Ch. 12 Higher Degree Equations – Rational Root Theorem

At this point, we cannot solve higher degree equations that cannot be factored. The Rational Root Theorem will help us with that. But before we get there, to make our lives easier, we will introduce Synthetic Substitution.

Sec 1. Synthetic Substitution ~ Division of Polynomials

This first section was covered in the chapter on polynomial operations. I'm reprinting it here because it applies to solving higher degree equations using the Rational Root Theorem.

Division of polynomials is done using the same procedure that was taught in 4th grade. That is, we use a trial divisor, divide, multiply, subtract, and bring down the next number in the dividend – repeat till finished.

Example 1 Divide $x^3 + x^2 - 3x - 2$ by $x + 2$

$$
\begin{array}{r}
x^2 - x - 1 \\
x+2\,\overline{\smash{)}\,x^3 + x^2 - 3x - 2} \\
\underline{x^3 + 2x^2} \qquad\qquad \\
-x^2 - 3x \qquad \\
\underline{-x^2 - 2x} \qquad \\
-x \quad -2 \\
\underline{-x \quad -2}
\end{array}
$$

It's important to notice how like powers are lined up vertically throughout the problem.

Synthetic Division

Since like terms are in the same column, that will allow us to write the polynomials without writing the variables. But to do that, we MUST remember to write in a placeholder of zero for the missing term.

Recopying Example 1 with only the coefficients results in the following:

$$
\begin{array}{r|l}
 & \quad 1\ -1-1 \\ \hline
1+2 & 1+1-3-2 \\
 & \underline{1\ +2} \\
 & \quad -1-3 \\
 & \quad \underline{-1-2} \\
 & \qquad\quad -1\quad -2 \\
 & \qquad\quad \underline{-1\quad -2}
\end{array}
$$

The way we do the division, the multiplication by 1 always subtracts out. That will allow me to leave out that step and copy all the products of (+2) on one line.

$$
\begin{array}{r|l}
 & \quad 1\ -1-1 \\ \hline
1+2 & 1+1-3-2 \\
 & \underline{1\ +2\ -2\ -2} \\
 & 1\ \ -1\ -1\ +0
\end{array}
$$

If we continue to examine this problem, we see that we really don't need to write the coefficient of the x in the divisor.* And since the three remainders are the same as the quotient, we can omit writing the quotient and use the remainders as coefficients of the quotient polynomial.

Since we started with a cubic equation, the quotient (depressed equation) will be one degree less – a quadratic with those coefficients. The last remainder is the final remainder.

Synthetic Division – Synthetic Substitution

To make long division less cumbersome, we are going to use the same process as above with a minor change. When we divide using the standard algorithms, we divide, multiply, subtract and bring down. With a little more thought, we might notice that if we change the sign of the constant in the divisor, rather than subtracting, we can add.

So, let's look at our original problem and write these with just coefficients and changing the sign of the constant in the divisor.

Procedure

1. Change the sign of the constant and write detached coefficients
2. Skip a space and draw a line
3. Bring down the 1st coefficient
4. Multiply the coefficient by the divisor and write the product under the next dividend coefficient and ADD
5. Repeat

Divide $x^3 + x^2 - 3x - 2$ by $x + 2$

Step 1

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -3 & -2 \end{array}$$

Steps 2 and 3

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -3 & -2 \\ & & & & \\ \hline & 1 & & & \end{array}$$

Step 4

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -3 & -2 \\ & & -2 & & \\ \hline & 1 & -1 & & \end{array}$$

Continuing

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -3 & -2 \\ & & -2 & +2 & +2 \\ \hline & 1 & -1 & -1 & 0 \end{array}$$

Write the quotient, depressed equation, with descending powers on the variable. Note that if ***n*** is the degree of the dividend, then ***n-1*** is the degree of the quotient.

- Since we are writing the dividends without the variables, it is VERY IMPORTANT that zero placeholders are inserted.
- In synthetic division, the coefficient MUST be 1.

Example 2 Use synthetic division to divide $x^3 - 2x - 21$ by $x - 3$

$$\begin{array}{r|rrrr} +3 & 1 & 0 & -2 & -21 \\ & & & & \\ \hline & 1 & & & \end{array}$$

$$\begin{array}{r|rrrr} +3 & 1 & 0 & -2 & -21 \\ & & 3 & 9 & 21 \\ \hline & 1 & 3 & 7 & 0 \end{array}$$

The quotient is $x^2 + 3x + 7$, no remainder.

The way we have learned to evaluate a polynomial was by substitution. Simply stated, we substituted a value for a variable and performed the indicated operations.

Example 1. If $P(x) = 3x + 2$, find $P(5)$

Letting $x = 5$, we have $P(5) = 3(5) + 2$ or $P(5) = 17$.

We say the value of P when $x = 5$ is 17. We can write that as an ordered pair (5, 17). If we graphed that, we would go over 5 and up 17 to find the point on the coordinate axes.

As always in math, we can never make the problems more difficult, only longer.

Example 2. If $f(x) = x^3 + 2x^2 + 3x + 4$, find $f(5)$

Letting $x = 5$, we have
$$\begin{aligned} f(5) &= 5^3 + 2(5)^2 + 3(5) + 4 \\ &= 125 + 2(25) + 3(5) + 4 \\ &= 194 \end{aligned}$$

The value of f when x is 5 is 194, written as an ordered pair (5, 194)

The process of direct substitution can become a real pain for higher degree polynomial expressions. There is another method called Synthetic Substitution that will make evaluating a polynomial a very simple process.

Given some polynomial $Q(x) = 3x^3 + 10x^2 - 5x - 4$ in one variable. You can evaluate Q when x = 2 by direct substitution of that value as we did before.

$$\begin{aligned} Q(x) &= 3x^3 + 10x^2 - 5x - 4 \\ Q(2) &= 3(2)^3 + 10(2)^2 - 5(2) - 4 \\ &= 3(8) + 10(4) - 10 - 4 \\ &= 24 + 40 - 10 - 4 \\ Q(2) &= 50 \end{aligned}$$

So the value of Q is 50 when x is 2. The ordered pair (2, 50)

Synthetic substitution is nothing more than a variation of the fourth-grade division algorithm.

You might remember when you divided a polynomial by another like $(x - 2)$ and there was no remainder, that meant $x - 2$ went into that evenly. If we had graphed it, we would have also noticed that the graph crossed the x-axis at (2,0).

If there was a remainder, that remainder would turn out to be the value of the polynomial at that particular value of x; an ordered pair.

To use Synthetic Substitution to evaluate a particular polynomial for a given value of x, we would write the coefficients of Q;

Example 3.

$$\begin{array}{cccccccc} Q = & 3x^3 & + & 10x^2 & - & 5x & - & 4 \\ & 3 & & 10 & & -5 & & -4 \end{array}$$

Now, we'll leave a space under those coefficients and draw a line. We will also write down the value of the variable to be substituted.

$$\begin{array}{cccc} 3 & 10 & -5 & -4 \\ \\ \hline \end{array}$$

Once we do that, we are set up to evaluate Q when x = 2.

To accomplish that, we bring down the first number, we will write the 2 in front of the leading coefficient with a half parentheses. Then bring down the leading coefficient 3, and multiply by the 2 and place that product (6) under the next number (10). Next, we add those numbers.

Now we start again, we multiply that result (16) by the 2, place that under the next coefficient, and add, we get 27.

Keep repeating this process. The last value will be the value of Q when x is 2.

$$\begin{array}{r|rrrr} 2) & 3 & 10 & -5 & -4 \\ & & 6 & 32 & 54 \\ \hline & 3 & 16 & 27 & 50 \end{array}$$

Notice, the last number we got was 50 as we did before when we directly substituted 2 into Q or found Q(2). – (2, 50)

Algorithm for Evaluating Polynomial using Synthetic Substitution

1. Write the coefficients of the polynomial (fill in placeholders)
2. Write the value you are substituting.
3. Skip a line and draw a line
4. Bring down the leading coefficient of the polynomial
5. Multiply the leading coefficient by the value being substituted
6. Add that result to the next coefficient in the polynomial

   Continue the process

Example 4: Evaluate $T(x) = 2x^4 - x^3 + 5x + 3$ when x = 3

In this example, notice there is no quadratic term, no x^2. When we write the coefficients, we'll need to write zero for the coefficient of that missing term as a placeholder.

So that expression would look like $2x^4 - x^3 + \mathbf{0x^2} + 5x + 3$

In this case, since I want to find the value when x = 3, I bring down the first number 2 and multiply that by 3, I write that result 6 under the next coefficient and add. I now multiply that by the 3, place that result under the next coefficient and add. I continue

3)	2	–1	0	5	3
		6	15	45	150
	2	5	15	50	153

Since the last result is 153, the value of that polynomial expression when $x = 3$ is 153. The ordered pair (3, 153)

Example 5. Let's evaluate $T(x) = 2x^4 - x^3 + 5x + 3$ when x is -3

This is the same polynomial from Example 4
Bring down the first number, multiply by –3, add and continue.

–3)	2	–1	0	5	3
		–6	21	–63	174
	2	–7	21	–58	177

So the value of T(x) when x = 3 is 177 or T(3) = 177

Remainder Theorem For every polynomial P(x) of positive degree *n* over the set of complex numbers, and for every complex number r, there exists a polynomial Q(x) of degree (*n* –)*1*, such that

$$\mathbf{P(x) = (x - r)\, Q(x) + P(r)}$$

What the Remainder Theorem says from the above example is:

$$2x^4 - x^3 + 5x + 3 = (x - (-3))[\, 2x^3 - 7x^2 + 21x - 58] + P(-3)$$

$$= (x + 3)[2x^3 - 7x^2 + 21x - 58] + 177$$

Notice we went from a degree 4 equation to a degree 3 and the numbers under the line are the coefficients of the depressed equation in red above.

Example 6: Let $y = 2x^4 + x^3 - 11x^2 - 4x + 12$, find the value of y when x = 2.

2)	2	1	–11	–4	12
		4	10	–2	–12
	2	5	–1	–6	0

The value of y when x = 2 is 0.

If we were to write this as an ordered pair, we'd have (2, 0). If we were to graph that, we'd also notice that (2, 0) is an x-intercept, where a graph crosses the x-axis.

In algebra, when we solve quadratic or higher degree equations, we set the equations equal to zero, then find values of the variable that will make the equation true.

Notice, in the last example x = 2 made y = 0. Oh wow! What that means is x = 2 is a solution to the equation, it's a zero, it is where the graph crosses the x-axis.

$$2x^4 + x^3 - 11x^2 - 4x + 12 = 0$$

In fact, anytime the last number in a synthetic substitution problem is zero, the value we substituted represents a solution, a zero, or what most of us might call an answer, if we had an equation.

If x = 2 is a zero, a solution, then x – 2 must be a factor of the polynomial expression.

Factor Theorem Over the set of complex numbers, (x – r) is a factor of a polynomial P(x) if and only if r is a root of P(x) = 0

Now don't you just love how math seems to come together.

Use synthetic substitution to find the value of the following polynomials. Fill in placeholders!

1. $P(x) = 2x^3 + 3x^2 - 4x + 5$, find P(2)
2. $Q(x) = 2x^4 - 3x + 6$, find Q(–1)
3. $T(x) = 5x^3 - 4x^2 + 3x - 10$, find T(1)
4. $H(x) = x^3 - x^2 + x - 1$, find H(5)
5. $J(x) = x^4 - x^2 + 1$, Find J(1)

Sec. 2 Rational Root Theorem

Factor Theorem **Over the set of complex numbers, x – r is a root of a polynomial P(x) if and only of r is a root of P(x) = 0**

Rational Root Theorem **If P(x) = 0 is a polynomial equation with integral coefficients of degree *n* in which a_0 is the coefficients of x^n, and a_n is the constant term, then for any rational root p/q, where p and q are relatively prime integers, p is a factor of a_n and q is a factor of a_0**

$$a_0x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1}x + a_n = 0$$

That's math talk. What that means is you have to start with an equation without fractions, and "if" there are rational answers (roots), the numerator of the answer must be a factor of the constant (a_n) and the denominator will be a factor of the leading coefficient (a_0).

If I looked at a simple quadratic equation solved by factoring, we can see how this works.

Let's look at $3x^2 - x - 2 = 0$ --≥ $(3x + 2)(x - 1) = 0$, solving that, we have $x = -2/3$ and $x = 1$. If we did enough of the problems, we might if there is a rational answer, its always seems to be made up of factors of the leading coefficient and the constant – AND the factors of the leading coefficient are in the denominator and the factors of the constant are in the numerator.

Example 1: Find the possible rational roots of $3x^2 - 5x - 2 = 0$

The factors of the leading coefficient 3 are $\pm 1, \pm 3$.
The factors of the constant are $\pm 1, \pm 2$

Now placing all the factors of the constant over all the factors of the leading coefficients, we have $\{\pm 1/1, \pm 1/3, \pm 2/1, \pm 2/3\}$as possible solutions to the equation.

Note, I picked prime numbers (convenient numbers) so there would not be a lot of factors to check later when I used synthetic substitution to find zeros. But the fact is, I cannot make these problems difficult – only longer

Example 2: Find the possible rational roots of $2x^2 + 5x - 12 = 0$

$a_0 = 2$ $a_n = -12$

Notice, no fractions, we have integral coefficients. The factors of the leading coefficient 2 are; $\pm 2, \pm 1$

The factors of the constant –12 are; $\pm 12, \pm 6, \pm 4, \pm 3, \pm 2, \pm 1$.

Now, we take each of the factors of the constant and put them over each of the factors of the leading coefficient (p/q). All of those fractions will be possible answers (roots) along with their negatives.

± **[12/1, 12/2, 6/1, 6/2, 4/1, 4/2, 3/1, 3/2, 2/1, 2/1, 1/1, ½]**

or simplifying the fractions ± **12, 6, 3, 4, 3, 2, 3/2 , 1, 1/2**

Oh yes – you are having fun. Math is your life.

The Rational Root Theorem says ***"if"*** there is a rational answer, it must be one of those numbers. Some of those possible answers repeat 6/2 is the same as 3/1

To find which, or if any of those fractions are answer, you have to substitute each one into the original equation to see if any of them make the open sentence true – hopefully using synthetic substitution.

It turns out 3/2 and – 4 are solutions. You know that because when you substitute either of those numbers into the polynomial, the result is 0. If you used synthetic substitution, the last number would have been 0. If the last number is NOT 0, then the number you are using is not a solution – is not a zero.

That's a lot of substituting in. However, if we were able to use Synthetic Substitution, our work would be much, much easier.

One side note, we could have solved this particular equation more efficiently by factoring or by using the Quadratic Formula.

The question is, why then am I solving it by the Rational Root Theorem? For practice. Typically, you use the Rational Root Theorem when you have higher degree equations - not quadratic equations or when you can't factor the polynomial.

The Rational Root Theorem is important because before knowing it, if you could not factor the polynomial, you could not solve the equation.

Example 3. Find the possible rational solutions of: $x^3 + 6x^2 + 3x - 10 = 0$

Can you use the quadratic equation?
Can you factor the polynomials?

The answer to both these questions is NO. The only other method we know at this time to solve this equation is by using the Rational Root Theorem.

The factors of the leading coefficient are; ± 1
The factors of the constant are; $\pm 10, 5, 2, 1$

Placing the factors of the constant over the factors of the leading coefficient, the possible solutions are:

$\pm$ **[10, 5, 2, 1]**

Again, to find out if any of these are solutions, I would have to substitute those back into the original equation or use synthetic substitution.

The Rational Root Theorem does not guarantee that there is a rational solution. So, there are times when none of the possible solutions will work. The equation will have a solution, it just won't be rational.

Example 4. Determine the possible rational roots of the following equation just by looking.

$$x^4 - 5x^3 + 9x^2 - 7x + 2 = 0$$

The equation looks longer, but it is not more difficult. The only possible rational solutions are $\pm$ **2, 1.**

The factors of the constant over the factors of the leading coefficient.

Example 5. Solve for x; $2x^3 + 3x^2 - 8x + 3 = 0$

Its not a quadratic, I can't factor it, so we'll use the Rational Root Theorem.

What are the factors of 2, the leading coefficient? $\pm 1, 2$
What are the factors of 3, the constant? $\pm\ 1, 3$

Therefore, the possible solutions are: $\pm$ **{1/1, ½, 3/1, 3/2}**

Is there a rational solution and if there is, what is it? Rather than substituting 8 possible solutions, let's try synthetic substitution.

I'm going to try whole numbers first only because they are easier to work with than fractions.

$$\begin{array}{r|rrrr} 1\} & 2 & 3 & -8 & 3 \\ & & 2 & 5 & 3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

Since the last number is 0, that means that x = 1 is a solution and that (x – 1) is a factor.

The numbers below the line are the coefficients of the depressed (factored) polynomial. The **2, 5, –3,** represent the coefficients

As importantly, I now have a reduced equation formed by those coefficients: $2x^2 + 5x - 3 = 0$ that I can solve using the Rational Root Theorem, the Quadratic Formula or by factoring.

If I solve the reduced equation by factoring, I have

$$(x - 1)(2x^2 + 5x - 3) = 0$$
$$(x - 1)(2x - 1)(x + 3) = 0$$

So $x = 1$, $x = \frac{1}{2}$ or $x = -3$. The solution set is $\{1, \frac{1}{2}, -3\}$

The Factor and Remainder Theorems are two very closely related theorems.

Graphically, when you look at Synthetic Substitution, you realize when the last number in Synthetic Substitution is zero, you have a root – a zero. In other words, you know where the graph crosses the x-axis. And you have factored the polynomial.

If the last number is not a zero, then you don't have a root. Using the Remainder Theorem, what you know is the value of the polynomial for that particular number which is a point on the graph – an ordered pair. That can help you locate roots – zeros.

If you substitute in one possible root and get a positive number at the end, the remainder, and the ordered pair is above the x-axis. Then, if you then substitute in another possible root and get a negative number, then the ordered pair is below the x-axis. That would suggest the graph has to cross the x-axis between those two points since the you are going from a positive to a negative number. Rather than just checking numbers sequentially from the possible roots, it might be wiser to choose possible rational solutions between those two numbers. That should cut down on your work.

Solve the following equations using the Rational Root Theorem. But first list the possible rational solutions.

1. $x^3 + 4x^2 + x - 6 = 0$

2. $x^3 + 2x^2 - 13x + 10 = 0$

3. $2x^3 + 15x^2 + 28x + 15 = 0$

4. $3x^3 - 4x^2 - 3x - 4 = 0$

5. $x^3 + 7x^2 + 7x - 15 = 0$

6. $3x^3 - x^2 - 8x - 4 = 0$