Chapter 16 Exponentials, Exponential Equations & Graphs

Sec. 1 Simplifying Exponentials

From earlier study, we derived and learned the 5 rules for exponents that were derived from the definition of exponents.

Rule 1. $A^m A^n = A^{m+n}$

Rule 2. $A^m \div A^n = A^{m-n}$

Rule 3. $A^0 = 1, A \neq 0$

Rule 4. $(A^m)^n = A^{mn}$

Rule 5. $A^{-n} = 1/A^n, A \neq 0$

So, what we can quickly see from Rule 1, when we multiply numbers with the SAME base, we add the exponents. Rule 2 states when we divide numbers with the SAME base, we subtract the exponents. Rule 5 indicates that when you have a negative exponent, we get a fraction, the reciprocal with a positive exponent. It's important that you know those rules and can identify which rules go with problems.

Example 1 Simplify $x^3 x^6$

Since its a multiplication problem, that's Rule 1, when you multiply exponentials with the same base, add the exponents.

Therefore, $x^3 x^6 = x^{3+6} = x^9$

Example 2 Simplify $y^8 \div y^3$

Since we are dividing exponentials with the same base, we subtract the exponents.

Therefore, $y^8 \div y^3 = y^{8-3} = y^5$

Example 3 Simplify T^0

Any number to the zero power, except zero, equals 1.

Therefore, $T^0 = 1$

I can't make these exercises more difficult, I can make them longer. All you do is apply the rules and simplify one step at a time.

Example 4 Simplify $\dfrac{(x^5y^3z^5)(x^4yz^3)}{x^6y^4z^{10}}$

First, notice the "y" in the second factor does not have an exponent written explicitly, we need to remember, that means the exponent is 1.

Now, simplifying the numerator, multiplying exponentials, we add the exponents in the numerator.

$$\frac{(x^5y^3z^5)(x^4y^1z^3)}{x^6y^4z^{10}} = \frac{x^9y^4z^8}{x^6y^4z^{10}}$$

Now, we have a division problem, we subtract the exponents.

$x^3\,y^0\,z^{-2} = x^3\,(1)\,(\,1/z^2) = x^3/z^2$

Section 1 should be a review, but it is a review you need to be comfortable with.

Sec. 2 Solving Exponential Equations

Most of us are familiar and comfortable with problems such as $5^2 = 25$ and $2^3 = 8$.

In algebra, we are sometimes asked to solve equations like $x^2 = 25$. Without too much fanfare, most of us would answer by saying $x = 5$ or $x = -5$. If you didn't know that, you could have solved that by factoring, then using the Zero Product Property or you could have solved by the definition as we did:

$$if\ x^2 = b, \text{ then } x = \pm\sqrt{b}$$

You'll notice the variables in the equations we have been solving for are not in the exponent. What would happen if the variable was in the exponent, a problem like

$$\mathbf{3^x = 81.}$$

That's called an exponential equation.

One way to solve it is by trying to plug a number in for x by trial and error that would make the equation true. Using intelligent guessing, trial and error is OK, but it's time consuming.

If we worked with enough of these, we might see a way of solving this same equation by rewriting 81 as a power of 3.

By substituting, we have $81 = 3^4$, rewriting the equation, we have

$$3^x = 3^4$$

Now, since the bases are equal, then the exponents must be equal. In other words, $x = 4$.

Mathematically, we write that understanding this way:

Theorem **For $b > 0$, $b \neq 1$, $b^x = b^y$ if and only if $x = y$**

Don't you just love how we write things mathematically? What's this ***b*** has to be greater than zero and not equal to one business?

Let's try ***b*** being negative, not greater than zero, and see what happens. $(-2)^2 = 2^2$, the exponents are equal, are the bases then equal? No!

How about when the base equals one: $1^5 = 1^{12}$ in this case, the bases are equal, do the exponents have to be? Again no, that why we have the restrictions in the theorem.

Now you know why $\boldsymbol{b} > 0$ and $\boldsymbol{b} \neq 1$.

This is key; **to solve exponential equations, equations with variables in the exponent, the bases MUST be the same! Then I can set the exponents equal by the previous theorem.**

Example 1 Find the value of x; $2^5 = 2^{2x-1}$

Since the bases are equal, then the exponents must be equal.
Therefore,

$5 = 2x - 1$ Solving

$6 = 2x$

$3 = x$

Substituting 3 makes the original equation true

Example 2 Find the value of x; $5^x = 125$

Notice, the 125 is not written as an exponential. Since the base on the left side of the equation has base 5, can I rewrite 125 as an exponential with base 5. Turns out that $125 = 5^3$, substituting, we have

$$5^x = 5^3$$

Now, by theorem, we have $x = 3$

Example 3 **Solve;** $2^{6x^2} = 4^{5x+2}$

Notice the bases are not the same, we therefore cannot set the exponents equal. That's too bad, things were working out so well, But alas! I've always wanted to use that expression.

Is it possible to make the bases the same? Can I write the number 4 as a power of 2? You wouldn't have asked if there was not a way you say.

$$2^{6x^2} = 4^{5x+2} \quad \text{Given}$$
$$2^{6x^2} = (2^2)^{5x+2} \quad \text{Substitution}$$
$$2^{6x^2} = 2^{10x+4} \quad \text{Power Rule, Exp.}$$
$$6x^2 = 10x + 4 \quad b^x = b^y \text{ Theorem}$$
$$6x^2 - 10x - 4 = 0$$

Once I had the bases equal, I set the exponents equal. In tis case, that resulted in a quadratic equation. Solving that, we have

$$2(3x^2 - 5x - 2) = 0$$
$$2(3x + 1)(x - 2) = 0$$
$$x = -1/3 \text{ or } x = 2$$

Yes, say it, you love math!!!

You might be thinking that the problem was more difficult than the first one; $3^x = 81$. But in reality, it was not. It was longer because we solved a quadratic equation instead of a linear equation– not harder.

Algorithm for Solving Exponential Equations

1. **Express each side of the equation as a power in the SAME base.**
2. **Simplify the exponents**
3. **Set the exponents equal**
4. **Solve the resulting equation**

Example 3 Solve for x. $9^{3x} = 27^{x-2}$

Since the bases are not the same, I cannot set the exponents equal. So, can I make the bases equal?

$$9 = 3^2 \quad \text{and} \quad 27 = 3^3$$

Making those substitutions into the original equation, we have:

$(3^2)^{3x} = (3^3)^{x-2}$	- Substitution
$3^{6x} = 3^{3x-6}$	- Exp. (power rule)
$6x = 3x - 6$	- $b^x = b^y$ Theorem
$3x = -6$	
$x = -2$	

Example 4 Solve for n. $9^{n-1} = (1/3)^{4n-1}$

I have to write each side of the equation using the SAME base. I can write 9 as 9^1 or as 3^2.

Remember, $\mathbf{3^{-1} = 1/3}$

Therefore, I can write both sides having base 3

$9^{n-1} = (1/3)^{4n-1}$	Given
$(3^2)^{n-1} = (3^{-1})^{4n-1}$	Make the bases the same
$3^{2n-2} = 3^{-4n+1}$	Simplify exponents
$2n - 2 = -4n + 1$	$b^x = b^y$ Theorem
$6n = 3$	
$n = ½$	

A longer equation - no problem. We isolate the exponential, make sure the bases are the same, and we are back to what we were doing.

Example 5 Solve: $4(2^x) - 6 = 58$

Isolate the exponential,	$4(2^x) = 64$	Add Prop =
	$2^x = 16$	Div Prop =
	$2^x = 2^4$	Bases =
Therefore,	$x = 4$	Solve

In all the problems we have solved, we were able to make the bases the same, then set the exponents equal. What happens when we can't make the bases the same? In the next chapter, we will discuss that.

Sec. 3 Graphing Exponentials of the form $y = b^x$, $b > 1$

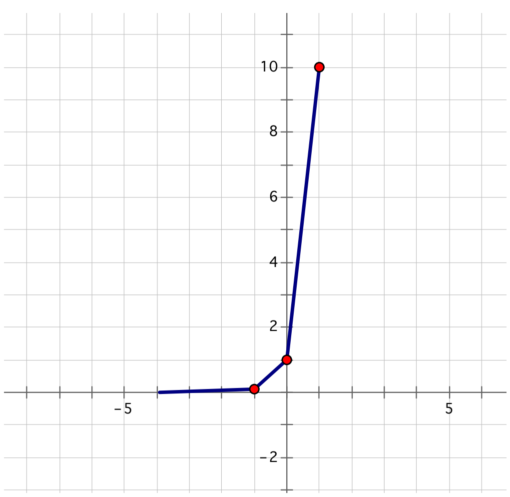
If I were to ask you to graph an exponential equation in two variables such as $y = 10^x$, my guess is you'd construct an x-y chart, plug in convenient values of x and find the corresponding values of y.

Example 1 Graph $y = 10^x$

x	-3	-2	-1	0	1	2	3
y	1/1000	1/100	1/10	1	10	100	1000

As you can see from the chart, the values of y get large very quickly. So quickly, it's almost impossible to actually plot the points. Who want to go over 3 and up 1000 to plot (3, 1000)?

So, to graph this function, I will plot a few key points to get an idea and extend the graph and

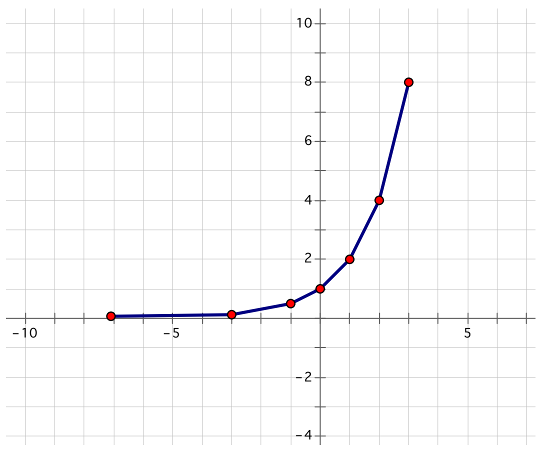


If I were to graph enough of these equations, we would begin to see an exponential equation of the form, $y = b^x$, all look pretty much the same when $b \geq 1$.

All the graphs would go through the point (0, 1), they would slide down to the left getting closer and closer to the x-axis but never touching it. The values of y are always positive no matter what values of x are chosen! If $x = 5$, then $y = 10^5$ or 100,000. If $x = -5$, the $y = 10^{-5}$ which is 1/10,000.

Example 2 Graph $y = 2^x$

Let x equal –3, –2, –1, 0, 1, 2, 3 and find the corresponding values of y. then plot those points



Since these are exponentials, just like in a geometric sequence, these numbers get very, very large – quickly.

Graph the following.

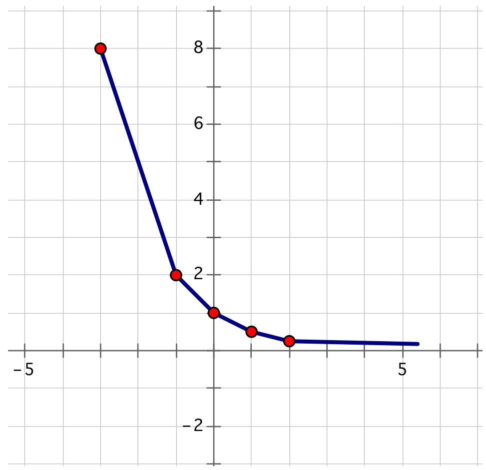
1. $y = 3^x$ 2. $y = 4^x$ 3. $y = 6^x$

Graphing those, we see they all pass through the point (0,1) and get closer and closer to the x-axis as shown in the two examples.

Sec. 4 Graphing Exponentials of the form $y = b^x$, $0 < b < 1$

Now, I mentioned that $b \geq 1$, the question becomes, what happens if $y = b^x$, $0 < b < 1$? In other words, what happens if b is a fraction between zero and one?

Well, we could graph $y = (1/2)^x$ and see what occurs. Since any number to the zero power, except 0, equals 1, the graph should go through the point (0, 1) just like it did before. How else is the graph different? Well, as x gets larger, the values of y get smaller. It appears the graph gets closer and closer to the x-axis, but never touches it.



The biggest difference in the graphs is that one graph slides to the left, the other graph slides to the right.

Graph the following

1. $y = (1/3)^x$ 2. $y = (1/5)^x$ 3. $y = (1/10)^x$

Sec. 5. Graphing Exponentials in the for $y = b^{x+h} + k$

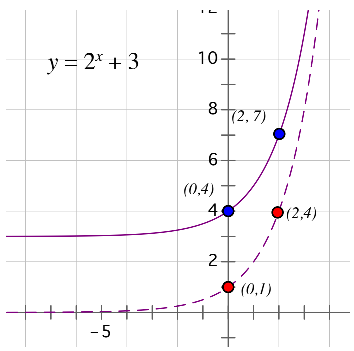
To graph equations in the $y = b^{x+h} + k$ format, I first graph the parent function, $y = b^x$, then move that graph using translations horizontally and vertically. Remember, the parent function always passes through the point (0, 1).

To make sure I have a good idea what the graph will look like, I typically plot a couple more convenient points. Since exponentials grow quickly, I try to use small numbers so they stay on the coordinate system drew.

Procedure

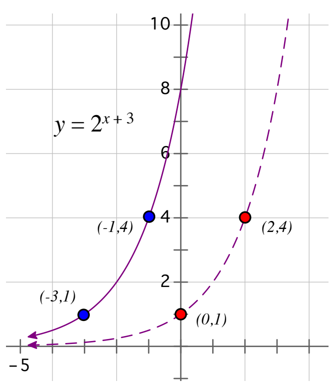
1. Graph the parent function with a dashed line thru (0,1)
2. From (0, 1) and another point, move the graph up/down k units
3. From there, move the graph horizontally h units
4. Connect the points.

Example 1 Graph $y = 2^x + 3$



Notice, From the parent function, the point (0,1) was moved UP 3, resulting in (0,4). (2,4) was also moved up 3 resulting in (2,7)

Example 2 Graph $y = 2^{x+3}$

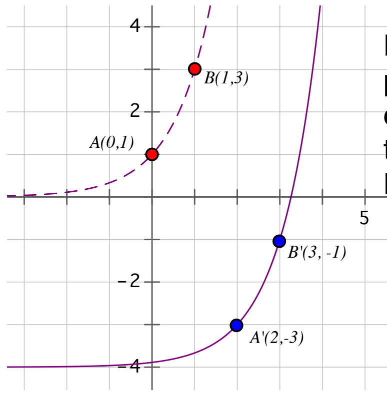


Notice, From the parent function, the point (0,1) was moved over 3 to the LEFT, resulting in (-3,1). (2,4) was also moved over 3 resulting in (-1,4)

So, we can see from these shifts caused by h and k on the parent function.

Now, let's put the h and k into one problem and make two shifts on each point; vertically and horizontally from the parent function.

Example 3 Graph $y = 3^{x-2} - 4$



Notice, with respect to the ordered pairs. 2 was added to the x-coordinates, and 4 was subtracted from the y-coordinates to get from A to A' and B to B'.

Based on the procedure on the preceding page, the points of the parent function should shift 2 to the RIGHT and 4 DOWN.

Sec. 6 Extending the Laws of Exponents to Rational Numbers

So far, we have learned the laws of exponents dealing with Integers, positive and negative Whole Numbers. However, when we graph exponential equations, we typically connect the ordered pairs and make a nice curve as we did in the last two examples. That suggests that x can take on other values besides the set of integers.

We have seen that that integers can be used as exponents, when the base is not equal to zero. That is

$5^1 = 5$ $(-3)^2 = 9$ $7^0 = 1$ $4^{-2} = 1/16$

When you first studied exponents, you learned an exponent tells you how many times to use the base as a factor. That is, $5^3 = 5x5x5$. From there, we saw that when we multiplied numbers with the same base, rather than writing that all out, we could add the exponents and shorten our work. We also noticed that when we divided numbers with the same base, we could subtract the exponents. But that introduced some complications. When we divided a number by itself, that resulted in having an exponent of zero – that did not make sense. So, we had a choice, drop the new rule or adapt it to fit the rest of the math we previously learned.

We noticed any time we divided a number by itself, by the Multiplicative Inverse, the answer was one and by using the rule of exponents, we had an exponent of zero. Using Substitution, it was easy to see that any number to the zero power, except zero, would always be equal to one as well. The reason we have except zero is because we can't divide by zero in the first place. So we made a third rule.

The rule for dividing exponents led to another rule that involved negative numbers. A negative exponent does not make sense by itself, we can not use a base a negative number of times. We did see some relationships develop when we simplified exponentials using the subtraction rule and when we did it out the long way. That lead us to the fourth rule, $a^{-n} = 1/a^n$.
We will continue with this line of reasoning for rational exponents –

Let's look at $5^{1/2}$. Using the definition of an exponent, this does not make sense either. But, if we played with it, we might notice we can make it fit the rest of the mathematics we learned just as we did for negative and zero exponents.

The following should be true using the laws of exponents we already know.

$$(5^{1/2})^2 = 5^{1/2 \cdot 2} = 5^1 = 5$$

This clearly suggests that if I square $5^{1/2}$, I get 5. Or the square root of 5 is $5^{1/2}$.

$$(4^{1/3})^3 = 4^{(1/3)\,3} = 4^1 = 4$$

This suggests that the third root of 4 can be written as $4^{1/3}$.

These observations will lead to an extension of the laws of exponents so the exponents can be rational expressions.

If $\boldsymbol{p}$ is an integer, $\boldsymbol{r}$ is a positive integer, and $\boldsymbol{b}$ is a positive real number, then

$$b^{\frac{p}{r}} = (\sqrt[r]{b})^p \qquad \text{and} \qquad (\boldsymbol{b}^{p})^{1/r} = (\boldsymbol{b}^{1/r})^{p}$$

Let's see what all that means:

Example 1 Write in exponential form $\sqrt{5y}$

The index, when not written, is understood to be two.
So , in the fraction, the denominator is 2.

$$(5y)^{1/2}$$

Example 2 Write in $\sqrt[3]{27x^2y5}$ exponential form

$$= 27^{1/3}\ x^{2/3}\ y^{5/3}$$
$$= 3\ x^{2/3}\ y^{5/3}$$

Example 3 Write $(7y)^{½}$ in radical form.

$$\sqrt{7y}$$

Example 4 Write $7^{1/3}\ x^{2/3}\ y^{1/3}\ z^{7/3}$ in radical form.

The index is 3. So all I have to do is label the index and place the appropriate exponents with each factor

$$\sqrt[3]{7^1x^2y^1z^7}$$

Now, we typically do not write an
exponent when it is 1. So the answer would look like

$$\sqrt[3]{7x^2 \cdot y \cdot z^7}$$

Sec 7 Exponential Growth and Decay

There are many variations for equations of growth and decay. While they are all the same basic formula, they are written differently. One equation that comes directly from graphing the parent function $y = b^x$ is $y = ab^x$.

In the equation $y = ab^x$, the "y" represents the final amount, the "a" represents the initial amount, the "b" is the rate of change, and the "x" is time.

Example 1 Write an equation to represent the following information. A population of a town is 20,000. The population is growing at a rate of 5% per year, find the population after t years.

The general equation for growth $\quad$ $y = ab^x$

a = 20,000 the original population

b = 1.05, 100% + 5% growth

x = t

Population in 10 years $\quad$ $y = 20{,}000(1.05)^t$

$y = 20{,}000(1.05)^{10}$

$y \approx 20{,}000(1.628)$

$y \approx 32{,}560$

The population after 10 years would approximate 32, 560 people.

Example 2 Bob places $10,000 in the bank and is paid 6% per year. How much money will be in the bank account after 5 years.

Again, the equation for growth/decay is $\quad$ $y = ab^x$

a - the original amount invested is $10, 000

b - the rate of growth is 1.06, 100% + 6%

x - the time is 5 years

$y = 10{,}000(1.06)^5$

$y \approx 10{,}000(1.338)$

$y \approx 13{,}380$

After 5 years, Bob would approximately $13,380 in his account.

Another way to write the exponential growth equations is to replace b, with (1 + r)

$$y = ab^x$$

$$y = a(1 + r)^x.$$

What that does is allows a quick way to find the growth - b. In the last example, we said the growth rate was 6%. We convert the percent to a decimal and add that to one. Notice, we still get 1.06. In other words, the new equation might look different, but it's really the same.

Example 3 Jack's base pay when he started his job was $30,000. If he was promised a cost of living increase of 2% per year for his first 10 years on the job, what would be his pay after 10 years.

$$y = a(1 + r)^x$$
$$y = 30{,}000(1 + .02)^x$$
$$y = 30{,}000(1.02)^{10}$$
$$y = 30{,}000(1.218)$$
$$y \approx 36{,}540$$

Jack's base pay would approximate $36,540.

Compound Interest

Compound interest is an application of exponential growth. Again, we have the same equation, written differently, and with different variables.

$$A = P(1 + r)^t$$

In compound interest problems, "A" represents the amount in the account, "P" represents the initial principal or investment, "r" the interest rate, and "t" time in years. So everything is the same except the variables.

Now the fact is most banks don't figure interest on a yearly basis. So, we need to tweak the equation $A = P(1 + r)^t$. So, if you were receiving 12% interest per year being compounded monthly, you would be earning 1% per month and the interest would be figured 12 times.

So the equation for compound interest is: $A = P(1 + \frac{r}{n})^{nt}$

So, looking at that "new" formula, r is replaced with $\frac{r}{n}$, $\frac{r}{n}$ is the interest rate received for each interest period. ***t*** was replaced with ***nt***, is the number of times the interest will be compounded.

Example 4 Juan's dad invested \$14,000 at 6% per year compounded monthly. How much money will be in his dad's account after 10 years.

Using the formula; $A = P(1 + \frac{r}{n})^{nt}$

P = 14,000, r = .06 and n = 12

t = 10

$$A = 14{,}000(1 + \frac{.06}{12})^{12(10)}$$

$$A = 14{,}000(1.005)^{120}$$

$$A \approx \$25{,}471$$

In this problem, his interest rate per month is .005 or 1/2%. His interest will be compounded 12 times per year for 10 years.

Now, how is exponential decay different from exponential growth? We noticed exponential growth, the rate of change was being added to 100%, so we had (1 + r) in the equation. With decay, we subtract the decay rate, (1 – r). Other than that, everything else is the same.

Example 5 Write an equation to represent the following information. A population of a town is 20,000. The population is decreasing at a rate of 5% per year, find the population after t years.

This is almost the same problem in Example 1. The only difference is this population is decreasing. Setting it up, we have

The general equation for growth $y = ab^x$

a = 20,000 the original population
b = .95, (100% – 5%) decay
x = t

$y = 20{,}000(.95)^t$

To find the population (y) after 10 years, $y = 20{,}000(.95)^{10}$
$y \approx 20{,}000(.598)$
$y \approx 11{,}960$

The population of the town after 10 years will approximate 11,960 people.