Chapter 1

Points, Lines & Planes

As we begin any new topic, we have to familiarize ourselves with the language and notation to be successful. My guess that you might already be pretty familiar with many of the terms about to be introduced in this section, the biggest difference is that we will formalize our understanding and introduce notation that will enable to express that knowledge quickly.

Undefined Terms & Basic Definitions

Let's look at one of our first elements in geometry, a point.

A *point* is pictured by a dot. While a dot must have some size, the point it represents has no size. Points are named by capital letters.

. X

A *line* extends indefinitely. A line, containing infinitely many points, is considered to be a set of points, hence it has no thickness. A line can be named by a lowercase letter or by two points contained in the line.

This line could be called line *k* **or** $\overleftrightarrow{RS}$, read "line RS". Notice that does not begin or end at either of the points R or S.

A *plane* is a flat surface. Such things as table tops, desks, windowpanes, and walls suggest planes. A plane, like the aforementioned, does not have thickness and extends indefinitely.

The terms point, line, and plane are undefined terms. Other terms in geometry are defined. Notice the following definitions are based on the undefined terms.

Space- The set of all points.

Collinear points – A set of points that lie on one line.

Coplanar points – A set of points that lie on a plane.

Axioms (postulates) – are basic statements, assumed true without proof.

Theorems– are statements that are proven.

Corollaries - theorems that follow directly from a previous theorem

Subsets of a Line

A point between two other points: Point B, on $\overleftrightarrow{AC}$ is said to be between points A and C.

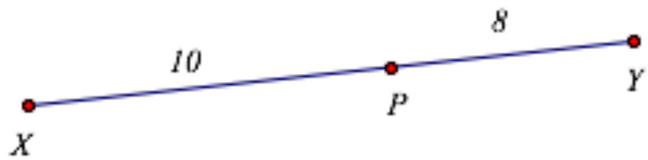
Segment: Given any two points A and B, segment AB is the set of points consisting of A and B and all the points that lie between A and B. Segment AB is denoted by $\overline{AB}$.

Segment Addition Postulate - A point between two other points: Point B, on $\overline{AC}$ is said to be between points A and C and can be written as
AB + BC = AC

Sum of the parts is equal to the whole line segment

Example 1 Given: $XP = 10$ and $PY = 8$, find XY

Always draw the picture and label any information given to you.



Now, ask yourself, what relationships have we studied about line segments? The Segment Addition Postulate states a line segment was equal to the sum of its parts.

Therefore we know, $XP + PY = XY$
Substituting, we have $10 + 8 = XY$
or $18 = XY$

Could this be any easier? Let's look at another example.

Example 2 Given: $AC = 12$ and $AB = 4$, find BC

Using the Segment Addition Postulate, we know
$AB + BC = AC.$
Let's substitute $4 + BC = 12$
$BC = 8$

The nice thing about math is its consistency, things don't change! What would problems involving the Segment Addition Postulate look like with some unknowns – algebra?

Example 3 Given: AT = x + 9, TB = 2x, and AB = 30, find the value of x.

Looking at the diagram and labeling the information given, we know that

AT + TB = AB by the Segment Addition Postulate

We also know that AB =30 was given to us. Substituting, we have

$$AT + TB = AB$$

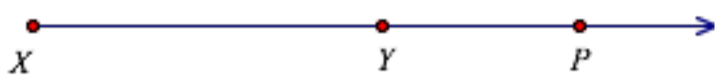
$$(x + 9) + 2x = 30$$

Solving the equation,

$$3x + 9 = 30$$
$$3x = 21$$
$$x = 7$$

If you were asked to find AT or TB, you would just substitue x = 7 into those expressions.

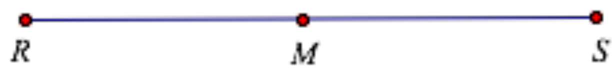
Ray: Ray XY, denoted by $\overrightarrow{XY}$, is the union of and the set of points P for which it is true that Y lies between A and P.



Opposite Rays: $\overrightarrow{SR}$ and $\overrightarrow{ST}$ are called opposite rays if S lies on $\overleftrightarrow{RT}$ between R and T.

Congruent Segments: Segments with equal lengths. If AB = XY, then AB is said to be congruent to XY, $\overline{AB} \cong \overline{XY}$

Midpoint of a segment: Point M is the midpoint of $\overline{RS}$ if M lies on $\overline{RS}$ and RM = MS.



Bisector of a segment: A line, segment, ray or plane that intersects a segment at its midpoint bisects the segment and is a bisector of the segment .

As you study geometry, I cannot stress enough the importance of you knowing vocabulary and notation to be successful. Also, as you study, be sure to take special note of information that describes relationships. The Segment Addition Postulate described a relationship that allows you to write an equation, is important to know. The same can be said of the definition of midpoint, those relationships allow you to write equations.

So, let's review the symbols-

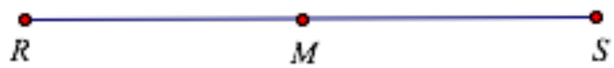
Symbols

$\overleftrightarrow{AB}$ - line containing A and B
$\overrightarrow{AB}$ - ray with endpoint A, passing through B
$\overline{AB}$ - segment joining A and B
$\overline{ABC}$ - line segment $\overline{AC}$ that contains B
AB - distance between A and B

It's very important that you take the time to memorize the notation for lines, line segments, rays, and distance. They will be used throughout the book and are meant to help you communicate mathematically. As we continue in our study, we will be introduced to more vocabulary and notation, if you learn it along the way, as it is introduced, then it will come naturally to you.

Now that we have the vocabulary and notation, let's see how we can use the definitions and postulates we learned to solve problems by using the relationships they described in the form of equations.

Example 4 Given M is the midpoint of $\overline{RS}$, RS = 10, find RM



We know by the Segment Addition Postulate that RM + MS = RS. By the definition of midpoint, we know that RM = MS.

By substituting RM for MS, we have

$$RM + RM = RS$$
$$2\ RM = 10$$
$$RM = 5$$

Rather than looking at that algebraically, it will be quicker to know that either segment would be half the distance of the line segment. We will look at midpoint more examples after we introduce the Ruler Postulate.

Three more terms that we will encounter quite often are axiom, theorem, and postulate.

Axiom (postulate) is a basic assumption in mathematics.

A **theorem** is a statement that is proved.

A **corollary** is a statement that can be proved easily by applying a theorem.

In a system of logic, math, the fewer assumptions (axioms) we make without proof, the better the system. But, we have to start somewhere, so let's look at some of our basic assumptions to begin our study of geometry.

Postulate A line contains at least two points, a plane contains at least three points not all on one line, and space contains at least four points not all in one plane.

Postulate Through any two points there is exactly one line.

Postulate Through any three points not on one line there is exactly one plane.

Postulate If two points lie in a plane, then the line joining them lies in the plane.

Postulate If two planes intersect, then their intersection is a line.

When a postulate or theorem is used quite frequently, we often give it a name. For instance, we introduced the **Segment Addition Postulate**, a postulate with a name because we will refer to again and again.

These postulates we accept as true because they seem to occur all the time and make sense to us. Based on these assumptions, postulates, we will then derive other information.

Theorem If two lines intersect, they intersect in exactly one point.

While that seems to make perfect sense, especially if we draw a picture, we call that a theorem rather than a postulate. The reason is, we can derive it, we can prove it.

Theorem If a point lies outside a line, then exactly one plane contains the point and the line.

Theorem If two lines intersect, exactly one plane contains both lines.

Ruler Postulate – The points on a line can be paired with the real numbers such that:

a. any point can be paired with zero
b. the distance between any two points is equal to the absolute value of the difference of the numbers paired with those points.

All the Ruler Postulate states is that if you have a line, you can pick any point on that line and call the coordinate 0. And the distance between any two points is equal to the absolute value of the of the difference of those coordinates.

Also AB = BA, then AB = |B – A| or |A – B| |12 – 4| = 8 or |4 – 12| = 8

Example 5 The coordinates of points R and S on a number line are –3 and 7, find RS.

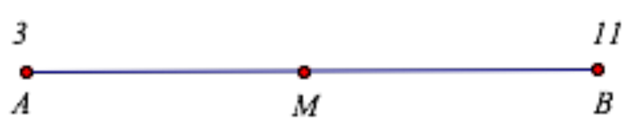
Using the Ruler postulate, we have |–3 – 7| = 10 or
|7 – (–3)| = 10, RS = 10

Example 6 Points Q and T lie on a number line, if QT = 10 and the coordinate of Q is 8, find the possible coordinates of T.

Knowing that QT = 10 and the coordinate of Q is 8, the line segment is 10 units long. If T is located to the right of Q, then T would be 8 + 10 = 18. If T is located to the left of Q, then T would be 8 + (–10) = –2. The possible coordinates of T would be 18 or –2.

Now that we introduced the Ruler Postulate, let's revisit midpoint and see how we might be able to solve problems.

Example 7 Given the diagram, find the midpoint of



Now, we are going to do this problem two ways, the long way and the easy way. Remember, the more you know in math, the easier it gets!

1st Method – By the Ruler Postulate, AB = |11–3| = 8.
Since AB = 8, half the distance from each end point would be 4 units. Starting from 3 and adding 4, the midpoint would be 7. Or I could have started from 11 and went 4 units to the left and the midpoint would still be 7.

2nd Method – If I did enough of these problems, it may dawn on me that a midpoint may be looked at as the average distance of two line segments, one segment being 3 units long, the other being 11 units. To find the average, midpoint, I add the two lengths 11 and 3, coordinates, and divide by 2, which is 7.

The second way leads us to the midpoint formula- which is finding the average of two numbers.

$$midpoint = \frac{x_1 + x_2}{2}$$

Let's look at another example. I will work it out using Method 2, the formula, using method 2, you can probably do the problem in your head.

Example 8 Given the diagram, find the midpoint of $\overline{CD}$

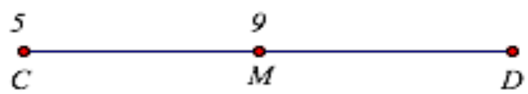
filling in the coordinates for C and D in the midpoint formula, we have

$$\text{midpoint} = \frac{3+15}{2}$$

which is $\frac{18}{2} = 9$. The coordinate midpoint of $\overline{CD}$ is 9.

The last two examples have been straight-forward applications for finding the midpoint. A slight variation of that might be a problem that gives you an endpoint of a line segment and the midpoint, and asks you to find the other endpoint. To do the problem, all you need to know is the midpoint formula and substitute in the numbers.

Example 9 If M is the midpoint of $\overline{CD}$ and the coordinate of C is 5 and the coordinate of the midpoint, M, is 9, find the coordinate of D.



Drawing the picture and labeling, we must now find the coordinate of D. Using the midpoint formula, we have

$$M = \frac{C+D}{2}$$

Now, let's fill in the numbers in the midpoint formula. That's easy enough, the coordinate of the midpoint is 9 and the end point is 5.

$$9 = \frac{5+D}{2}$$

Now, let's multiply both sides of the equation by the common denominator (2) to get rid of the fraction.

18 = 5 + D, so D = 13; the coordinate of D is 13.

Example 10 Points P, Q and R lie on a number line, Q lies between P and R, if PR = 15 and PQ is 9, find QR.

So far, we know two pieces of information that result in relationships, the Segment Addition Postulate and the Midpoint of a line segment. Since nothing was mentioned about a midpoint, it seems logical to try the Segment Addition Postulate to solve this problem.

Since PR is longer than PQ, Q must be between P and R, so we have PQ + QR = PR.

Substituting the numbers given to us in the problem, we have
9 + QR = 15
QR = 6

The Ruler Postulate leads us to the next couple of theorems.

Theorem On a ray, there is exactly one point a given distance ***d*** from the endpoint of the ray.

Theorem A segment has exactly one midpoint.

We have introduced a lot of new terminology and notation so far in this chapter that you need to be comfortable using. We are not quite done.

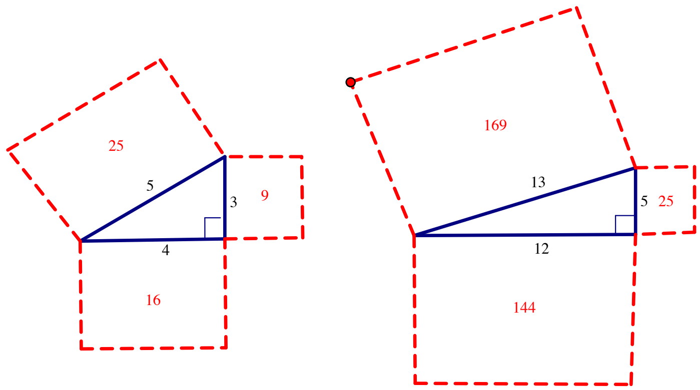
There are three pieces of information that lead to relationships in this section, you need to know those relationships. The Segment Addition Postulate can be broken down to the whole equals the sum of the parts. The definition of midpoint divides a line segment into two equal parts AND you should know the midpoint formula.

Pythagorean Theorem

In middle school, you were introduced to the Pythagorean Theorem: The square of the measure of the hypotenuse of a right trinagle is equal to the sum of the squares of the measures of the legs – commonly written as $\mathbf{c^2 = a^2 + b^2}$. c is the hypotenuse, the side opposite the right angle, the other sides are called legs, labeled a and b.

That's an important theorem in math that is used in many different contexts. In later chapters, we will see how it was developed and proved. In this chapter, we will use this theorem to introduce the distance formula.

Let's look at a couple of right triangles.



Notice the relationship in the areas of the squares formed by the legs to the area of the square formed by the hypotenuse.
That led us to the Pythagorean Theorem.

We can generalize these observations by labeling the legs a and b and the hypotenuse c to: $c^2 = a^2 + b^2$

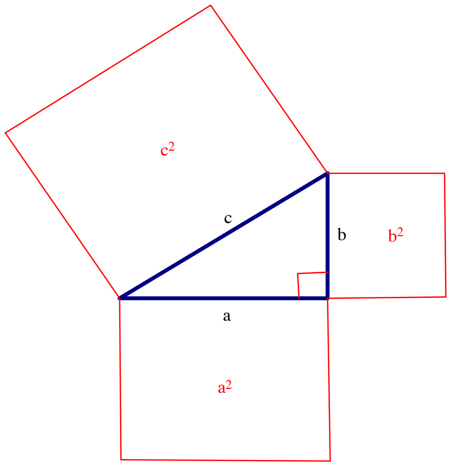
Since the Pythagorean Theorem comes up so often, it's nice to know combinations of numbers that could be sides of right triangles. Those triples are called Pythagorean Triples along withtheir multiples.

Examples of Pythagorean Triples:	3, 4, 5	x2	6, 8, 10
	5, 12, 13		
	7, 24 25		
	9, 40, 41		

Knowing those and some of their multiples will often cut down on the arithmetic in solving problems later.

To determine if 3 sides will form a right triangle, use the longest side as the hypotenuse - c, then determine if the relationship $c^2 = a^2 + b^2$ is true. If so, the 3 sides form a right triangle. If not, then the 3 sides do not form a right triangle.

The generalization below lead us to the Pythagorean Theorem. And we will use that to find the distance formula.



If a is the distance along the x-axis, then $a = x_2 - x_1$

If b is the distance along the y-axis, then $b = y_2 - y_1$

And rather than calling the length of the hypotenuse c, we can call it "d" for distance.

$\mathbf{c^2 = a^2 + b^2}$ making those substitutions, we have $\mathbf{d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2}$

taking the square root of both sides $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Distance Formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

That's read, *d is equal to the square root of the difference of the x's squared plus the difference of the y's squared.*

Example 11 Find the distance between the points A(2, 3) and B(8, 11).

$$d = \sqrt{(8-2)^2 + (11-3)^2}$$
$$d = \sqrt{6^2 + 8^2}$$
$$d = \sqrt{100}$$
$$d = 10$$

Example 12 Find the distance between (7, –6) and (4, 3)

$$d = \sqrt{(7-4)^2 + (-6-3)^2}$$
$$d = \sqrt{3^2 + (-9)^2}$$
$$d = \sqrt{9 + 81}$$
$$d = \sqrt{90}$$
$$d = \sqrt{9(10)}$$
$$d = 3\sqrt{10}$$

As usually happens in math, I can't make these problems any more difficult – only longer.

Example 13 For instance, I could ask if a triangle with vertices of A(–2, 2), B(10, –3) and C(5, 9) form an isosceles triangle.

To do that problem, I would have to find AB, AC and BC – three separate distance problems. I would also need to know an isosceles triangle has 2 equal sides. If the distances of two sides were equal, then the triangle would be isosceles.

Using the distance formula and substituting the numbers, we have

$$AB = \sqrt{(10-(-2))^2+(-3-2)^2} = \sqrt{12^2+5^2} = \sqrt{149}$$

$$BC = \sqrt{(5-10)^2+(9-(-3)^2} = \sqrt{(-5)^2+12^2} = \sqrt{149}$$

$$AC = \sqrt{(5-(-2))^2+(9-2)^2} = \sqrt{7^2+7^2} = \sqrt{98}$$

We can see that AB = BC, ∴ΔABC is isosceles.

We are all familiar with the Properties of Real Numbers; Commutative, Associative and Distributive Properties, let us finish this chapter with the **Properties of Equality**

Reflexive Property	a = a
Symmetric Property	if a = b, then b = a
Transitive Property	if a = b and b = c, then a = c
Addition Property	if a = b and c = d, then a + c = b + d
Subtraction Property	if a = b and c = d, then a – c = b – d
Multiplication Property	if a = b and c = d, than ac = bd
Division Property	if a = b and c = d, c ≠ 0, then a/c = b/d

You will be using these properties quite a bit in geometry – while they make sense, make sure you are familiar with them.

In algebra, you learned how to solve equations and justify your answers by using those Properites of real Numbers.

For instance, you might have been asked to solve 3x – 2 = 13 for x.

This is how we might write that problem in geometry.

Statements	Reasons
1. 3x – 2 = 13	Given
2. (3x – 2) + 2 = 13 + 2	Add. Prop of Equality
3. 3x + (–2 + 2) = 15	Assoc.Prop/ Math Fact
4. 3x + 0 = 15	Add. Inverse
5. 3x = 15	Prop.of Zero
6. 3x/3 = 15/3	Div. Prop. Of Equality
7. x = 5	Mult. Inverse/ Mult Fact

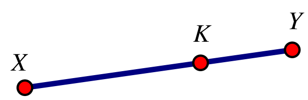
What we wrote is exactly what your teacher said when solving equtions such as this one. Notice, the problem was "given" to us, so we write it was "Given" fort he reason.

The good news is nothing changes in geometry. We will be given a problem, then as we do it, we will justify our steps using our knowledge of algebra and geometry.

Let's look at a couple of problems.

Ex. 1 Given: $\overline{XKY}$

Prove: XK = XY – KY



Statements	**Reasons**
1. $\overline{XKY}$	Given
2. XK + KY = XY	Segment Add. Postulate
3. (XK + KY) – KY = XY – KY	Subtract Prop. of Equality
4. XK + (KY–KY) = XY – KY	Assoc. Prop.
5. XK + 0 = XY – KY	Add. Inverse
6. XK = XY – KY	Prop of Zero

To be frank, when you do these proofs, we need to have only enough steps so a person can follow what you did to see if the conclusion followed from your argument.

If I were asked to do that same problem, I would have left out some steps thinking you could follow my reasoning:

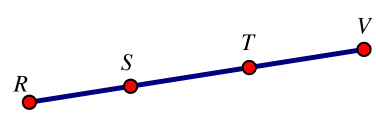
Statements	**Reasons**
1. $\overline{XKY}$	Given
2. XK + KY = XY	Segment Add. Postulate
3. XK = XY – KY	Subtract Prop of Equality

So, I wrote what was given, then from that used the Segment Addition Postulate to get an equation. From there, I simply use the Subtraction Property of Equality to get the desired result.

Here's the rub, you have to put in enough steps so a reasonable person can follow wihtout having questions.

Ex. 2 Given: $\overline{RSTV}$ and RT = SV

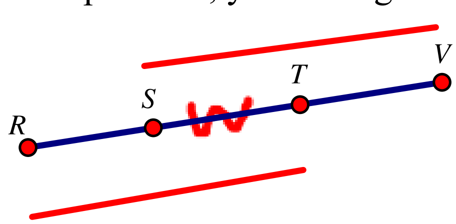
Prove: RS = TV



IMPORTANT!!!

When doing problems with pictures, use "tick" marks to show any equalities that are either given explicitly to you or implied, such as a midpoint or Segement Addition Postulate. After you do that, and ONLY after do that, then write what was given, then just do the math!

In this problem, you were given RT = SV explicitly. Mark it on the drawing.



Looking at the problem and what you want to show, it looks like we will be subtracting ST from both RT and SV to get RS and TV.

Let's do it.

Statements	Reasons
1. $\overline{RSTV}$ and RT = SV	Given
2. RT = RS + ST SV = ST + TV	Segment Add. Postulate
3. RS + ST = ST + TV	Substitution
4. RS = TV	Subtract Prop. of Equality

In the next chapter, we will delve deeper into logic and proofs. But keep in mind, in order to be successful doing proofs, you will have to know definitions, theorems and postulates that result in equations. So far, those include the Segment Addition Postulate, midpoint definition, Pythagorean Theorem, Distance Formula and Properties of Real Numbers.