Ch. 14 Solving Quadratic Systems

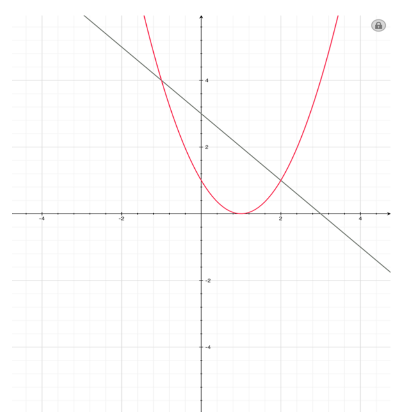
Sec 1. Graphing

We saw earlier when graphing systems of linear equations how graphing gave us a visual of what was occurring. We could see where lines intersected and approximate the solution. Graphing is also a great way to determine real number solutions of systems of equations in 2 variables in which one or both equations are quadratic.

The directions to solving quadratics might be, *find the solution set of a system of quadratics by graphing*. What should be clear is we can only find real number solutions by graphing. If we were directed to find solutions over the complex numbers, then, as we will see, the graphs don't necessarily intersect. So, while graphing will give us an idea of how many solutions a system will have and where those points of intersection are located, graphing is little help when solving over the complex numbers.

Example 1: Find the approximate solution sets by graphing

$y = x^2 - 2x + 1$
$x + y = 3$



Graph both equations on the same coordinate axes and determine points of intersect. The points of intersection are points (ordered pairs) that satisfy both equations.

The solution, the points, appear to be {(–1, 4), (2, 1)}

When solving systems of 2 equations containing quadratics, we could have no real solutions or as many as four. There would be no real solutions if the graphs did not intersect.

Graphing the equations will give you an idea, a visual, if the graphs will intersect and the number of times. It's not a great way to accurately solve systems.

Sec 2. Linear Quadratic Systems – Substitution

The most effective way to solve a system of equations that contain a linear and quadratic equations is by substitution. This substitution method is exactly the same substitution method learned for systems of linear equations.

Procedure for Solving Linear – Quadratic Systems by Substitution

1. Solve for one of the variables in the linear equation
2. Substitute that expression into the quadratic equation
3. Solve the resulting quadratic equation in one variable
4. Substitute those values back into the linear equation
5. Write the possible solutions as ordered pairs.
6. Check EACH ordered pairs in both equations

Be aware that systems of equations that involve quadratic equations many have complex roots as well as real roots. Using the graphing method only yields real roots.

Example 2. Solve the following system by substitution

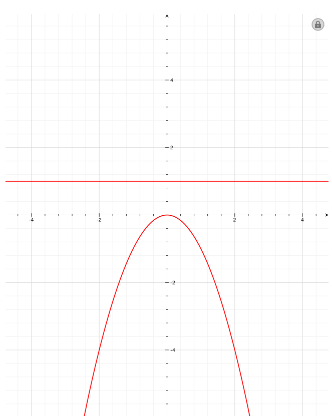
$$y = -x^2$$
$$y = 1$$

Substituting y = 1 into the first equation, we have $1 = -x^2$

$$-1 = x^2$$

so $x = i$ or $x = -i$

Substituting those values, we have the following possible solutions are **(i, 1) or –i, 1)**



You can see by graphing, there are ***no real solutions*** – the graphs do not intersect.

Example 3. Solve the following system by substitution

$$9x^2 + 16y^2 = 2$$
$$3x + 4y = 0$$

Using our procedure, I will solve for one of the variables in the linear equation and substitute that expression into the quadratic.

$$3x + 4y = 0$$
$$3x = -4y \rightarrow x = \frac{-4y}{3}$$

Substituting $\frac{-4y}{3}$ into the quadratic

$$9x^2 + 16y^2 = 2$$

$$9\left(\frac{-4y}{3}\right)^2 + 16y^2 = 2$$

$$9\left(\frac{16y^2}{9}\right) + 16y^2 = 2$$
$$16y^2 + 16y^2 = 2$$
$$32y^2 = 2$$
$$y^2 = 1/16$$

$$y = \pm \frac{1}{4}$$

$$x = -4y/3$$

$$x = -4\left(\frac{1}{4}\right)/3 = -1/3$$

and

$$x = -4(-1/4)/3 = 1/3$$

{(–1/3, 1/4), (1/3, –1/4)}

This problem did not get harder, but the computation was cumbersome and did take longer to find the solutions.

Before I go on, let me make a point. As always, you can follow an algorithm to help you find answers. That’s good. But, it is always better to think before starting.

If I looked more closely at the two equations above, I may have noticed a relationship in one or both of the variables that would allow me to simplify my work.

Example 3a. Solve the following system of equations.

This is the same problem in example 3

$9x^2 + 16y^2 = 2$
$3x + 4y = 0$

$9x^2 + 16y^2 = 2$
$3x + 4y = 0 \quad \rightarrow \quad 3x = -4y$

So, we still have $3x = -4y$ as before, but if I square both sides, I get $9x^2 = 16y^2$.

If I look at the first equation, do you see a nice substitution?

Now substitute $16y^2$ for $9x^2$, we get $16y^2 + 16y^2 = 2$, that could gave saved us a lot of work, don't you think? Yes, thinking before doing is good.

Solving that equation, we have

$$32y^2 = 2$$
$$y^2 = \frac{1}{16}$$
$$y = \pm\frac{1}{4}$$

Look familiar? That was a lot less work.

Example 4 Find the solution set of the system

$x^2 + 2y^2 = 12$
$2x - y = 2$

To get a visual, you might recognize these as equations as an ellipse and a line.

Do you see anything that might make your work easier than just substituting? No, then just solve for y in the second equation, and substitute that into the quadratic.

$2x - y = 2 \quad \rightarrow \quad 2x - 2 = y$

$x^2 + 2y^2 = 12$
$x^2 + 2(2x - 2)^2 = 12$ Substitution
$x^2 + 8x^2 - 16x + 8 = 12$ Expanding
$9x^2 - 16x - 4 = 0$ Simplifying
$(x - 2)(9x + 2) = 0$ Factoring
so $x = 2$ or $x = \frac{-2}{9}$

Substitute those values into: $2x - 2 = y$

When $x = 2$, $2x - 2 = y$, $y = 2$ $(2, 2)$
When $x = \frac{-2}{9}$, $2x - 2 = y$, $y = \frac{-22}{9}$ $(-\frac{-2}{9}, \frac{-22}{9})$

If we checked **each** ordered pair by substituting those in both equations, we'd find they worked.

So, the solution set is $\{(2, 2), (-\frac{-2}{9}, \frac{-22}{9})\}$

Sec 3. Quadratic – Quadratic Systems

You can solve quadratic-quadratic systems in two variables by graphing, substitution or by the elimination method. To choose which method is most appropriate and then determine which variable to solve for or eliminate means you should "think" in advance. That will cut your work down considerably. Remember, there could be no solutions or as many as four in quadratic-quadratic systems. We sure don't want to be needlessly bogged down in arithmetic.

Example 1. Find the solution set of the system

$$x^2 + 2y^2 = 17$$
$$2x^2 - 3y^2 = 6$$

We have an ellipse and a hyperbola.

To solve this by substitution, there is a quadratic term whose coefficient is one. That would be the easiest equation to solve for because it would eliminate fractional equations. So, that's what I will do, solve for x^2 in the first equation.

$$x^2 = 17 - 2y^2$$

Substituting that expression into the second equation, we have:

$$2(17 - 2y^2) - 3y^2 = 6$$
$$34 - 4y^2 - 3y^2 = 6$$
$$34 - 7y^2 = 6$$
$$-7y^2 = -28$$
$$y^2 = 4$$

So, we have $y = 2$ or $y = -2$

Substituting those values into the quadratic for BOTH those values of y

For $y = 2$	For $y = -2$
$x^2 = 17 - 2y^2$	$x^2 = 17 - 2y^2$
$x^2 = 17 - 2(2)^2$	$x^2 = 17 - 2(-2)^2$
$x^2 = 17 - 8$	$x^2 = 17 - 8$
$x^2 = 9$	$x^2 = 9$
so $x = 3$, $x = -3$	so $x = 3$, $x = -3$
$(3, 2), (-3, 2)$	$(3, -2), (-3, -2)$

$\rightarrow \{(3, 2), (-3, 2), (3, -2), (-3, -2)\}$

Remember to check these ordered pairs in BOTH equations!

Math is about decision-making, thinking. Whether it's about how to find a common denominator or which of the 4 methods you'd use to solve a quadratic equation, it's smart to look and think first, then proceed.

We just solved the last example by substitution. Linear combination can also be used to solve quadratic systems of equations. Let's look at the next example.

Example 2. Find the solution set of the system

$$3x^2 + 4y^2 = 16$$
$$x^2 - y^2 = 3$$

To use linear combination, we made one of the coefficients the same but opposite in sign on one of the variables, then added the equations together to eliminate a variable.

This problem is set up to solve using linear combination – but it is your choice. My choice is to solve it the easiest way – by linear combination – not substitution.

$$\begin{array}{llll} 3x^2 + 4y^2 = 16 & & 3x^2 + 4y^2 = 16 & \\ x^2 - y^2 = 3 & \rightarrow \text{x4} & \underline{4x^2 - 4y^2 = 12} & \text{Now add} \\ & & 7x^2 \qquad = 28 & \end{array}$$

$$x^2 = 4$$

$x^2 = 4$, so $x = 2$ or $x = -2$, now substitute those values for x into one the equations as we have been doing.

We get $\{(2, 1), (2, -1), (-2, 1), (-2, -1)\}$

Check your solutions!

Example 3 Find the solution set over the set of complex numbers.

$$4x^2 + y^2 = 25$$
$$y^2 = x^2 + 5$$

Which method should we use, substitution or linear combination?

I could substitute $x^2 + 5$ for y^2 in the first equation and solve the subsequent quadratic equation. ***Or***

I could rewrite the second equation as $x^2 - y^2 = -5$ and use linear combination. To be frank, in this particular case, the work appears to be about the same – so either one will do.

I'll choose linear combination.

$$4x^2 + y^2 = 25$$
$$y^2 = x^2 + 5$$

$4x^2 + y^2 = 25$	
$\underline{x^2 - y^2 = -5}$	**Rewriting the equation**
$5x^2 \quad = 20$	Add
$x^2 = 4$	Simplify
$x = \pm 2$	Solve

Substitute ± 2 into the equations, and checking our answers, we have $\{(2,3), (2,-3), (-2,3), (-2,-3)\}$

Without redoing the problem, we could have used substitution replacing y^2 in the top equation with $x^2 + 5$ resulting in the quadratic $4x^2 + (x^2 + 5) = 25$, then solving.

Example 4 Find the solution set over the complex numbers.

$$y = x^2 + 2x - 3$$
$$y = 2x^2 - x - 1$$

Since both equations are solved for y, I'll use substitution to set them equal and solve.

$2x^2 - x - 1 = x^2 + 2x - 3$	Substitution
$x^2 - 3x + 2 = 0$	
$(x - 2)(x - 1) = 0$	
$x = 2$ or $x = 1$	

When $x = 2$, $y = 5$ ….(2, 5)
When $x = 1$, $y = 0$ ….(1, 0)

The solution set is $\{(2, 5), (1, 0)\}$

In example 4, there were only two solutions, the other problems had more than two solutions, why? If we sketched graphs of each of these systems, we might see where they would intersect, giving us the number of solutions.

In example 4, we would have sketched two parabolas and saw they would have only intersected in two places, hence two solutions. As I have mentioned before, always think before you start a problem. Making good decisions makes math a lot easier. If you just start "doing" before thinking, well … math can be time consuming.

Let's look at one last example.

Example 5 Solve the following system over the Real Numbers

$$x^2 + y^2 = 25$$
$$x^2 + y^2 = 3^2$$

DONE! The answer is the null set. { } – no solution

Both of these equations are circles with center at the origin, the top equation has radius 5, the bottom has radius 3. These circles do not intersect.