (c)
3. Find which of those add up to the linear term (b) and circle.
4. Write those as part of your binomial factors in parentheses

Example 5   Factor $x^2 + 7x + 12$ **12**

Write all the factors of 12   12, 1
4 and 3 also add to 7   6, 2
                             4, 3

so $x^2 + 7x + 12 = (x + 4)(x + 3)$

Example 6   Factor $x^2 + 25x + 24$ **24**

Write all the actors of 24   24, 1
24 and 1 also add to 25   12, 2
                             8, 3
                             6, 4

so $x^2 + 25x + 24 = (x + 24)(x + 1)$

Example 7   Factor $x^2 - 3x - 10$ **−10**

                             −10, 1
− 5 and 2 add to − 3   10, −1
                             −5, 2
so $x^2 - 3x - 10 = (x - 5)(x + 2)$ 5, −2

When finding factors of c whose sum is b by guessing, a more systematic approach is to use the idea of multiplying by 1 in the form of a fraction. That is, 1/2 x 2, 1/3 x 3, etc. That works by taking c x 1, then dividing by 2 and multiplying by 2, etc.

Sample: $x^2 + 19x + 60$ **60**
                             60, 1   ½ x 2
                             30, 2   ½ x 2
                             15, 4   ½ x 2
                             20, 3   1/3 x 3
                             10, 6   ½ x 2
                             5, 12

$(x + 15)(x + 4)$

The addition method can be used for computation too. What a surprise!

When you have the leading coefficient 1, like in (x+5), that can be viewed as (10+5). You might recall, to multiply that out, we added the numbers, then multiplied those numbers. So, $(x+5)(x+4)$ was $x^2 + 9x + 20$. So to multiply 14(13) would be the same as $(10+4)(10+3) = 10^2 + 7(10) + 12 = 100 + 70 + 12 = 172$. Using $16(12) = (10+6)(10+2) = 100 + 80 + 12 = 192$.

Factor using the Addition Method

$x^2 + 9x + 20$	$x^2 + 8x + 12$	$x^2 + 13x + 42$
$x^2 + 10x + 16$	$x^2 + 5x + 6$	$x^2 + 11x + 10$
$x^2 - x - 20$	$x^2 + x - 20$	$x^2 - 2x - 24$
$x^2 - 4x - 21$	$x^2 - x - 2$	$x^2 + 7x - 30$
$x^2 - 7x + 10$	$x^2 - 9x + 20$	$x^2 - 3x + 2$
$x^2 - 7x + 6$	$x^2 - 12x + 32$	$x^2 - 10x + 24$
$x^2 + 11x + 28$	$x^2 - 3x - 28$	$x^2 - 15x + 50$
$2x^2 + 18x + 40$	$3x^2 - 6x - 72$	$5x^2 - 5x - 10$

Sec. 5   Trial & Error; $a \neq 1$

If the coefficient of the quadratic term is not one $(a \neq 1)$, we use **Trail & Error** to factor the trinomial.

The name Trial and Error suggests we try different factors to see if they work, if they don't we try others.

Pick factors that work for the quadratic and constant terms, then check to see if when multiplied out, we get the linear term

Example 8   Factor $12x^2 + 56x + 9$

Picking factors for 12 and 9, I have the following choices.

(12x 9)(x 1)   (12x 1)(x 9)
(6x 1)(2x 9)   (4x 1)(3x 9)
(4x 3)(3x 3), etc

All these choices work for the quadratic and constant terms, we need to check to determine if any of these will result in the linear (middle) term.

Because of space considerations, I did not list all the possibilities. Which of those will add up to 56 – the coefficient of the linear term? Hopefully you identified $(6x + 1)(2x + 9)$ since the sum of the Outer and Inner numbers is 56.

So $12x^2 + 56x + 9 = (6x + 1)(2x + 9)$

Factoring by Trial & Error can be time consuming, with practice you will realize that while you pick factors that work for the first and third terms, you can quickly determine the middle term by taking the sum of the products of the outside terms and the inside terms.

Example 9   Factor $6x^2 + 19x + 10$.

Since the coefficient of the quadratic term is not one, I should use Trial & Error.

I find factors for the quadratic term $6x^2$, 3x and 2x, then I find factors for the constant 10, 5 and 2. Because I picked those factors, I know I will have $6x^2 + ?x + 10$. So I have the first and last term. But will those factors give me a 19 for the linear term?

Checking the sum of the products of the outers and inners using , I do get 19.

Therefore $6x^2 + 19x + 10 = (3x + 2)(2x + 5)$

Sec. 6   ac Method – Trial & Error Alternative; *a ≠ 1*

There is an alternative to that guess and check method we call Trial & Error.

Stay with me. On the last example, Factor $6x^2 + 19x + 10$,   ac = 6(10) = 60

To factor using the ac Method:
1. Multiply the constant term (10) by the coefficient of the quadratic term (6), I get 60.
2. Find all the factors of ac; 60
3. Determine which of those factors add up to the linear term (b), which is 19. 15 and 4
4. Rewrite the linear term of the original trinomial using those two factors. Hopefully, you identified 15 and 4. Now, I will rewrite my original trinomial as a polynomials with four terms.

$$6x^2 + \underline{\mathbf{19x}} + 10 \quad \text{as} \quad 6x^2 + \underline{\mathbf{15x + 4x}} + 10$$

5. Factor the first two terms using the Distributive property and then the second two terms using the Distributive Property.

$$6x^2 + 15x + 4x + 10$$

$$3x(2x + 5) + 2(2x + 5)$$

There is a $(2x + 5)$ in both terms, taking that out using the distributive property, we have

$$3x(2x + 5) + 2(2x + 5) = (2x + 5)(3x + 2)$$

Example 10 Factor completely $10x^2 + 21x + 9$ ac = 90

90

90, 1	÷2x2
45, 2	÷3x3
15, 6	÷3x3
5, 18	x2÷2
10, 9	

Multiply the constant (9) by the leading coefficient (10), you get 90.

Which of those factors multiply together give you 90 and add up to the coefficient of the linear term (21)? If you answered 15 and 6, we are in business.

We'll rewrite 21x as 15x + 6x

$10x^2 + 21x + 9$ as $10x^2 + 6x + 15x + 9$

Factoring the first two term and the last two terms using the Distributive Property, we have

$10x^2 + 6x + 15x + 9 = 2x(5x + 3) + 3(5x + 3)$

Now I have a $(5x +3)$ in each term, factoring out the $(5x + 3)$, we have

$2x(5x + 3) + 3(5x + 3) = (5x + 3)(2x +3)$

Example 11   factor $6x^2 - x - 12$

Step 1. Multiply constant by leading coefficient - $\underline{-72}$
Step 2. Find all the factors of -72 72, –1
                                                          36, –2
                                                          18, –4
                                                           9, –8
                                                          3, –24

Step 3. Factors whose sum is –1   –9, 8
Step 4. Rewrite polynomial $6x^2 - 9x + 8x - 12$
Step 5. Factor $3x(2x - 3) + 4(2x - 3)$
                     $(2x - 3)(3x + 4)$

Factor using Trial & Error or the ac Method.

A   B
1) $6x^2 + 9x + 3$ $8x^2 + 14x + 5$

2) $6x^2 + 19x + 10$ $12x^2 + 20x + 3$

3) $12x^2 + 28x - 5$ $6x^2 - 5x - 21$

4) $5x^2 + 58x - 24$ $5x^2 - 2x - 24$

Sec. 5   Factoring, combined

We have learned to factor polynomials using the Distributive Property, Difference of Two Squares, Addition Method, Trial & Error and the ac Method. Now, do you know which method to use by looking at the polynomial?

The greatest problem students encounter while factoring is determining which method to use. Teachers need to take the time to teach the students to compare and contrast. To some students, all polynomials look alike. They have to be explicitly taught to differentiate between the problems so they can use the correct methodology.

If we go back to our diagram, we see we should try to factor first by using the Distributive Property. That will make the numbers more manageable.

Look at the following polynomials and select the method that should be used to factor it.

      a. $8y^2 + 2y - 3$
      b. $x^2 - 25$
      c. $z^2 - 7x - 12$
      d. $2x^2 + 18x$
      e. $a^3 - 3a^2 + 9a - 27$

Let's see how you did.

Answer a. is Trial & Error or the ac Method because the coefficient of the quadratic term is NOT one.
Answer b. is the Difference of Two Squares because you have a binomial, both terms are perfect squares, and it is a difference.
Answer c, is the Addition Method because the coefficient of the quadratic term is one.
Answer d. is the Distributive Property and
Answer e. is Grouping since it is not a binomial or trinomial.

If you can discriminate between the polynomials, that's half the battle. Because once you know what method to use, you just follow the procedures you have already learned.

Now let's see how you go about factoring polynomials by **GROUPING**. Oh, we just factored by grouping when we used the ac Method.

To be able to factor using Grouping requires you to know how to factor using other methods. Then using those methods, we group terms together.

Example 11   Factor $a^3 - 3a^2 + 9a - 27$

I cannot use the Distributive Property, nor do I have a binomial of trinomial, that's a pretty good indication that I have to factor by Grouping.

The question is, how do I group? The first two and the last two terms, the first three terms and the fourth term? The first and third term and the second and fourth?

Well, we are going to group them to try and find common factors.

Notice, if I took an a^2 out of the first two terms and a 9 out of the third and fourth terms, I have a common factor of $(a - 3)$

$a^3 - 3a^2 + 9a - 27 = a^2 \,(a - 3) + 9 \,(a - 3)$

Factoring out the $(a - 3)$ from both terms, we have **$(a - 3)$**$(a^2 + 9)$

Example 12   Factor $x^2 - 6x + 9 - y^2$

This again is a Grouping problem because these is more than 3 terms. In this case, I recognize $x^2 - 6x + 9$ as a perfect square. Since I see this pattern, I will group the first three terms together and factor.

Here is where experience comes in to play. Looking at that polynomial, I could group the first and second, then the third and fourth terms.

$$x^2 - 6x + 9 - y^2 = x^2 - 6x + 9 - y^2.$$

I see I can factor and x out of the first two terms using the Distributive Property and I also recognize the $9 - y^2$ as the Difference of 2 Squares. Doing that does not change my polynomial into an expression that is essentially a product.

If I grouped the first three terms together by recognizing the perfect square (special product), we can factor this using the Difference of 2 Square

$$
\begin{aligned}
x^2 - 6x + 9 - y^2 &= x^2 - 6x + 9 - y^2 \\
&= (x-3)^2 - y^2 \\
&= [(x-3) + y][(x-3) - y]
\end{aligned}
$$

Factor the following expressions using the appropriate method.

1. $2x^2 + 7x + 5$
2. $3x^2 + 10x + 7$
3. $5x^2 + 7x + 2$
4. $8x^2 - 10x + 3$
5. $6x^2 - 17x + 5$
6. $10x^2 - 19x + 7$
7. $x^2 + x - 6$
8. $x^2 + x - 12$
9. $x^2 + x - 30$
10. $2a^2 - 7a - 4$
11. $3x^2 - 10x - 25$
12. $2x^2 - x - 3$
13. $x^2 + 7x + 10$
14. $b^2 + 5b + 6$
15. $a^2 + 4a + 3$
16. $y^2 - 10y + 16$
17. $m^2 - 9m + 20$
18. $d^2 - 11d + 30$
19. $5y^2 - 5$
20. $4x^2 - 100$

Sec. 7   Sum and Difference of 2 Cubes

The patterns we came up with to factor polynomials up to this point were pretty much based on our experiences multiplying polynomials. However, we rarely multiplied numbers that resulted in the sum or difference of two cubes. So, the pattern I'm going to introduce will not be familiar to you now, but it needs to be after reading this section.

Example 1   Multiply $(x + 2)(x^2 - 2x + 4)$

$$= x^3 - 2x^2 + 4x + 2x^2 - 4x + 8$$

$$= x^3 + 8$$

Notice most of the terms subtracted out. Now, to help see a pattern develop, I am going to write 4 as 2^2 and 8 as 2^3.

In doing so, we have $x^3 + 2^3 = (x + 2)(x^2 - 2x + 2^2)$

Let's hold onto that for a minute, and do a very similar problem.

Example 2   Multiply $(x - 3)(x^2 + 3x + 9)$

$$= x^3 + 3x^2 + 9x - 3x^2 - 9x - 27$$

$$= x^3 - 27$$

Again, you can see most terms subtracted out. As I did before, I am going to write 9 as 3^2 and 27 as 3^3.

In doing so, we have $x^3 - 3^3 = (x - 3)(x^2 + 3x + 3^2)$

If I did a few more of these, we would see the following two patterns; one for the sum of two cubes, the other for the difference of two cubes.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

and

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

There are a couple of things I would like you to note:

1. The pattern I want you to see is if I have the sum of two cubes, then the binomial factor is also a sum. If I have the difference of two cubes, the binomial factor is also a difference.
2. The sign of the linear term in the trinomial is opposite of the binomial being factored.

Using those patterns, let's try a few.

Example 3 Factor $x^3 + 4^3$

Using pattern for the sum of two cubes; $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$$x^3 + 4^3 = (x + 4)(x^2 - 4x + 4^2)$$

$$x^3 + 4^3 = (x + 4)(x^2 - 4x + 16)$$

Example 4 Factor $x^3 - 125$

The cube in the first term might suggest that this will be the difference of two cubes. Rewriting the problem to match the pattern, we have $x^3 - 5^3$.

$$x^3 - 5^3 = (x - 5)(x^2 + 5x + 5^2)$$

$$x^3 - 5^3 = (x - 5)(x^2 + 5x + 25)$$

Those two examples were pretty straight forward and recognizable. Let's look at one in disguise and apply the pattern.

Example 5 Factor $8x^3 + 27y^3$.

What we should quickly realize because of the cubes, we will be using the sum of two cubes to try and factor. So, our first job is to rewrite $8x^3$ and $27y^3$ as cubes so we can apply the pattern.

$8x^3 = (2x)^3$, a in my pattern and $27y^3 = (3y)^3$, b in my pattern

Rewriting the original problem, we have $(2x)^3 + (3y)^3$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$(2x)^3 + (3y)^3 = (2x + 3y)[(2x)^2 - 6xy + (3y)^2]$$

$* \text{ the 6 came from } ab = (2x)(3y)$

$$= (2x + 3y)(4x^2 - 6xy + 9y^2)$$

Example 6 Factor $125x^3 - 64y^3$

Using the pattern $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Rewrite $125x^3$ as $(5x)^3$ and $64y^3$ as $(4y)^3$

$$(5x)^3 - (4y)^3 = (5x - 4y)[(5x)^2 + 20xy + (4y)^2$$

$* \text{ the } 20xy \text{ came from } ab = (5x)(4y)$

$$= (5x - 4y)(25x^2 + 20xy + 16y^2)$$

Factor completely.

A	B
1. $x^3 - 8$	$x^3 + 8$
2. $x^3 - 27$	$x^3 + 27$
3. $y^3 - 64$	$y^3 + 64$
4. $x3 - 125$	$t^3 + 125$
5. $8x^3 - 125$	$27x^3 + 64y^3$