Chapter 2

Reasoning

Suppose you know the following two statements are true.

1. Every board member read their back-up material
2. Tom is a board member

You can conclude:

3. Tom read his back-up material.

Once a reasonable person accepts statements 1 and 2 as true, that person knows statement 3 is also true. Statement 3 is a conclusion that follows from statements 1 and 2.

This kind of reasoning is called *deductive reasoning*.

Deductive reasoning is the process of reasoning from accepted facts to a conclusion.

In contrast to deductive reasoning is inductive reasoning (thinking). Consider a golfer who tries a new grip on his clubs. The next 4 times he plays golf, he strokes the ball better, he decides it's because of the new grip. That's an example of inductive reasoning.

Inductive reasoning is the process of observing individual cases, then stating a general principle suggested by those observations.

As we go inot tis chapter, let's review some math we already know and accept.

Postulates & Theorems of Equality

Reflexive Property	$a = a$
Symmetric Property	if $a = b$, the $b = a$
Transitive Property	if $a = b$ and $b = c$, then $a = c$
Addition Property	if $a = b$ and $c = d$, then $a + c = b + d$
Subtraction Property	if $a = b$ and $c = d$, then $a - c = b - d$
Multiplication Property	if $a = b$ and $c = d$, then $ac = bd$
Division Property	if $a = b$ and $c = d$, $c \neq 0$ ^ $d \neq 0$, then $a/c = b/d$

Substitution Principle – if $a = b$, then a may be replaced by b in any equation or inequality.

Existence Properties

Additive Inverse – If $a \in R$, then $(-a) \in R$ and $a + (-a) = 0$

Multiplicative Inverse – If $a \in R$, then $(1/a) \in R$ and $a \times (1/a) = 1$

The existence properties tell you if one number exists, then another number must also exist. Specifically, the Additive Inverse states that if 5 is a Real Number, then (–5) is also a real Number and 5 + (–5) = 0. We actually used that quite frequently in algebra when solving equations.

These postulates and theorems are used often in math. The symmetric property often causes students concern. All it states is if 10 dimes is equal to 1 dollar, then 1 dollar is equal to 10 dimes. That makes sense.

Quite frequently in math, you will see a teacher solve an equation, get a solution like $10 = y$, then they write, $y = 10$. The Symmetric Property allows for that. The Symmetric property can be used when the expressions on each side of the equal sign are longer. For instance $10 = 3x - 2$ could be rewritten as $3x - 2 = 10$.

Statements

We use statements like, "*If* you get your homework done before 6 pm, *then* we can watch television after dinner" all the time. In math, we refer to those statements as ***"if-then"*** statements.

If-then statements are called **conditional** statements. They are called conditionals because the end result depends on, is conditional on, that something was already done. A conditional in math is made up of two parts, the part that follows the *if* is called the **hypothesis**, it states what is given. The end result, the **conclusion**, is what we typically try to prove in geometry.

A conditional looks like this; If ***a***, then ***b;*** where ***a*** and ***b*** are both statements, ***a*** is the hypothesis (what is given to you), and ***b*** is the conclusion (what needs to be proved).

An example of a conditional (if-then) statement is: *If it rains outside, then the sidewalks will be wet.*

Rains outside is what is given to us as true, the hypothesis, we will call that A. The sidewalks will be wet is the conclusion, what we would like to prove, we will call that B.

Mathematically, we would write that conditional as $A \rightarrow B$. That is read *"A implies B"* or we could say "*if A, then B.*"

Converse of a statement

If you change the order of the hypothesis and conclusion, you can create another conditional statement. The new conditional is referred to as the **converse** of the original conditional.

That is; $A \rightarrow B$ is the statement, the converse of that statement is $B \rightarrow A$.

If it rains outside, then the sidewalks will be wet is an example of a conditional statement. If that turns out to be true, would the converse also be true? In other words, *if the sidewalks are wet, then it rained*, is that necessarily true? The answer is no. Maybe the sprinklers were on, so be careful, don't think that just because a conditional statement is true, the converse has to be.

Another example of a converse of a conditional statement is if you are given the following as a true statement.

If you make an A on your math test, then we will go to the movies. We will write that as A→M. What is the converse of that statement? M →A. That is, if you went to the movies, you made an A on your math test.

Now again, and this is important, if a statement is true, the converse of that statement does <u>not</u> have to be true!!!

For instance, on the last example I could have decided to take you to the movies whether you made an A on your math test or not. But, if you made an A, then I would clearly have to take you to the moves to fulfill my promise.

Negation of a statement

To negate a statement, we write its opposite. *If it is raining outside* can be negated by saying *if it is NOT raining outside.* So, if R represents it is raining, then either of the symbols, ~R or R', would represent that it is not raining.

Contrapositive of a statement

The contrapositive, like the converse, changes the order of the hypothesis and conclusion and negates each. The new conditional is referred to as the contrapositive.

That is; $A \to B$ is the statement, the contrapositive is $\sim B \to \sim A$. That is read, Not B implies not A.

Let's go back to our original conditional statement; if it is raining outside, then the sidewalks are wet. Symbolically we write R → W. The contrapositive is ~W →~R. In other words, if the sidewalks are not wet, then it is not raining.

Another example of a conditional statement was, if you make an A on your math test, then we will go to the movies. $A \to M$. The contrapositive, $\sim M \to \sim A$ says that if you don't go to the movies, then you did not make an A.

It turns out that a statement and its contrapositive have the same truth values. We say they are logically equivalent. If statements are logically equivalent, they can be substituted for one another. It means that if the original statement is true, then the contrapositive is also true. That's important for you to know.

Later, we will find there are some things we can not prove directly. So, like Sherlock Holmes, we will solve them indirectly. And the way we will do this is by an indirect proof sometimes called proof by contradiction. Those proofs use the contrapositive and that's why we need to know it.

Inverse of a statement

The inverse of a statement keeps the same order as the original conditional statement, but negates each statement within the conditional. The new conditional is called the inverse.

That is; $A \to B$ is the statement, the inverse is $\sim A \to \sim B$.

So again, let's go back to our original conditional statement; if it is raining outside, then the sidewalks are wet. Symbolically we write R →W. The inverse is ~R →~W. In other words, if it is not raining, then the sidewalks are not wet.

You can see the inverse doe not have to be true if the original statement is true. The sidewalks could be wet because a sprinkler is on.

Another example using a conditional statement, if you make an A on your math test, then I will take you to the movies; $A \rightarrow M$. The inverse is if you do not make an A on your math test, then I will not take you to the movies; $\sim A \rightarrow \sim M$.

Bi-conditional statement

A bi-conditional statement is a statement in which the statement and its converse are both true. Sometimes these are referred to as ***if and only if*** statements, abbreviated by ***iff.***

The Pythagorean Theorem states i*f we have a right triangle, then the square of the hypotenuse is equal to the sum of the squares of the two legs.* That mathematical relationship is often written as $c^2 = a^2 + b^2$. It turns out the converse is also true. That means if we have a triangle where this relationship is true, $c^2 = a^2 + b^2$, then we have a right triangle. We can combine those two statements into one by using a bi-conditional.

Mathematically, we will use a double arrow (↔) to show a bi-conditional symbolically.

$$\text{Rt } \Delta \leftrightarrow c^2 = a^2 + b^2$$

Writing that out in long hand, we have

It is a right triangle ***if and only if*** the square of the hypotenuse is equal to the sum of the squares of the two legs.

Again, in this chapter, there are a lot of new vocabulary words and mathematical notation. To be successful, you need to be comfortable using these.

Quick symbolic review:

$A \to B$	Conditional Statement
$B \to A$	Converse
$\sim B \to \sim A$	Contrapositive
$\sim A \to \sim B$	Inverse

* If the statement is true, the contrapositive is logically equivalent and is also true.

$A \leftrightarrow B$ Bi-conditional

Presenting an Argument

An argument consists of two or more related premises (statements) and a conclusion based which is based on those premises.

There are a number of valid argument forms, the one we will use mostly is the Law of Detachment (Modus Ponens).

Law of Detachment. If we have established p —> q is true and the antecedent p is true, then we can conclude the conclusion q is true.
Mathematically we write; p —> q
p
∴ q

While statements can either be true or false, arguments are either valid or invalid.

Example	If a boy is an athlete, then he is healthy	p —> q
	John, a boy, is an athlete	p
	Therefore, John is healthy	∴ q

Law of Syllogism If p —> q and q —> r, then p —> r is always true.
Mathematically we write: p —> q
q —> r
∴ p —> r

We will use the Laws of Detachment and Syllogism to prove theorems in geometry.

A valid argument is an argument in which the conclusion must be true whenever the hypothesis is true following our laws in logic. An argument is a set of statements in which one of the statements is the conclusion and the rest of the statements make up the hypothesis.

Under the Math Resources, there is a chapter on Logic which includes Truth Tables.

Proving Theorems – Deductive Reasoning

Parts of a T-Proof

A proof has 5 parts: the statement, the picture, the given, the prove, and the body of the proof.

Playing with the picture and labeling what you know will be crucial to your success. What's also crucial is bringing in your knowledge of previous definitions, postulates, and theorems. This is always the FIRST thing you do when constructing a geometric proof.

Always write out the proof and identify what is being given. Next, and this is extremely important, draw the picture and label information given (using tick marks) to you onto your drawing. Sometimes you will look at your drawing and be able to see more relationships – that's good, label those! Next, write down what you want to prove. The body of your proof should come directly from the picture you have drawn and labeled and what you were able to add to that with your prior knowledge of math.

The biggest difference you will immediately notice is in algebra, you were asked to find a solution. In geometry, you will be given the answer (conclusion) and you have to show each step on how you could arrive at that conclusion.

So, let's start where you are most comfortable – solving equations from last year. Only this time, I will give you the answer.

Example 1 Given: $3x + 2 = 14$
Prove: $x = 4$

There is no picture to be drawn, so we will go directly to the T-Proof.

	STATEMENTS	REASONS
1.	$3x + 2 = 14$	Given
2.	$(-2) \in R$	Additive Inverse
3.	$3x + 2 + (-2) = 14 + (-2)$	Add Prop Equality
4.	$3x + 0 = 12$	Add Inverse/ Arithmetic
5.	$3x = 12$	Identity for Add
6.	$\frac{3x}{3} = \frac{12}{3}$	Division Prop Equality
7.	$1x = 4$	Mult Inverse/Arithmetic
8.	$x = 4$	Identity for Mult

A proof is nothing more than an argument in which the conclusion follows, but each statement has to be backed up with a reason!

In algebra, chances are the way you solved that equation would look more like this:

$$3x + 2 = 14$$
$$3x + 2 - 2 = 14 - 2$$
$$3x = 12$$
$$x = 4$$

The big difference in a proof is you are showing almost every step to arrive at the result/conclusion that was given to you. I would guess most students would have left out steps 2, 4, and 7 if they only applied what they normally did in an algebra class.

Now, here is the good news, your proof does not have to look exactly like mine to be correct. Someone else might do the proof in 10 steps and another person might have taken 6 steps. In a proof, we are determining does your conclusion follow from your argument.

Let's look at a proof in geometry based upon line segments.

Example 2

Given: $\overline{RONY}$
$\overline{RO} \cong \overline{NY}$

Prove: RN = OY

Looking at this picture, we start off with a line segment and $\overline{RO} \cong \overline{NY}$. I want to prove RN = OY, but I don't have an RN or an OY in the problem. So, I have to ask myself, how can I get them in the problem? If I used the Segment Addition Postulate in the picture, I have RO + ON = RN. That gives am the RN I need.

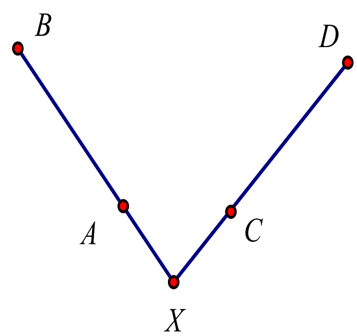
So, we want to add $\overline{ON}$ to both segments. But we don't have a theorem or postulate that allows us to add segments together – only distances associated with those segments. The other hint that I had to get rid of the segment notation was I had to prove the distances were equal.

	STATEMENTS	REASONS
1.	$\overline{RONY}$, $\overline{RO} \cong \overline{NY}$	Given
2.	RO = NY	Def of Congruence
3.	ON = ON	Reflexive Property
4.	RO+ON= NY+ON	Add Prop Equality
5.	RO+ON = RN ON+NY= OY	Segment Add Postulate
6.	RN = OY	Substitution into step 4

Notice in these proofs, I like to write the given information as my first step, my last step is always what I am supposed to prove. That's a matter of preference.

Step 3, ON was not given to me in the problem, so I had to get in it there somehow, so I set it equal to itself using the Reflexive Property.

Example 3



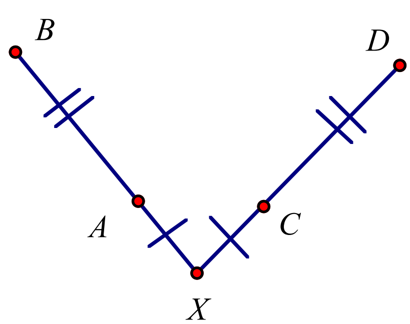
Given: $\overline{XAB}, \overline{XCD}$

$XA = XC$

$AB = CD$

Prove: $XB = XD$

Looking at what I want to prove, $XB = XD$, which are two line segments suggests that I might want to use the Segment Addition Postulate. Before I begin the T-Proof, let's mark on the diagram the segments that are equal.



Now we can see on the diagram XA=XC, and AB=CD. Knowing the sum of the parts is equal to the whole, we are ready to do the proof. We'll write what was given as the first step, then fill in the equalities we have on the diagram.

Now, as we do the proof we will keep referring back to the picture that was labeled and the equal parts identified.

	STATEMENTS	REASONS
1.	$\overline{XAB}, \overline{XCD}$ XA = XC, AB = CD	Given
2.	XA + AB = XC + CD	Add Prop Equality
3.	XA + AB = XB XC + CD = XD	Segment Add Postulate
4.	XB = XD	Substitution into step 2

Now, let's be clear, proving a theorem demands that you know and can visualize the definitions, postulates and theorems previously introduced. And while you should know them, you should take special note of any of those that lead to a mathematical relationship, an equation or inequality. It is those that will allow you to write the equations in a proof and use the Properties of Real Numbers to build further relationships.

You should also play very close attention to any diagrams that associated with the problem. There will be things to look for in the diagram that result in you being able to write equations: Sharing a line for instance allows us to use the Reflexive Property. Later, as we learn more, we will know if lines intersect, we can use the fact that vertical angles are congruent; if lines are parallel, then alternate interior angles are congruent, corresponding angles are congruent; if there is a perpendicular bisector, then right angles are formed and segments are congruent; angle bisectors result in two angles being congruent; and the segment and angle addition postulates result in the equation the sum of the parts equal the whole. etc. The point, a diagram will provide clues to what can be used in a proof.