Chapter 8

Applying Congruent Triangles

In the last chapter, we came across a very important concept. That is, corresponding parts of congruent triangles are congruent - ***cpctc***. In this chapter, we will look at polygons we have not studied and, using construction, create triangles within those polygons so we can use our knowledge of congruence to prove relationships.

A **parallelogram** is a **quadrilateral in which both pairs of opposite sides are parallel**.

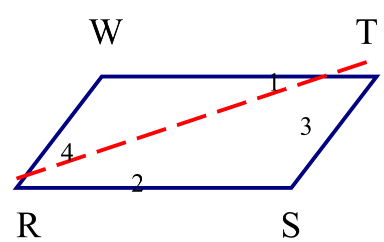
The most common way of finding these relationships is turning something that we are not familiar with into something we have more knowledge and understanding – triangles. In our first theorem, we are given a parallelogram. The only thing we know about a parallelogram comes from the definition. However, if I can introduce triangles into the picture, I have a better chance of finding relationships.

Drawing a diagonal in a parallelogram introduces parallel lines being cut by a transversal. In this case, we know the alternate interior angles will be congruent, setting the diagonal equal to itself because of the Reflexive Property, the result will be the triangles will be congruent by ASA.

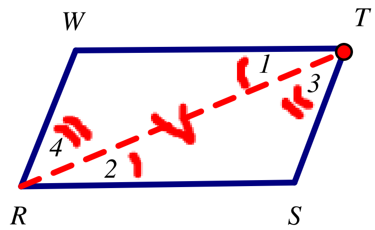
Theorem **A diagonal of a ||ogram separates the ||ogram into 2 ≅ Δ's**

Given: ▱RSTW

Prove: ΔRST ≅ ΔTWR



The best thing I can do after drawing the picture described in the theorem is to mark up the picture so we can see the relationships.



Statements		Reasons
1.	RSTW is a ||ogram	Given
2.	$\overline{RS} \parallel \overline{WT}$	Def - ||||ogram
3.	$\angle 1 \cong \angle 2$	2 || lines cut by t, alt int$\angle$'s $\cong$
4.	$\overline{\underline{RT}} \cong \overline{\underline{RT}}$	Reflexive
5.	RW || ST	Def - ||ogram
6.	$\angle 3 \cong \angle 4$	2 || lines cut by t, alt int$\angle$'s $\cong$
7.	$\Delta RST \cong \Delta TWR$	ASA

Knowing the diagonal separates a parallelogram into 2 congruent triangles suggests some more relationships.

Looking at the congruent triangles formed by the diagonal, we can see other relationships using the **cpctc**.

Corollary **The opposite sides of a parallelogram are congruent.**

Corollary **The opposite angles of a parallelogram are congruent.**

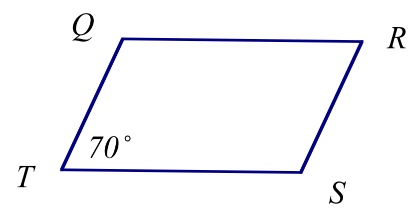
Both of these corollaries can be proven by adding **CPCTC** to the last proof.

Many students mistakenly think the opposite sides of a parallelogram are equal by definition. That's not true, the definition states the opposites sides are parallel. This theorem allows us to show they are also equal or congruent.

The idea of using **cpctc** after proving triangles congruent by SSS, SAS, ASA, and ASA will allow us to find many more relationships in geometry.

The last corollary stated the opposite angles of a parallelogram are congruent. Use that in the next example

Example 1 Find the $m\angle$R, given ▱ QRST



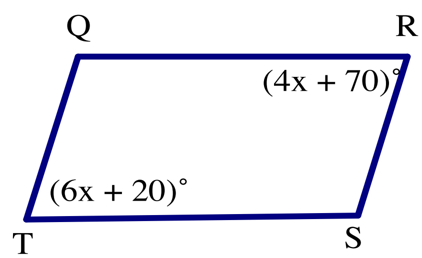
The opposite $\angle$s of a ||ogram are $\cong$.

$\therefore \longrightarrow$ $m\angle\text{R} = 70^\circ$

If I did not know that corollary, I could have used my knowledge of parallel lines being cut by a transversal and used the same side interior angles. That's a whole lot more work when you can almost just look at the problem and find the answer.

Let's look at another problem. Same concept, use algebra.

Example 2 QRST is a llogram, Find the $m\angle$R,



Setting the opposite angles equal

$$\angle \text{T} = \angle \text{R}$$

$$6x + 20 = 4x + 70$$
$$2x = 50$$
$$x = 25$$

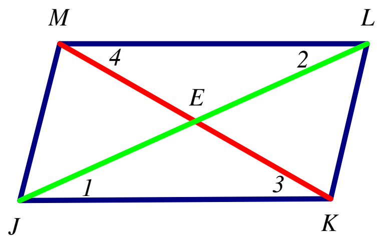
Substituting x = 25 into $m\angle\text{R} = 4x + 70$

$m\angle\text{R} = 170^\circ$

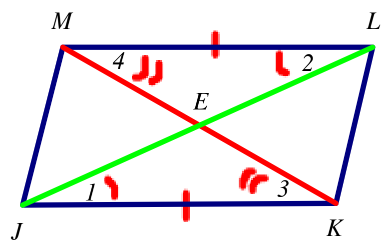
Theorem **The diagonals of a ||ogram bisect each other.**

Given: ▱ JKLM

Prove: $\overline{JE} \cong \overline{LE}$; $\overline{KE} \cong \overline{ME}$



I drew the diagonals and marked some angles formed by those diagonals. Now, marking up the diagram will help me see the congruent triangles.

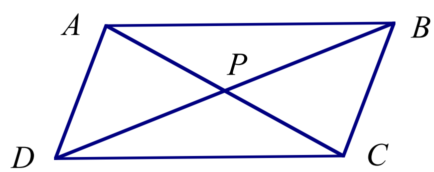


Statements		Reasons
1.	JKLM is ||ogram	Given
2.	$\overline{JK} \parallel \overline{ML}$	Def - ||ogram
3.	$\angle 1 \cong \angle 2$	2 || lines cut by t, alt int∠'s
4.	$\overline{JK} \cong \overline{ML}$	opposite sides ||ogram ≅
5.	$\angle 3 \cong \angle 4$	2 || lines cut by t, alt int∠'s
6.	$\Delta JEK \cong \Delta LEM$	ASA
7.	$\overline{JE} \cong \overline{LE}$ $\overline{KE} \cong \overline{ME}$	cpctc

Notice to prove this theorem, I first drew the parallelogram, then I drew in the diagonals. In order to prove triangles congruent, I had to add angles to the picture, so I labeled angles 1, 2, 3, and 4 and developed the relationships based upon my previous knowledge of geometry – alternate interior angles.

Drawing and labeling the information given to you is important. It is also important to label other information in your picture from your previous knowledge of geometry. You need to remember and be able to visualize your definitions, postulates and theorems.

Example 2 Given the diagram, ABCD is a ||ogram, AC = 16, DP = 7, find PC and BD.



Segments AC and BD are diagonals.

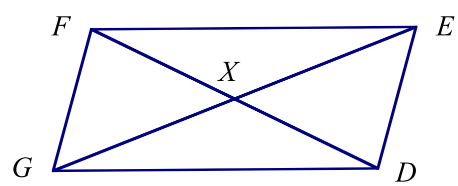
AP = PC and DP = PB

Since AC = 16, AP and PC must be 8.

Since DP is half the diagonal and is 7, then BD = 14.

The problem would be done the same way using algebra, let's look.

Example 3 Given DEFG is a ||ogram, FX = 7y – 6, FD = 16, find the value of y.



If FD = 16, then FX = 8

FX = 8, FX = 7y – 6, by substitution, we have

$$7y - 6 = 8$$
$$7y = 14$$
$$y = 2$$

Sometimes we can be given information about a quadrilateral and we could develop more information if we knew the quadrilateral was a parallelogram. So being able to show a quadrilateral is a parallelogram can be important to us. The way that is done is by having the quadrilateral satisfy the definition of a parallelogram.

Theorem **If two sides of a quadrilateral are congruent and parallel, then the quadrilateral is a parallelogram.**

In order to show you have a parallelogram, you have to prove you have parallel lines. In order to show you have parallel lines, you would have to show the corresponding angles congruent, alternate interior angles congruent or the same side interior angles are supplementary.

So, to prove that theorem, you'd draw a picture of a quadrilateral, construct a diagonal, show two triangles congruent, name he corresponding parts – specifically the angles, then you will find the other lines parallel. That satisfies the definition.

Theorem **If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.**

Theorem **If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.**

Using my triangle congruence knowledge, I can further develop my knowledge of parallelograms, rectangles and rhombi.

Rectangle

A **rectangle** is a parallelogram with one right angle.

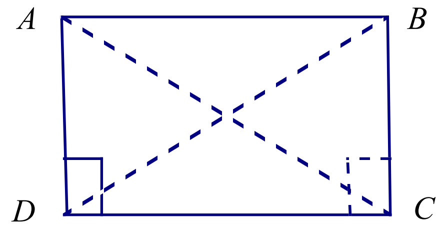
width

length

Some might ask, don't rectangles have four right angles? The answer is yes. So why are we saying in the definition that it has one right angle? Well, we know when we have parallel lines, the same side interior angles are supplementary. So, if one angle is 90˚, then all the angles are 90˚.

And since we proved the opposite sides of a parallelogram are congruent, then the opposite sides of a rectangle are congruent because a rectangle is a parallelogram.

Theorem **The diagonals of a rectangle are congruent.**



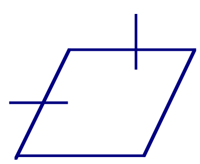
To prove this theorem, we need to show $\Delta ADC \cong \Delta BCD$. We can do that by SAS or LL. Once we know the triangles are congruent, then their corresponding parts are congruent by cpctc. That means $\overline{AC} \cong \overline{BD}$, the diagonals are congruent.

Remember, since a rectangle is a parallelogram, all the properties and relationships we learned about parallelograms work for rectangles.

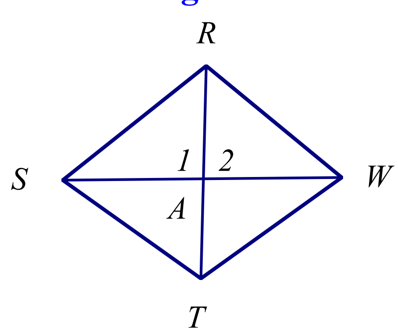
In other words, rectangles have their opposite sides congruent, opposite angles congruent, and the diagonals bisect each other.

Rhombus

A rhombus (rhombi) is a parallelogram with congruent sides.



Theorem: **The diagonals of a rhombus are $\perp$**



Given: Rhombus RSTW

Prove: $\overline{RT} \perp \overline{SW}$

Statements	Reasons
1. RSTW – Rhombus	Given
2. $\overline{RS} \cong \overline{RW}$	Def – Rhombus
3. $\overline{SA} \cong \overline{WA}$	Diagonals ||ogram bisect each other
4. $\overline{RA} \cong \overline{RA}$	Reflexive
5. ΔRSA $\cong$ ΔRWA	SSS
6. $\angle 1 = \angle 2$	cpctc
7. $\overline{RT} \perp \overline{SW}$	2 intersecting lines form $\cong$ adj $\angle$'s

The following theorem follows from this proof and the theorem that states if two sides of a triangle are congruent, the angles opposite those sides are congruent.

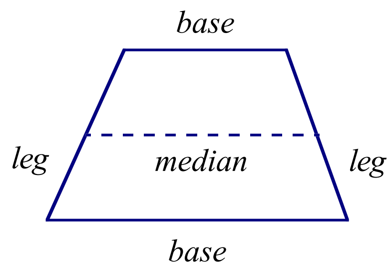
Theorem **Each diagonal of a rhombus bisects a pair of opposite angles.**

As you can see, the congruence theorems allow us to determine other mathematical relationships by going one step further and using cpctc.

There are other strategies we can use to show other relationships. We can combine our knowledge of algebra and demonstrate other relationships.

A **trapezoid** is a quadrilateral with exactly one pair of parallel lines. The parallel lines are called the **bases** and the non-parallel lines are called **legs**.

The line segment that joins the midpoints of the legs is called the **median**.



The next two theorems would be a little tough to prove using the current method. Proofs by the methods used in coordinate geometry will make these a lot easier to prove.

Coordinate Geometry

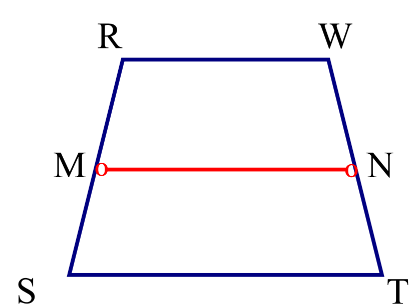
We've been using deductive reasoning in a t-proof to prove theorems thus far. Sometimes, theorems might be better proven using coordinate geometry. In a nutshell, coordinate geometry allows us to express our knowledge of geometry using algebra.

We are going to look at a couple of proofs using coordinate geometry.

Theorem **The median of a trapezoid is || to the bases and is equal to half the sum of the bases.**

Given: RSTW is a trap

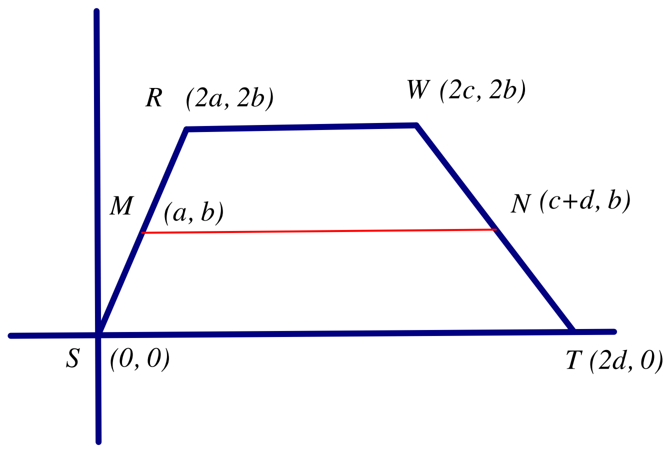
Prove: MN || ST
MN || RW
MN = 1/2 (ST + RW)



In this theorem, we are required to prove two things; first lines are parallel and second a mathematical relationship.

From the geometry we have already learned, we know how to show lines are parallel by looking at angles formed by parallel lines. In your algebra class, you learned that parallel lines have the same slope

Place the trapezoid on coordinate axes, label points, carefully keeping relationships as simple as possible. Find slopes, || lines have = slopes. Find distances. That sounds easy enough, but it takes a little extra thinking to label things so the arithmetic does not get in the way of what we are trying to prove.



I have very conveniently placed the trapezoid on the coordinate axis having coordinates (0, 0). Now, I could have labeled the coordinates of R as (a, b). The reason I did not do that was because I know I have to find the midpoint of RS and that would have led to a fraction. So, I got a little tricky, thought ahead, I labeled R as (2a, 2b). That way the midpoint M is (a, b).

Using the same type of logic, I will label W as (2c, 2b) and T as (2d, 0).

The slope is the change in y over the change is x, so the slope of

Slope of MN = (b–b)/(c+d–a) or 0.

The slope of $\overline{RW}$ is (2b–2b)/(2c–2a) or 0.

The slope of $\overline{RT}$ is (0–0)/(2d–0) or 0.

Since $\overline{MN}$, $\overline{RW}$, and $\overline{ST}$ have the same slope, those lines are all ||. Or more precisely

MN || ST and MN || RW

We have shown the lines are parallel. Now we have to show the median is half the sum of the bases. Since these are all horizontal lines, all I have to do is subtract to find their distances.

$MN = c + d - a$ Look at the diagram

$RW = 2c - 2a$
$ST = 2d$

$RW + ST = (2c - 2a) + 2d$
$= 2(c - a + d)$

From the algebra we can see that RW + ST is twice MN. Another way to say that is MN is half of RW + ST

$$\mathbf{MN = \frac{1}{2}(STW + RW)}$$

Now we have shown both parts, the lines being parallel and the median being half the sum of the bases.

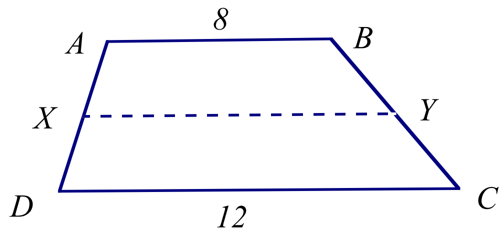
I cannot stress enough the importance of setting this up by labeling points conveniently so the arithmetic does not become a distraction.

We have proven another theorem, but this time we used coordinate geometry. As you become more comfortable with t-proofs and coordinate geometry, you will have to decide which method to use.

When you are not able to prove a theorem using one method, you now have another way at getting at the proof.

So, the good news, now we can use that theorem to find measures of a trapezoid.

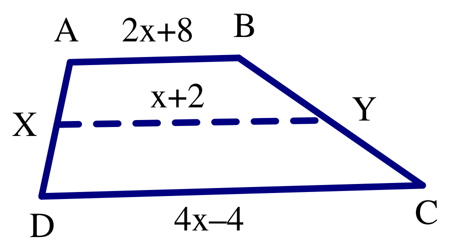
Example 1 Given trap ABCD, segment XY is the median, AB = 8 and CD = 12, find XY.



By the previous theorem, we know that $XY = \frac{AB + CD}{2}$

Substituting, we have $XY = \frac{8 + 12}{2} = 10$

Example 2 Given trap ABCD, segment XY is the median, AB = 2x–8, CD = 4x–4, and XY = x+2. Find XY.



Setting the problem up exactly the same way as in Example 1, we have

$$XY = \frac{AB + CD}{2}$$

Substituting, we have $$x + 2 = \frac{(2x - 8) + (4x - 4)}{2}$$

Multiplying both sides by 2 to clear the denominator, we have

$$2x + 4 = (2x–8) + (4x - 4)$$
$$2x + 4 = 6x - 12$$
$$16 = 4x$$
$$4 = x$$

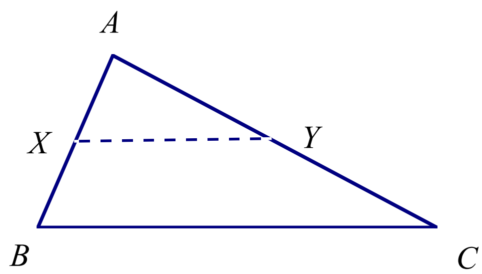
So, we have x = 4, but we were asked to find XY which was given to us as x + 2. Substituting, we have

$$XY = 6$$

N.B. If this was a high-stakes test multiple choice test, one distractor would have been 4 and another would have been 6. If you solved the problem correctly for x, you might be tempted to choose the wrong answer. Be careful!

Let's look at another theorem and prove it using coordinate geometry.

Theorem **The segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is half the length of the third side.**



XY = 1/2 (BC)

Example 2 In the figure above, $\overline{XY}$ is the midsegment and BC = 12, find XY

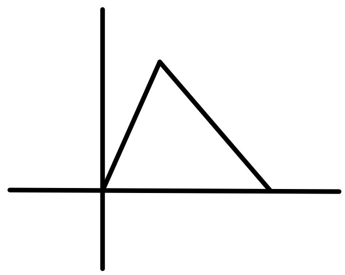
By the previous theorem, we know the median is ½ the length of the third side.

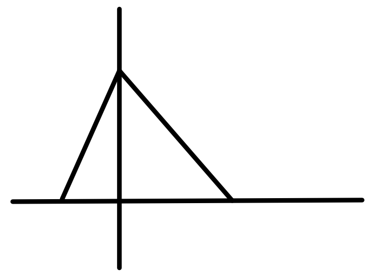
XY = ½ (BC)
XY = ½ (12)
XY = 6

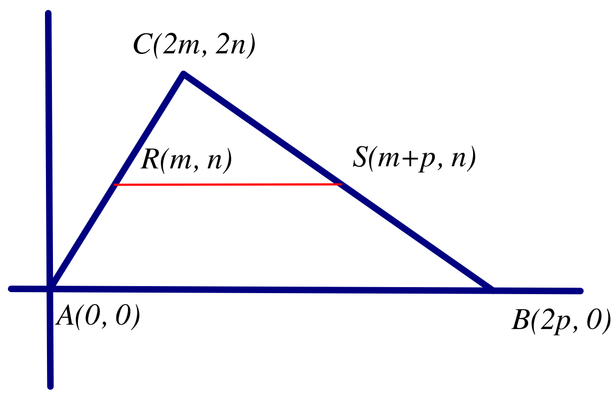
The last proof might suggest to show lines are parallel in coordinate geometry, we need to show they have the same slope.

We also found distances in the last proof, knowing how to find that will help us find the distances in this problem.

Let's look how I positioned the triangles on the coordinate axes. Which positioning do you think might help us with our proof. Since we are going to be looking for distances and midpoints again, using the origin comes in handy because the coordinates are (0, 0). Let's label the diagram using the positioning on the left.







By labeling C (2m, 2n), the midpoint R is easy to find for $\overline{AC}$. The same is true finding the midpoint S for BC.

The slope of $\overline{RS}$ and $\overline{AB}$ are zero. Since they have the same slope, the lines are parallel.

Let's find the distances -

$$\begin{aligned} RS &= (m + p) - m \\ &= p \end{aligned}$$

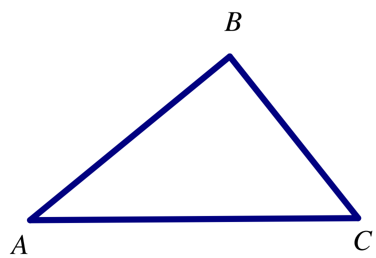
$$\begin{aligned} AB &= 2p - 0 \\ &= 2p \end{aligned}$$

AB is twice RS, that means that RS is half AB. Mathematically, we have RS = ½ AB

The proof of this will be done again in Chapter 9 using Similar Δ's.

Triangle Inequalities – One Triangle

Theorem **If one side of a triangle is longer than a second side, then the angle opposite the first side is larger then the angle opposite the second side.**

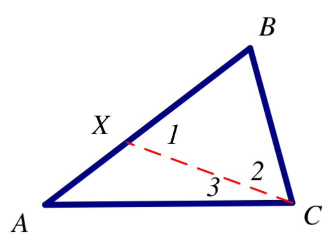


If AB > BC,
then $m\angle C > m\angle A$

The converse of that theorem is also true. In other words, if one angle of a triangle is larger than a second angle, then the side opposite the first is greater than the side opposite the second.

To prove this theorem, we need to remember if a = b + c and c > 0, then a > b. In other words, if 5 = 3 + 2, then 5 > 3.

Given: ΔABC, AB > BC
Prove: $m\angle C > m\angle A$



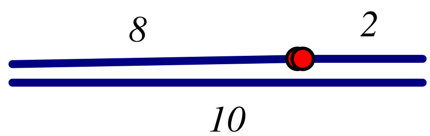
As always, drawing auxiliary lines is helpful in many proofs. So I'm going to draw $\overline{CX}$ so $\overline{BX} \cong \overline{BC}$, thus giving me an isosceles triangle.

Statements		Reasons
1.	On $\overline{BA}$, take X so $\overline{BX} \cong \overline{BC}$	Exactly one point a given distance
2.	Draw $\overline{CX}$	Thru any 2 pts
3.	$m\angle 1 = m\angle 2$	Base $\angle$s isos Δ
4.	$m\angle 1 = m\angle A + m\angle 3$	Ext $\angle$ of Δ
5.	$m\angle 1 > m\angle A$	if a = b + c
6.	$m\angle 2 > m\angle A$	Substitution
7.	$m\angle ACB = m\angle 2 + m\angle 3$	Angle Add Postulate
8.	$m\angle ACB > m\angle 2$	if a = b + c
9.	$m\angle C > m\angle A$	Transitive Prop, 6 & 8

Triangle Inequality Theorem

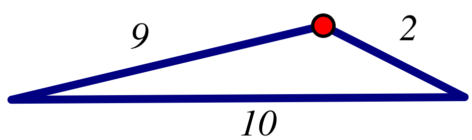
The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

To see this must be true, let's just look at three sticks that measure 2, 8 and 10 inches. Try and make a triangle.



If I placed the 8 and 2 inch sticks on top of the 10 inch stick, they would fit right on top of each other forming the same line.

The only way I would form a triangle is having two sticks longer then the third stick so when I raise them, they would form a triangle.



Example 1 Is it possible for a triangle to exist with the given sides?

a) 4, 5, 6
b) 8, 8, 10
c) 1, 2, 3
d) 8, 2, 5
e) 7, 15, 6

If I laid the longest leg horizontally, would the sum of the other two legs be greater then the length of the horizontal leg? In other words, would they be able to touch when rotated off the line?

a) a triangle could exist
b) a triangle could exist
c) No
d) No
e) No

Using this same Triangle Inequality Theorem, if two sides of a triangle were given, could you find the possible values for the third side?

Example 2 Find the domain (possible values for a third side) if two sides were given.

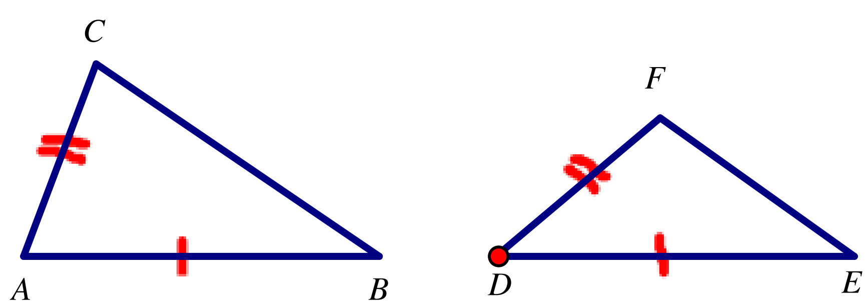
a) 7 and 8

The 3rd side would clearly have to be less then 15. Could it be 1? If it was, then two sides would not be greater than 8. So the list of possible numbers that satisfy that theorem are:
$1 < x < 15$. Any number between 1 and 15.

b) 4 and 6
To find the largest side, add the two lengths 4 and 6; 10
To find the shortest length, subtract the two lengths, 4,and 6; 2
So, the possible lengths of the 3rd side of the triangle is between 2 and 10, described as $2 < x < 10$

Triangle Inequalities – Two Triangles

Theorem **If two sides of a triangle are congruent to two sides of another triangle, but the included angle of the first triangle is larger than the included angle of the second triangle, then the third side of the of the first triangle is longer than the third side of the second triangle.**



$$\text{Given the two sides congruent}, \angle A > \angle D, \text{then } BC > EF$$

Theorem **If two sides of a triangle are congruent to two sides of another triangle, but the 3rd side of the first triangle is larger than the third side of the second triangle, then the included angle of the of the first triangle is greater than the included angle of the second triangle.**

Notice the parallel of these two theorems to help you remember them. The diagram makes it more clear.