Chapter 9 Solving Quadratic & Higher Degree Equations

Sec 1. Zero Product Property

Back in the third grade students were taught when they multiplied a number by zero, the product would be zero. In algebra, that's an extremely important property, so important it has a name, the Zero Product Property.

While teachers may not have used that name, the point they tried to get across was that the product of any number and zero is zero. 5 x 0 = 0, 123 x 0 = 0, and 0 x 4.32 = 0. It's a pretty straight forward concept.

Extending the thinking, we found that in order for the product of 4 and some number ***n*** to be zero, n would have to be zero.

$$4\boldsymbol{n} = 0$$
$$\boldsymbol{n} = 0$$

If I multiplied two numbers, x and y, and found their product to be zero, the one of those two numbers would have to be zero. Mathematically, we'd write

Zero Product Property If xy = 0, then x = 0 or y = 0.

One of the two numbers would be equal to zero.

So, a list of all possible answers would include, x = 0 or y = 0.

An application of the Zero Product Property will allow us to solve higher degree equations by changing them into multiplication problems whose product is zero.

Let's look at one; if (x –2)(x + 5) = 0,
then by the Zero Product Property, one of those 2 factors must be zero.
So, (x – 2) = 0 or (x + 5) = 0, solving, we have x = 2 or x = –5.

Example 1 Find the solution $(x + 6)(x - 7) = 0$

Using the Zero Product Property, set each factor equal to zero.

$$\begin{array}{lcl} x + 6 = 0 & \text{or} & x - 7 = 0 \\ x = -6 & & x = 7 \end{array}$$

Example 2 Solve, $x(x - 1) = 12$

Notice, this product is not equal to zero.

A basic strategy used in math is to change something we don't recognize or know how to do into something we do recognize and can do using the Properties of Real Numbers.

So, in order to use the Zero Product Property, I will need to rewrite this equation so the product is zero, then I can use the Zero Product Property.

Multiplying the left side, we have $x^2 - x = 12$

Rewriting that in General Form results in $x^2 - x - 12 = 0$

Factoring, I would have the product of two numbers equaling zero rather than 12.

$$x^2 - x - 12 = 0$$

$$(x - 4)(x + 3) = 0.$$

I have a product of two numbers equaling zero, that means one or both of them of them has to be zero.

$$\begin{array}{lcl} x - 4 = 0 & \text{or} & x + 3 = 0 \\ x = 4 & \text{or} & x = -3. \end{array}$$

Substituting those back into our original equation $x(x - 1) = 12$ will make that statement true.

A reason we learned how to factor is so we could solve quadratic and higher degree equations. Those factoring skills are not just important, they are extremely important.

You not only need factoring skills for solving the following quadratic and higher degree equations, you'll also need to factor for adding and subtracting rational expressions, simplifying expressions and graphing.

A quadratic equation is an equation of degree 2, that is, the exponent on the variable is 2.

$$3x^2 + 5x - 4 = 0$$

There are a number of different methods that can be used for solving quadratic equations, we'll look at a few of these methods. We'll solve them by Factoring (Zero Product Property), $x^2 = \boldsymbol{n}$, Completing the Square, and the Quadratic Formula.

Let's make sure we know what factoring is;

Factoring is the process of changing an expression that is essentially a sum into an expression that is essentially a product.

Depending upon what the expression looks like, we may factor using the Distributive Property, Difference of 2 Squares, Linear Combination, Trial & Error (ac Method), Grouping and the Sum or Difference of 2 Cubes. It is very, very important that you can recognize these patterns for factoring.

When solving quadratic or higher degree equations, the answers could be described as roots, zeros, or solution sets.

Algorithm Solving Quadratic Equations by Factoring.

1. Put everything on one side of the equal sign, zero on the other side.

2. Factor the expression completely.

3. Set each factor equal to zero.

4. Solve the resulting equations.

Example 4 Solve for x, $x^2 + 7x + 12 = 0$

We want to change this addition problem into a multiplication problem by factoring.

$$x^2 + 7x + 12 = 0$$
$$(x + 3)(x + 4) = 0$$
$$x + 3 = 0 \quad \text{or} \quad x + 4 = 0,$$
$$x = -3 \quad \text{or} \quad x = -4$$

Example 5 Find the roots, $x^2 - 35 = 2x$

1. Put everything on one side, zero on the other side. $x^2 - 2x - 35 = 0$
2. Factoring $(x + 5)(x - 7) = 0$
3. Set each factor equal to zero $x + 5 = 0 \quad x - 7 = 0$
4. Solve $x = -5$ or $x = 7$

Now remember, if I substitute these values back into the original equation, the result is a true statement.

Let's look at the reason why this method works. By putting everything on one side and factoring, we are looking for two numbers when multiplied together equal zero – **Zero Product Property.**

Example 6 Find the zeros, $8y^2 + 2y = 3$

1. Put everything on one side $8y^2 + 2y - 3 = 0$
2. Factor $(4y + 3)(2y - 1) = 0$
3. Use the Zero Product Property $4y + 3 = 0$ or $2y - 1 = 0$
4. Solve: $4y = -3 \quad 2y = 1$

$y = -\frac{3}{4}$ or $y = \frac{1}{2}$

If we had a cubic equation to solve, an equation of degree 3, factoring is an appropriate method to use.

Example 7 **Find the solution set,** $x^3 + 12x = 7x^2$

Putting everything on the side we have:

$$x^3 - 7x^2 + 12x = 0$$

Factoring $x(x^2 - 7x + 12) = 0$

$$x(x - 3)(x - 4) = 0$$

Now I have three numbers multiplied together that equal zero. That means at least one of those numbers; x, $(x - 3)$ or $(x - 4)$ must be zero.

That's an extension of the Zero Product Property. Setting each factor equal to zero we have:

$x = 0$ $x - 3 = 0$ $x - 4 = 0$

$x = 0$ or $x = 3$ or $x = 4$

We can write those answers as a solution set $\{0, 3, 4\}$

Substitute these values into the original equation and you see they work.

Solve by the Zero Product Property (factoring).

$x^2 + 7x + 12 = 0$	$x^2 + 9x + 20 = 0$	$x^2 + 6x + 5 = 0$
$x^2 - 5x + 6 = 0$	$x^2 - x - 20 = 0$	$x^2 - 11x + 30 = 0$
$3x^2 - 21x = 0$	$x^2 - 6x = 0$	$6x^2 + 10x - 4 = 0$
$2x^2 + 5x + 2 = 0$	$x^2 + 5x = -4$	$x^2 + 9x = -18$
$x^2 + 2x = 3$	$5x^2 = 40x$	$x^2 + 12x = -27$

Sec. 2 Simplifying Radicals

From grade school, you can probably remember how to take square roots of numbers like $\sqrt{25}$, $\sqrt{64}$, and the $\sqrt{100}$. The number on the inside of the radical sign is called the radicand. When there was not an index, it was understood to be a square root, a second root – $\sqrt[2]{25}$

Finding those answers are easy when the radicands are perfect square. But what happens if the radicand is not a perfect square? Knowing you were thinking just that very thought, you must be very pleasantly surprised to know I am going to tell you how to simplify those expressions.

Let me just tell you how, then I'll use an example to make it more clear.

To simplify a square root (second root)

1. rewrite the radicand as a product of a perfect square and some other number,
2. take the square root of the perfect square you know.
3. The number you don't know the square root of stays inside the radical.

Examples of Perfect Squares; 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …

Example 1. Simplify $\sqrt{75}$

We don't know the $\sqrt{75}$, it's not a perfect square. We'll rewrite the radicand as a product of a perfect square and some other number. Since 25 is a perfect square that is a factor of 75, we'll rewrite.

$$\sqrt{75} = \sqrt{25x3}$$

$$= \sqrt{25} \text{ x } \sqrt{3}$$

$$= 5\ \sqrt{3}$$

That's pretty straight forward. If you are not familiar with perfect squares, you should write them down. By multiplying the counting numbers by themselves.

Example 2. **Simplify** $\sqrt{72}$

$$\sqrt{72} = \sqrt{36x2} = 6\sqrt{2}$$

What would have happened if you rewrote 72 as 9 x 8 instead of 36 x 2

$$\sqrt{72} = \sqrt{9 \cdot 8}$$
$$= 3\sqrt{8}$$

Oh, oh, that's not the same answer we got before! How can that be? Well, one reason is we're not finished, we can simplify $\sqrt{8}$.

$$3\sqrt{8} = 3\sqrt{4 \cdot 2}$$

$$= 3(2)\sqrt{2}$$

$$= 6\sqrt{2}$$

That's the same answer we got before. Whew! You would not have able to go to bed tonight thinking about that if we didn't get the same answer.

When simplifying radicals, it would be a good idea to become comfortable with perfect squares. And just as importantly, when you rewrite the radicand as the product of a perfect square and some other number, use the **LARGEST** perfect square you can. Otherwise, you'll end up doing more steps.

Remember simplifying fractions, you'd use the largest factor that divided into both the numerator and denominator because that meant you ended up with less work. The same applies here, the larger the perfect square you use, the shorter the problem.

Try these on your own.

1. $\sqrt{48}$ 2. $\sqrt{175}$ 3. $\sqrt{32}$ 4. $\sqrt{200}$

Sec 3. Solve by taking the Square Root, $x^2 = n$

If I look at a couple of special cases for solving quadratic equations by the Zero Product Property (factoring), we might find a shorter way to do some problems.

Let's quickly look at a few and see if we see a pattern that would allow us to do some problems in our head.

Example 1. Solve for x, $x^2 - 4 = 0$

$(x + 2)(x - 2) = 0$

So $x = -2$ or $x = 2$

Example 2. Solve for x; $x^2 - 36 = 0$

$(x + 6)(x - 6) = 0$

So $x = -6$ or $x = 6$

Example 3. Solve for x, $x^2 - 100 = 0$

$(x + 10)(x - 10) = 0$

So $x = -10$ or $x = 10$

Those were relatively easy equations to solve. We can make them easier if we look at just the problem and the answers. Isolating the x^2, then taking the square root, we have the following.

$$x^2 - 4 = 0 \longrightarrow x^2 = 4 \longrightarrow x = \pm\sqrt{4} \longrightarrow x = \pm 2$$
$$x^2 - 36 = 0 \longrightarrow x^2 = 36 \longrightarrow x = \pm\sqrt{36} \longrightarrow x = \pm 6$$
$$x^2 - 100 = 0 \longrightarrow x^2 = 100 \longrightarrow x = \pm\sqrt{100} \longrightarrow x = \pm 10$$

Note that these solutions match the solutions we just did by factoring.

Recognizing this pattern allows me to generalize how to solve equations when one side is squared and the other side is a number.

If $x^2 = n$, then $x = \pm\sqrt{n}$

Notice, because the equations are quadratic, we have to have two solutions. So, the "$\pm$" is very important.

Example 4. Solve for x, $x^2 = 49$

$$x = \pm 7$$

Example 5. Solve for x, $x^2 = 50$

$$x = \pm\sqrt{50}$$
$$x = \pm\sqrt{25 \cdot 2}$$

$$\mathbf{x =} \pm 5\sqrt{2}$$

Example 6. Solve for x, $(x - 1)^2 = 100$

$$x - 1 = \pm\sqrt{100}$$
$$x - 1 = \pm 10$$
$$x = 10 + 1 \quad \text{or} \quad x = -10 + 1$$
$$x = 11 \quad \text{or} \quad x = -9$$

Solve the following by the x^2 Method

1. $x^2 = 81$ 2. $x^2 = 45$ 3. $x^2 = 20$ 4. $x^2 = 98$

Recognizing this pattern will help me in the next section, completing the square. So, let's look at a few more problems like example 6 and see what that might suggest.

Sec. 4 Completing the Square

Seems like we should finish a picture when we say complete the square, doesn't It?

We know, that if $x^2 = \boldsymbol{n}$, then $x = \pm\sqrt{n}$, that means if $x^2 = 4$, then $x = \pm\sqrt{4}$

Well, let's expand that just a little. If $(x + 1)^2 = 9$, then taking the square root of both sides, we have $x + 1 = \pm\sqrt{9}$.

Solving that resulting equation, we have $x + 1 = \pm 3$ which translates to

$$x + 1 = 3 \quad \text{or} \quad x + 1 = -3$$

$$\text{so } x = 2 \quad \text{or} \quad x = -4$$

Example 1 Solve $(x - 5)^2 = 64$ by the x^2 Method

$x - 5 = \pm\sqrt{64}$
$x - 5 = \pm 8$
So $x - 5 = 8$ or $x - 5 = -8$
$x = 13$ or $x = -3$

Example 2 Solve $(x - 3)^2 = 50$ by the x^2 Method

$x - 3 = \pm\sqrt{50}$
$x - 3 = \pm 5\sqrt{2}$
So $x - 3 = 5\sqrt{2}$ or $x - 3 = -5\sqrt{2}$
$x = 5\sqrt{2} + 3$ or $x = -5\sqrt{2} + 3$

Remember when we memorized special products back when we were working with polynomials.

One of those special products was $(a + b)^2 = a^2 + \mathbf{2}ab + b^2$. $a^2 + \mathbf{2}ab + b^2$ is called a perfect square because it came from squaring $(a + b)$, Using that pattern, let's look at some special products.

$(x + 3)^2 = x^2 + 6x + 9$
$(x + 5)^2 = x^2 + 10x + 25$
$(x + 10)^2 = x^2 + 20x + 100$
$(x + 4)^2 = x^2 + 8x + 16$
$(2x + 3y)^2 = 4x^2 + 12xy + 9y^2$

There maybe a benefit to rewrite polynomials as perfect squares
When solving quadratic equations based on the examples we just did.

Now, if we looked and played with these long enough, we might see some relationships that really don't just jump out at you. But the relationships are important, so I am going to tell you what they are.

Looking at the linear term (middle term) of a polynomial whose leading coefficient is 1 and its relation to the constant, (see how important vocabulary is in math), do you see any way to get from the 6 to the 9 in the first special product from 10 to 25 in the second product? Chances are you won't see it immediately.

$$x^2 + 6x + 9 \text{ factoring } (x + 3)^2$$

If I took half of the 6, and squared it, I'd get 9.

$$x^2 + 10x + 25 \text{ factoring } (x + 5)^2$$

If I took half the 10, and squared it, I would get 25. Getting excited? Looking at;

$$x^2 + 20x + 100 \text{ factoring } (x + 10)^2$$

Taking half of 20 and squaring it gives me 100. Neat, huh? And they are all perfect squares.

So, when we have a perfect square, the relationship that we see is between the linear term and the constant of a polynomial with leading coefficient 1. Well, let's see how recognizing a perfect square can make our work easier.

Up to this point, I can solve quadratic equations by the Zero Product Property (factoring) or by the $x^2 = \boldsymbol{n}$ Method

Recognizing patterns in math has always been important to making our work easier. Now, with this attest observation, we can make trinomials perfect squares if you use the patterns we discussed. The constant term is always half the linear term squared. Let's say that again, the constant term is always half the linear term squared.

Now if I ask, is $x^2 + 6x - 11$ a perfect square? By taking half of 6 (linear term) and squaring, you get 9 not11. Therefore, it's not a perfect square.

Let's say I was asked to solve the following equation:

Example 3 Solve for x, $x^2 + 6x - 11 = 0$

I find I cannot factor the trinomial – there are no factors of –11 whose sum is 6. So, it looks like I can't solve it by the Zero Product Property. The trinomial is not a perfect square, so I can't solve it that way either.

But, since I understand how to make a perfect square and I know the Properties of Real Numbers, I could transform that equation into a perfect square.

Again, we know in order to have a perfect square, the constant term comes from taking half the linear term and squaring it.

So, let's push the (-11) out of the way and complete the square.

$$x^2 + 6x - 11 = 0$$

How do we do that? Adding 11 to both sides of the equation using the Addition Property of Equality we have.

$$x^2 + 6x = 11$$

Now, let's make a perfect square. Take half of 6 and square it, we get 9. This is important, if we add 9 to the left side, we must add 9 to the right side.

$x^2 + 6x + \_ = 11 + \_$	
$x^2 + 6x + 9 = 11 + 9$	Add Prop Equality
3	half linear term
$(x + 3)^2 = 20$	Factor
$(x + 3) = \pm\sqrt{20}$	$x^2 = \boldsymbol{n}$
$x + 3 = \pm 2\sqrt{5}$	Simplify Radical
$x = \pm 2\sqrt{5} - 3$	Subtract Prop Equality

We took an equation we could not solve and changed into an equation we could by completing the square. We can use the method of Completing the Square to solve quadratics we can't factor. All we are doing is making an equivalent equation by adding a number to both sides of the equation that will make the polynomial a perfect square. Then we solve the resulting equation.

Example 4 Solve for x, $x^2 + 10x - 3 = 0$

The first thing we might notice is we can't factor that trinomial. So, let's try completing the square. We quickly realize that 3 is not a result of taking half the linear tem and squaring. So, let's use what we know and transform the equation so we can use $x^2 = n$, then $x = \pm\sqrt{n}$ by completing the square.

$x^2 + 10x - 3 = 0$	Given
$x^2 + 10x = 3$	Add Prop Eq.
5	half linear term
$x^2 + 10x + \underline{25} = 3 + \underline{25}$	Add Prop Eq.
$(x + 5)^2 = 28$	Factor
$x + 5 = \pm\sqrt{28}$	$x^2 = n$
$x = \pm\sqrt{4 \cdot 7}$	
$x + 5 = \pm 2\sqrt{7}$	Simplify Radical

so $x = -5 + 2\sqrt{7}$ or $x = -5 - 2\sqrt{7}$

In all the polynomials that we completed the square, the coefficient of the quadratic term was one. That's important to recognize. The pattern of taking half the linear term and squaring <u>only</u> works when *a* = 1

Sec. 5 Quadratic Formula

The reason we learned to simplify radicals is because not all polynomials can be factored over the set of Rational Numbers. We saw in the last two examples we had irrational answers – radicals in the answer.

What that means is that you might be asked to solve a quadratic equation where the polynomial cannot be factored – as just happened.

Well, that poses a small problem, but not one we can't handle. If we can't factor it, we can solve the quadratic by "Completing the Square". If we generalize the process for completing the square, we get a formula.

So, another method for solving quadratic equations is the Quadratic Formula

That formula is derived directly from "Completing the Square", what we were just doing.

Derivation of Quadratic Formula

Completing the square works when the coefficient of the quadratic term is 1. So, let's look at a quadratic equation in general form.

$$ax^2 + bx + c = 0$$

We will use our knowledge of completing the square to solve for x by rewriting $ax^2 + bx + c = 0$.

$$ax^2 + bx = -c$$

Since the coefficient of the quadratic term **must be 1**, I will divide both sides of the equation by a.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now, to complete the square, I take half the linear term ($\frac{b}{a}$) and square it. Half of $\frac{b}{a}$ is $\frac{b}{2a}$, square it and add the result to both sides.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

The left side is a perfect square – as we planned.

$$(x + \frac{b}{2a})^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Simplify the fraction on the right by finding a CD, we have

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

Take the square root of both sides

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Simplify the radical

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Subtract $\frac{b}{2a}$ from both sides

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Writing the fractions as a single fraction, we have the

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The solutions set for $ax^2 + bx + c = 0$ is

$$\left\{ \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad , \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right\}$$

Rather than write it as one fraction, I am going to write it as two factions

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

That will eliminate some of the difficulties in reducing algebraic fractions that some of the students experience.

You need to memorize that formula, say it once, say it twice, say it 20 times if you have to - but know it!

The General Form of a Quadratic Equation looks like this: $ax^2 + bx + c = 0$, ***a*** is the coefficient of the quadratic term (squared term), $\underline{b}$ is the coefficient of the linear term (coefficient of x)) and $\underline{c}$ is the constant (the number without a variable).

To use the Quadratic Formula, everything must be on one side and zero on the other side, then use those coefficients to substitute into the Quadratic Formula. Watch your signs.

Example 1 Solve for x; $2x^2 + 3x - 5 = 0$

In this example $a = 2$, $b = 3$, and $c = -5$

Substituting those values in $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(3)}{2(2)} \pm \frac{\sqrt{3^2 - 4(2)(-5)}}{2(2)}$$

$$x = \frac{-3}{4} \pm \frac{\sqrt{9 - (-40)}}{4}$$

$$x = \frac{-3}{4} \pm \frac{\sqrt{49}}{4}$$

$$x = \frac{-3}{4} \pm \frac{7}{4}$$

Therefore, one answer is $\frac{-3}{4} + \frac{7}{4}$ or 1 and the other answer is

$\frac{-3}{4} - \frac{7}{4} or - \frac{10}{4} = \frac{-5}{2}$. $\{-5/2, 1\}$

Example 2 Solve using the Quadratic Formula, $x^2 + x = 12$

To use the Quadratic Formula, everything must be on one side, zero on the other side. Rewriting the equation, we have

$$x^2 + x - 12 = 0$$
$$a = 1,\ b = 1,\ c = -12$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-12)}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{49}}{2} = \frac{-1 \pm 7}{2}$$

so $x = 3$ or $x = -4$

The Quadratic Formula allows you to solve ***any*** quadratic equation. The bad news is if you use it to solve an equation, it takes 2 or 3 minutes to do. The good news is to solve a quadratic by the Quadratic Formula; it takes 2 to 3 minutes to do.

What that means to you is this, if you can see the factors immediately – Solve the problem by factoring – the Zero Product Property, its quicker. This last problem could have been done much easier and quicker by factoring.

However, if you don't see the factors and you don't think the polynomial can be factored, use the Quadratic Formula:

That way you won't spend forever trying to factor the problem.

Do the next three problems both ways.

1. $x^2 - 6x - 55 = 0$

2. $x^2 + 3x = 40$

3. $2x^2 - 12 = 5x$

SOLUTIONS:

1. $x = 11$ or -5 **2.** $x = -8$ or -5 **3.** $x = -\frac{3}{2}$ or 4

Use of the quadratic formula, when necessary, to solve each of the following equations. Check.
Assume R = {all real numbers}.

1. $x^2 - 8x + 15 = 0$
2. $x^2 + 7x - 8 = 0$
3. $x^2 + x - 42 = 0$
4. $x^2 - 11x + 30 = 0$
5. $2x^2 - x - 1 = 0$
6. $6x^2 - x - 15 = 0$
7. $4x^2 - 23x = 6$
8. $15x^2 - 16x = 15$
9. $8x^2 - 6x = -1$
10. $3x^2 - 20x = 7$
11. $x^2 - 4x + 1 = 0$
12. $x^2 + 10x + 21 = 0$
13. $4x^2 - 12x + 7 = 0$
14. $9x^2 + 6x - 4 = 0$
15. $x^2 + 10x + 19 = 0$
16. $3x^2 + 12x + 8 = 0$