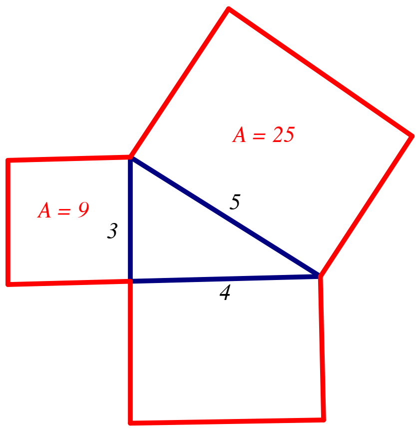
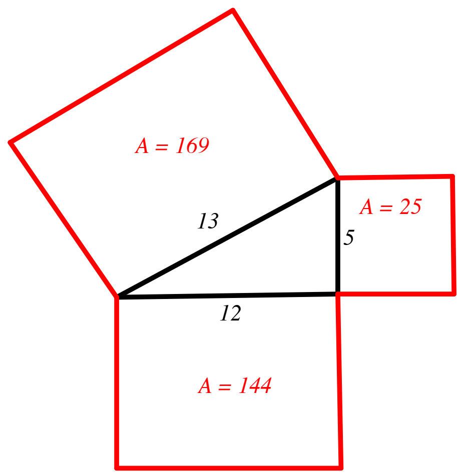
Chapter 10

Right Triangle Theorems

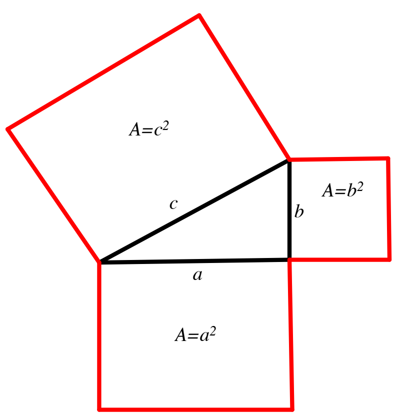
If we looked at enough right triangles and experimented a little, we might eventually begin to notice some relationships developing. For instance, if I were to construct squares formed by the legs of a right triangle as shown below, we may see a relationship in the areas of squares formed.





Try looking at the areas formed by a right triangle whose sides measure 6, 8, and 10. By looking at these limited examples, it appears in each right triangle the area of the square formed by the hypotenuse is always equal to the sum of the areas of the squares formed by the two legs.

Generalizing by replacing the numbers with letters, we have the following.



The relationship we saw with the numbers suggest that in our generalization;

$$c^2 = a^2 + b^2$$

This is an important relationship in mathematics, and since it is important, we will give this a name, the Pythagorean Theorem.

Pythagorean Theorem **In any right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.**

A minute ago, we were talking about the area of the square formed by the hypotenuse being equal to the sum of the areas of the squares formed by the two legs. Notice, the Pythagorean Theorem does not mention areas of squares, so how do those relationships lead us to the Pythagorean Theorem?

Well, the square of the length of the hypotenuse – $\mathbf{c^2}$, is equal to the area of the square formed by the hypotenuse. The same is true of the other two legs. So there is a connection between our observation and the Pythagorean Theorem.

The converse of the Pythagorean Theorem is also true. That is, if the square of longest leg of a triangle is equal to the sum of the squares of lengths of the other two sides, the triangle is a right triangle.

Now I can prove the Pythagorean Theorem by recalling what we have already learned about similar triangles. The relationships in the areas of the squares led us to the Pythagorean Theorem, but those were examples – not a proof. To prove the theorem, I will need to more information about similar triangles and set up some proportions using the **Angle-Angle Postulate**.

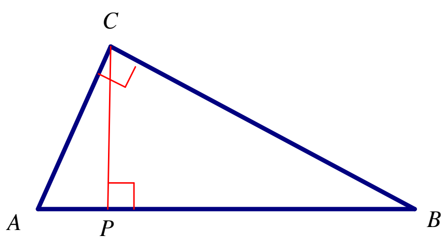
Hopefully, you see the importance of continually reviewing information. Without a body of knowledge to draw from, it would be very difficult to think critically or to prove subsequent theorems.

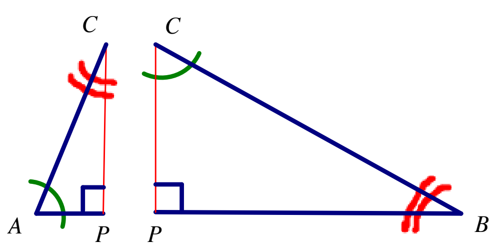
Let's look at the right triangle ACB, pictured below, and draw an altitude from the right angle to the hypotenuse. That results in two smaller triangles being formed, ΔPCA and ΔPBC. By the AAP, those two triangles are similar to ΔCBA, the large triangle and therefore similar to each other.

The theorem states:

Theorem **If the altitude is drawn to the hypotenuse of a right triangle, the two triangles formed are similar to the given triangle and to each other.**

That's pretty easily proven using the AA Postulate.





$\Delta APC \sim \Delta ACB$

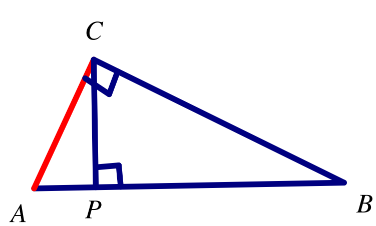
$\Delta CPB \sim \Delta ACB$

Since $\Delta APC \sim \Delta ACB$ and $\Delta CPB \sim \Delta ACB$ by the AA Postulate, then the two smaller triangles must be similar.

$\Delta APC \sim \Delta CPB$

A corollary follows directly from $\Delta APC \sim \Delta ACB$ that is called the **Geometric Mean Leg Corollary**

Corollary **If the altitude is drawn to the hypotenuse, the length of the leg of the right triangle is the geometric mean between the length of the hypotenuse and the length of the segment of the hypotenuse adjacent to that leg.**



$$\frac{AB}{AC} = \frac{AC}{AP}$$

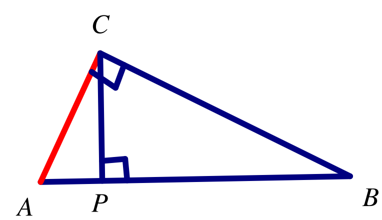
That sounds interesting, but all that is happening here is I am setting up the proportions as a result of the two triangles being similar and introducing a new term – **geometric mean**. A **geometric mean** occurs when the second and third term of a proportion are the same. You've seen one before when we worked with equivalent fractions, we just never gave it a name. Let's look at one.

The 3 is the **geometric mean** between 1 and 9 as AC is the geometric mean between AP and AB.

Now, back to the corollary. Since those two triangles are similar, I can pick any leg of the right triangle, either AC or BC. In the example above I picked AC. If I picked BC, the other leg, then BC is the geometric mean between the length of the hypotenuse AB, and the segment of the hypotenuse adjacent to that leg XB.

Remember, the leg of the right triangle is the geometric mean between the hypotenuse and the segment of the hypotenuse adjacent to the chosen leg!

Example 1. Find AC if AP = 4 and AB = 16



$$\frac{AB}{AC} = \frac{AC}{AP}$$

By corollary, we know AC is the geometric mean between the hypotenuse and the segment of the hypotenuse adjacent to the leg. So, we have

$$\frac{AB}{AC} = \frac{AC}{AP}$$

Substituting our values, or $(AC)^2 = 4\ (16)$

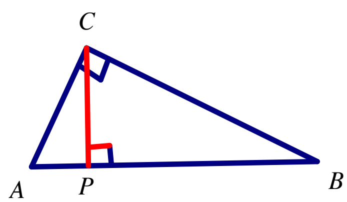
$$(AC)^2 = 64$$
$$AC = 8$$

Now, going back to the same right triangle with the altitude drawn to the hypotenuse, we found the two smaller triangles were similar to the original larger right triangle, and therefore similar to each other.

That resulted in the **ΔPCA ~ ΔPBC** – same as before. But, let's look at a different proportion.

Which leads us to another corollary called the **Geometric Mean Altitude Corollary**

Corollary **The altitude drawn to the hypotenuse of a right triangle separates the hypotenuse into two segments. The length of the altitude is the geometric mean between the lengths of the two segments.**



$$\frac{AP}{CP} = \frac{CP}{PB}$$

Example 2 Find CP if AP = 2 and PB = 8, Use diagram above.

By corollary we know

$$\frac{AP}{CP} = \frac{CP}{PB}$$

Substituting our values, we have

Cross multiplying $(CP)^2 = 16$

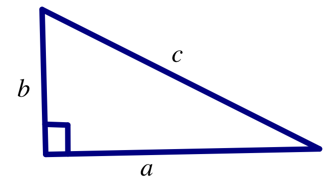
$CP = 4$

Those two corollaries are a direct result of showing triangles similar using the AA Postulate with an altitude being drawn to the hypotenuse of a right triangle. The resulting proportions coming from the similar triangles allowed us to describe in words (corollaries) those relationships.

Here's more good news. Using that same information we gained from similar polygons and the two corollaries, we can now prove the **Pythagorean Theorem**.

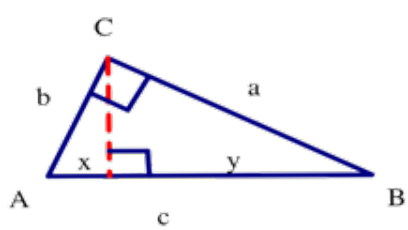
Pythagorean Theorem **In any right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.**

$$c^2 = a^2 + b^2$$



To prove the theorem, we start off with a right triangle that was given and we draw the picture. Like most of the theorems we have proved, we then have to add more information to our picture using our knowledge of geometry. In this case, we will draw an altitude to the hypotenuse of the right triangle. We learned the altitude formed similar triangles. Similar triangles results in sides being in proportion, and more specifically, one leg of the right triangle turns out to be the geometric mean between the hypotenuse and the segment of the hypotenuse adjacent to that leg.

Let's draw the picture, construct the altitude, and label the picture.



Statements	Reasons
1. ΔACB, C is rt $\angle$	Given
2. Draw alt. to AB	Construction
3. $\frac{c}{a} = \frac{a}{y}$; $\frac{c}{b} = \frac{b}{x}$	Altitude drawn, leg is geo mean
4. $x + y = c$	Segment Addition Postulate
5. $cy = \mathbf{a^2}$ **;** $cx = \mathbf{b^2}$	Prop of proportion from step 3
6. $cy + cx = a^2 + b^2$	Add prop =, from step 5
7. $c(y + x) = a^2 + b^2$	Distrib prop (factoring)
8. $\boldsymbol{c^2 = a^2 + b^2}$	Substitution

The converse of the Pythagorean Theorem is also true. That is, if you are given lengths of three sides of a triangle, if the square of the longest side is equal to the sum of the squares of the other two sides, then it must be a right triangle.

Pythagorean Triples

I find it helpful to know the following Pythagorean Triples. Knowing these cuts down on some arithmetic later. You should also note if you know these triples, multiples of the triples also are Pythagorean Trifles.

3, 4, and 5 (x2) —> 6, 8, and 10
5, 12, and 13
8, 15, and 17
7, 24, and 25

Pythagorean Theorem: Applications

To use the Pythagorean Theorem, we need to know how to simplify radicals. We'll look at simplifying square roots.

Simplifying expressions such as the and are pretty straight-forward.

$$\sqrt{25} = 5$$
$$\sqrt{64} = 8$$

The question that we need to consider is what happens if we want to take the square root of a number that is not a perfect square.

To simplify a square root, **you rewrite the radicand as a product of a perfect square and some other number. You then take the square root of the perfect square.**

If I square the numbers 1 through 10, the result will give me perfect squares.

Perfect Squares

1^2	=	1
2^2	=	4
3^2	=	9
4^2	=	16
5^2	=	25
6^2	=	36
7^2	=	49
8^2	=	64
9^2	=	81
10^2	=	100

Example Simplify $\sqrt{50}$

Now 50 can be written as a product of 5 and 10. Should I use those factors?

$$\sqrt{10 \bullet 5}$$

Hopefully, you said no. We want to rewrite the radicand as a product of a perfect square. Neither 5 nor 10 are perfect squares. So, looking at my list of perfect squares, which, if any, are factors of 50?

That's right, 25 is a factor of 50 and it is a perfect square.
Simplifying, I now have

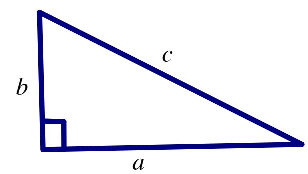
$$\sqrt{25 \cdot 2}$$

$$5\sqrt{2}$$

Simplify the following radicals.

1. $\sqrt{75}$
2. $\sqrt{12}$
3. $\sqrt{98}$
4. $\sqrt{32}$
5. $\sqrt{72}$

Example 1 Find c if a = 12 and b = 5



Using the relationship; $c^2 = a^2 + b^2$, substitute the given values.

$$c^2 = 12^2 + 5^2$$
$$c^2 = 144 + 25$$
$$c^2 = 169$$
$$c = 13$$

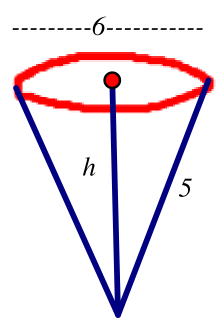
Example 2 Find b if c = 10 and a = 5

Using $c^2 = a^2 + b^2$

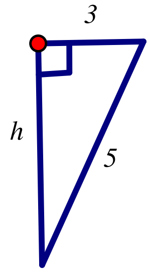
$$10^2 = 5^2 + b^2$$
$$100 = 25 + b^2$$
$$75 = b^2$$

$$\sqrt{75} = b$$
$$5\sqrt{3} = b$$

Example 3 Find the height of a cone knowing its diameter is 6 feet and the slant height is 5 feet.



Drawing the picture is always helpful. I know the radius of the circle is half the diameter, that's 3'. I can also see a right triangle being formed.



Using the Pythagorean Theorem, I know the square of the hypotenuse is equal to the sum of the squares of the lengths of the other two legs.

Therefore, I have $5^2 = 3^2 + h^2$

$25 = 9 + h^2$

$16 = h^2$

$4 = h$

The height of the cone is 4 feet.

Example 4 Determine if a triangle with sides 5, 8, 10 is a right triangle.

If it is a right triangle, the longest side would have to be the hypotenuse. Using $c^2 = a^2 + b^2$, substitute the given values and see if it true.

10^2 ? $a^2 + b^2$

100 ? 25 + 64

$100 \neq 89$, therefore a 5, 8, 10 triangle is not a right triangle.

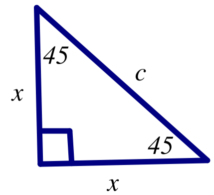
Special Right Triangles

45-45-90° Δ

The first thing that comes to mind when working with right triangles is the Pythagorean Theorem. It turns out, with special conditions, I can actually find sides of a right triangle in my head without substituting numbers into $c^2 = a^2 + b^2$

To get to those theorems, I will use the Pythagorean Theorem.

The diagram below is an isosceles right triangle, which means the base angles are 45˚ and the two legs have the same length..



We know in any right triangle; $c^2 = a^2 + b^2$.

$c^2 = x^2 + x^2$

$c^2 = 2x^2$

$$c = \sqrt{2} \cdot x$$

Now what that means is any time I have a 45-45-90˚ Δ, the length of the hypotenuse is the side multiplied by the $\sqrt{2}$

That leads us to a theorem.

Theorem If each acute angle of a right triangle measures 45˚, the hypotenuse is $\sqrt{2}$ times as long as a leg.

Example 1. In a 45-45-90 Δ, if one side is 6, find the length of the other leg and the hypotenuse.

Essentially I have two choices, I could use the Pythagorean Theorem or the last theorem. Clearly the last theorem makes our lives easier. If one leg has length 6, then the other length has length 6 and the hypotenuse is $6\sqrt{2}$.

These problems can not be made difficult. To get from a leg to the hypotenuse you multiply by the $\sqrt{2}$. To get from the hypotenuse to a leg, you would divide by the $\sqrt{2}$.

Example 2 In a 45-45-90˚Δ, if a leg had length 7, find the other leg and the hypotenuse.

The other leg's length is also 7 and the hypotenuse is $7\sqrt{2}$

Example 3 In a 45-45-90˚ Δ, if the hypotenuse has length $15\sqrt{2}$, find the length of the legs.

Divide by the hypotenuse by $\sqrt{2}$, the leg have length 15.

Example 4 In a 45-45-90˚ Δ, if the hypotenuse has length 8, find the lengths of the other two sides.

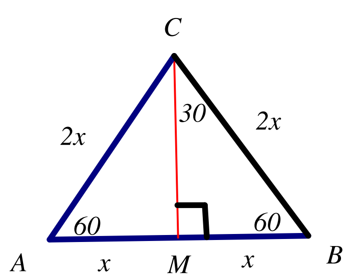
Divide the hypotenuse by the $\sqrt{2}$.

$$\frac{8}{\sqrt{2}} = \frac{8}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{8\sqrt{2}}{2}$$
$$= 4\sqrt{2}$$

Each side measures $4\sqrt{2}$

30-60-90˚ Δ

Now, let's continue looking at special right triangles. If we were to draw an equilateral triangle, we know through theorem those ∠'s are also equiangular. If I draw a perpendicular bisector to one of the sides, I can find some interesting relationships.

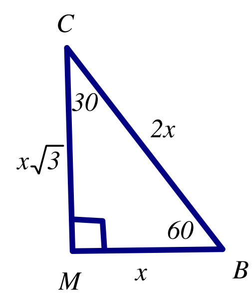


As always, think before we get going. Notice I labeled each side 2x of the equilateral triangle. I did that because I knew a bisector would divide the base in half and half of 2x is x. If I labeled each side of the triangle x, then I would have introduced fractions.

Let's just look at ΔCBM, which is a 30-60-90˚ Δ. Using the Pythagorean Theorem as I did in the 45-45-90˚ Δ, we have

$$c^2 = a^2 + b^2$$
$$(BC)^2 = (CM)^2 + (MB)^2$$
$$(2x)^2 = (CM)^2 + \quad x^2$$
$$4x^2 = (CM)^2 + \quad x^2$$
$$3x^2 = (CM)^2$$
$$\sqrt{3} \cdot x = CM$$

Now, let's see what that means by generalizing what we see in the diagram.



The length of the shortest side, opposite the smallest angle, 30˚ is x. The length of the hypotenuse is twice the shorter leg, 2x. And the length of the longer leg is the $\sqrt{3}$ times the shorter leg. That's the nice thing about algebra, we can see these generalizations! And that leads us to another theorem.

Theorem If the acute angles of a right triangle have measure 30 and 60˚, then
a) the hypotenuse is twice as long last the shorter side, and
b) the longer side is the $\sqrt{3}$ times as long as the shorter leg

s – short side, l – long side, h – hypotenuse

s = x, then ***l*** = $\sqrt{3}$x and ***h*** = 2x

The last two theorems can really cut down the amount of work we do to find sides of special right triangles. If you don't know those, then you will need to use the Pythagorean Theorem or use similar triangles.

Example 5 In a 30-60-90˚ Δ, if the length of the side opposite the 30˚ angle (short side) measures 5, find the length of the other side and the hypotenuse.

By theorem, the side opposite the 60˚ angle (long side) is 5 √3 and the hypotenuse is 10

When doing these problems, always try to find the short side first, the side opposite the 30˚ angle. then find the other side or hypotenuse.

Example 6 In a 30-60-90˚ Δ, if the hypotenuse has measure 12, find the lengths of the other two sides.

If the hypotenuse is 12, then the short side is half that or 6. If the short side is 6, then the long side is 6√3

Example 7 In a 30-60-90˚ Δ, if the side opposite the 60˚ angle measures 7√3, find the length of the shorter side and hypotenuse.

The side opposite the 60˚ angle is the long side, that means the short side measures 7 and the hypotenuse is 14.

1. In a right triangle, if the length of the hypotenuse is 10" and the length of a leg is 5", find the length of the other leg.
2. In a right triangle, if the length s of the legs are 5" and 9", find the length of the hypotenuse.
3. Determine if a triangle with sides of the given lengths is a right triangle.
   {5, 12, 13}; {5, 7, 8}; {8, 10, 13}
4. The length and width of a rectangle are 6 ft and 4 ft respectively. Find the length of the diagonal of the rectangle.