Other Methods

for

Adding, Subtracting & Multiplying

&

Shortcuts

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Whole Number Operations

Standard and Non-Standard Algorithms

There are multiple procedures for operating with whole numbers. There are what are called the "standard" algorithms; those are the algorithms that were typically taught to older generations. Under the common core standards, they are still supposed to be taught. In addition to those algorithms, there are many, many different ways to add, subtract, multiply and divide.

Some of the "new" algorithms are actually hundreds and even thousands of years old. The lattice methods and repeated doubling are just a few examples of those. The use of procedures in algebra, such as adding from left to right is another example of a "new" algorithm.

The importance of which algorithms are taught under different operations can and will impact student understanding, comfort levels and achievement. Those decisions are important, but so is knowledge commonality. When students go from one grade to the next or school to school, having commonality in backgrounds doesn't leave students in the dust. And, clearly, what is taught in one grade or course has an impact on future learning.

For instance, the common algorithm for division is used in algebra to divide polynomials, for synthetic division, and synthetic substitution in algebra when solving higher degree equations using the Rational Root Theorem. Secondary teachers have a clear expectation that students know how to divide using the standard algorithm.

To be quite frank, the algorithms I choose are often based on the numbers presented to me. For people that know math, they know that math is about decision-making. We continually ask students to stop and think before actually doing a problem. An axiom in math is: *"the more math you know, the easier math gets."*

Many students would approach the following problem by multiplying left to right using the Order of Operations, another student might see relationships that would allow this computation to be done mentally. Unfortunately, those students who can do it mentally are seen as smart. I would suggest they are not smarter than the other students, they just think and know more, then apply their knowledge.

Multiplying 35 x 18 using the standard algorithm is perfectly acceptable and will result in a correct answer. But, … multiplying by 1 is used quite frequently in math to rewrite equivalent fractions, simplifying radicals or conversions. We can also use it for computing.

35 x 18 Using my knowledge of multiplying by 1 and the Multiplicative Inverse, I will rewrite 35 x 18 as

35 x (2 x ½) x 18

(35 x 2) x (½ x 18)

Using the Commutative & Associative Properties, that results in

70 x 9 = 630

No pencil hit the paper.

When I'm factoring polynomials, I use the multiplicative inverse to help me find factors, that is I multiply and divide.

For example, to find the factors of 48; I can divide by 2 and multiply by 2 by the multiplicative inverse, or divide by 3 and multiply by 3, etc.

Example Factors of 48

48	x	1
24	x	2
12	x	4
6	x	8
3	x	16

Notice all I did was multiply and divide by 2 to find the other factors. There are 10 factors.

Since the prime factorization of 48 = $2^4 \times 3^1$, adding one to each exponent and multiplying (5x2) tells me I have a total of 10 factors.

Addition Algorithms

Linking

These fact strategies and arithmetic procedures taught in elementary school are the building blocks used throughout mathematics. The greatest difference in the math taught in elementary school and in later math is vocabulary, how we say it, and the notation. The procedures pretty much stay the same unless another pattern is recognized that makes our work easier. It's often described differently because it is being used in different contexts.

Let's look at these next two problems that clearly require students to know their arithmetic facts, procedures and understand place value and see how elementary math is connected to algebra.

Here we have two problems, look and you will see the actual numbers in the problems are the same. Problem **A** will be taught in elementary, Problem **B** will be taught in a pre-algebra or algebra class.

A) $341 + 256 = (3+2)100 + (4+5)10 + (1+6)1$
$= 5(100) + 9(10) + 7(1)$
$= 597$

B) $(3x^2 + 4x + 1) + (2x^2 + 5x + 6) = (3+2)x^2 + (4+5)x + 1 + 6$
$= 5x^2 + 9x + 7$

The 341 in arithmetic corresponds to the $3x^2 + 4x + 1$ in algebra, the 256 corresponds the $2x^2 + 5x + 6$. Notice the coefficients of the sums – the 597 in arithmetic corresponds to the $5x^2 + 9x + 7$ in algebra.

Problem A was written out in expanded notation when terms were combined. Problem B is written out in polynomial notation. Notice the language change. Clearly, while the numbers in each problem are the same, the notation has also changed.

When adding using the standard algorithm, we ask students to add the hundreds column to the hundreds, the tens to the tens, etc. In adding or subtracting polynomials, we ask students to use the same concepts but describe it by saying “combine like terms”. Notice the notation AND language changes to describe the sum, but the math is the same. Students need to know their arithmetic facts or the students will struggle in algebra.

In initial instruction, choosing simple straight-forward examples that work, that clarify and don’t bog students down in arithmetic allows students to more readily understand and succeed in math. So, I purposely chose numbers in my examples that had no regrouping.

By using this linkage, teachers can introduce “new” concepts and skills in more familiar language, which makes students feel more comfortable in their experiences, knowledge and understanding.

Linkage also provides teachers an opportunity to review and reinforce previously learned math or address student deficiencies as they follow their assigned curriculum – remediating along the way.

Different Procedures for Addition

There are many different ways to add numbers. The standard algorithm (procedure), Pictures/modeling, Lattice Method, Left to Right, Partial Sums and a whole host of other options.

These other procedures can be used for remediation or for enrichment, depending upon individual student needs. The procedure emphasized however, should be the procedure that will be used as a building block, a link, that will be used in later math. Knowing the standard procedure is important for smoother transitions for students.

The difficulties students often face is what we take for granted. For instance, would you ever write the number twelve like this:

1

2

Your immediate response to that is: No, of course not. But, let’s look at our standard algorithm.

Well, we actually do exactly that with the common procedure for addition, let's see how we do this problem:

Standard Algorithm (Procedure)

Adding from right to left.

$$\begin{array}{r} {}^{1}3\ 7 \\ +\ \ 5 \\ \hline 2 \end{array}$$

And there it is, we actually do write 12 that way.

5 plus 7 is 12, we put down the 2 and regrouped the 10 over the 3.

In the standard Algorithm, we typically teach students to add from Right to Left with regrouping.

My point is, for new learners, they are still memorizing and reinforcing their addition facts, and now are using those facts with regrouping while learning procedures for multi-digit numbers. That's called multi-tasking and not as simple as some might think. Patience and practice should be practiced.

Partial Sums

Let's see how we can add numbers using Partial Sums. The benefit for partial sums is it emphasizes understanding of place value and it eliminates regrouping as we did with the standard algorithm.

4 2 3	
1 6 4	
+ 3 5 8	
1 5	Adding the units' column, we have 1 ten + five - 15
1 3	Adding the tens' column, we have 13(10) or 1(100) and 3(10) - 13
8	Adding the hundreds' column, we have 8(100) – 8
9 4 5	Combining the partial sums

The importance of place value is illustrated again by the columns in which we placed the partial sums (numbers).

Adding Left to Right

While I typically use the standard algorithm to add larger numbers with students, if I'm in a rush and more concerned with a really good approximation, I add Left to Right being keenly aware of place value.

4 2 3 + 1 6 4 + 3 5 8 =	add the hundreds	(4 + 1 + 3) –	8 0 0
	add the tens	(2 + 6 + 5) –	130
	add the ones	(3 +4 + 8) –	1 5
			9 4 5

While that looks like Partial Sums from Left to Right, the way it is spoken would sound like 400 500, 800, by adding the hundreds, now go to the tens column, 820, 880, 930, by adding on the tens, and finally adding the ones column we have 933, 937 and 945.

Another method for adding is called the Lattice Method. In essence, the Lattice Method is often used to help students who might be having difficulty with regrouping.

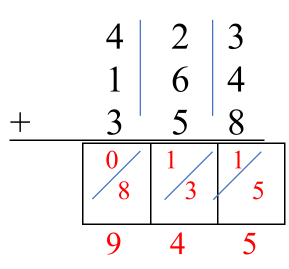
Lattice Method

It's very much like Partial Sums, but written with a Lattice to keep the place values together.

Let's look at the same example done using the Lattice Method – notice it's a form a Partial Sums with boxes. First, draw a box under each column.

Second, place diagonal lines in each box as shown below. Third, add the numbers in each column and write their sum in each box. And finally, add the numbers in each diagonal and place it below the box.

Example



Compare this to Partial Sums

The sum is 945

Let’s do another addition using the lattice method. Please note, the great thing about math is I can never make more difficult – only longer.

Example

8 4 6
6 5 2
7 3 9

2/1	1/2	1/7

2 3 7 - 2237

Different Procedure for Subtraction

Like addition, there are many methods to subtract numbers. The standard algorithm is to line up the digits using place value, subtract from right to left in the minuend with regrouping (borrowing) if needed.

Examples

A) 8 6 7 – 4 3 1 could be written as

$$\begin{array}{r} 8\ 6\ 7 \\ -\ \underline{4\ 3\ 1} \\ 4\ 3\ 6 \end{array}$$

B) 7 6 3 – 2 1 7 —≥

$$\begin{array}{r} 7\ 6\ 3 \\ \underline{2\ 1\ 7} \end{array} \quad \longrightarrow \quad \begin{array}{r} 7\ {}^{5}6\ {}^{1}3 \\ \underline{2\ \ 1\ \ 7} \\ 5\ \ 4\ \ 6 \end{array}$$

Subtract Left to Right – Using Decomposition

Another way to subtract with those examples is using place value and subtracting from left to right.

In example A, 867 – 431, I will subtract 400 from 800, that is 467, then subtract 30 more, that's 437, and finally subtract 1, the difference is 436 just like before – but done mentally.

In example B, 763 – 217, subtract 200 first. That leaves 563. Now subtract 10, that results in 553, and finally subtract 7; the difference is 546.

A simpler example, 90 – 38, to do that quickly in your head, subtract 30 from 90, that's 60, now subtract 8 more. The answer is 52.

Like anything else, with practice, these become very easy mental math problems.

Same Change Rule

Another method of subtracting is using the Same Change Rule. That is changing a problem into a simpler mental math problem.

Example 92 – 36 can be changed into subtracting an easier number. I do that by adding 4 to 36 make the subtrahend 40. To keep equality, I add 4 to the minuend, that becomes 96.
Now the problem becomes 96 – 40 = 56

Example 87 – 59, make the subtraction easier by changing the subtrahend to 60 by adding 1. Now add 1 to the minuend. The new problem is 88 – 60 = 28

So, we are making the same change to the subtrahend and minuend.

Example 93 – 47. Add 3 to both the minuend and subtrahend. The problem becomes 96 – 50 = 46

Subtracting from the Base

Another method of subtracting is Subtracting from the Base. Again, we will change the problem into an equivalent problem. Subtracting from the base translates to subtracting from 10.

Example 82 – 57,

$$\begin{array}{r} 8\ 2 \\ \underline{-5\ 7} \end{array}$$

written vertically

In this problem, using the standard procedure, regrouping is required.

But using the Subtract from the Base procedure, I will subtract 7 from 10 and add the digit in the minuend. That is 3 + 2 = 5

$$\begin{array}{r} {}^{7}8\ \ 2 \\ \underline{-5\ \ 7} \\ 2\ \ 5 \end{array}$$

Using subtracting from the base, we don't have to regroup and we only need to know subtraction facts to 10.

Example 94 – 18

$$\begin{array}{r} {}^{8}9\ 4 \\ \underline{-1\ 8} \\ 7\ \ 6 \end{array}$$

Subtract 8 from 10 and add 4; 2 + 4 = 6

And there are more methods.

When I’m subtracting relatively small numbers, like 180 – 63, I tend to use subtracting from Left to Right mentally. That is 180 – 60 is 120 – 3 is 117. Otherwise I use the standard algorithm. That blending, decision-making makes math a lot easier.

Different Procedures for Multiplication

Like addition and subtraction, there are number of different multiplication procedures. I tend toward emphasizing the "standard" algorithms because they relate so closely with algebra.

Look at the two problems below, both using the standard algorithm,

$$\begin{array}{rr} & 3\,2 \\ \times & 2\,1 \\ \hline & 3\,2 \\ 6\,4 & \\ \hline 6\,7\,2 & \end{array} \qquad\qquad \begin{array}{rr} & 3x + 2 \\ \times & 2x + 1 \\ \hline & 3x + 2 \\ 6x^2 + 4x & \\ \hline 6x^2 + 7x + 2 & \end{array}$$

The 32 corresponds to the 3x + 2, the 21 corresponds the 2x + 1. Both the two 2-digit numbers and the polynomials are multiplied using the same standard algorithm. Note, the partial products 32 and 3x + 2 correspond as do 64 and $6x^2 + 4x$. The final products have the same coefficients 672.

Recognizing patterns, I could rewrite the problem horizontally, then use the Distributive Property.

$$\begin{array}{rr} & 3x + 2 \\ \times & 2x + 1 \\ \hline & 3x + 2 \\ 6x^2 + 4x & \\ \hline 6x^2 + 7x + 2 & \end{array} \qquad\qquad \begin{array}{l} (2x + 1)(3x + 2) \\ \\ 6x^2 + 4x + 3x + 2 \\ \\ 6x^2 + 7x + 2 \end{array}$$

Notice the partial products are the same (color coded) as is the final product. 2x is distributed over the (3x + 2), then the +1 is distributed over the (3x + 2).

FOIL

If I multiply two two-digit numbers, I can use FOIL (First, Outer, Inner, Last). That is often used in algebra to multiply binomials quickly. Using the same example as above.

(2x + 1) (3x + 2)

1. Mult the First #s in each parenthesis $6x^2$
2. Mult the Outer #s in each parenthesis $4x$
3. Mult the Inner #s in each parenthesis $3x$
4. Mult th Last #s in each parenthesis 2
5. Combine like terms $6x^2 + 7x + 2$

Let's see how FOIL works in arithmetic.

```
    3  2
x   2  1
    6  2
```

Multiply the First #s 2x3 = 6, then do the Last #s 1x2 = 2
Place those under those factors as shown.

Now we will do the Inners and Outers and add them together by diagonally multiplying

```
    3  2
x   2  1
    6 7 2
```

3x1 = 3, 2x2 = 4, 3 + 4 = 7. Place that between the 6 and 2

```
    4   1            4   1
x   5   2        x   5   2
   20   2           20 13 2

                    21 3  2
```

Notice when you diagonally multiply and add, we get 13, so what we will do is write down the 3 and carry the one. So the answer is 2132.

With practice, you'd be able to multiply two digit numbers very quickly.

Repeated Doubling

To multiply using repeated doubling, you write down the number 1 and beside it write the multiplicand. Then begin the doubling process on each number until we can find a combination of numbers whose sum equals the multiplier.

13 x 25,

$$\begin{array}{r} 25 \\ \underline{\times\ 13} \end{array}$$

using 25, we start doubling 25

1	25
2	50
4	100
8	200

We look to find combinations of numbers whose sum is 13. That is 1, 4 an 8.

Now add the numbers whose sum corresponds to 13; 25 + 100 + 200 = 325

Example 26 x 63

1	63
2	126
4	252
8	504
16	1008

Now add the numbers whose sum corresponds to 26; 126 + 504 + 1008

Lattice Method

The lattice method uses diagonals very much like was used in addition.

The procedure is to write the multiplicand horizontally and the multiplier vertically forming boxes. Then within those boxes, draw diagonals. After which, we multiply the numbers as shown below to fill in each box. Finally, we add the numbers between the diagonals to get the final product.

Example 45 x 26

	4	*5*	
			2
			6

	4	*5*	
0 + 1	0 8	1 0	*2*
11 - 1	2 4	3 0	*6*
	7	**0**	

Special Products

Multiplying by Powers of 10

When multiplying by powers of 10, you move the decimal point to the right the same number of places as there are zeros. Some people refer to this as adding zeros. This is an easily recognizable pattern.

Example 72 x 10 = 720

Example 543 x 10,000; add 4 zeros; 543 0000 = 5,430, 000

Multiplying by 5

When multiplying by 5, you multiply by 10 ÷ 2

Example 16 x 5 = 16 x 10 ÷ 2
160 ÷ 2 = 80

Multiplying 2-digit number by 11.

When multiplying a 2-digit number by 11, you write down the first and last digits, then add them to find the middle digit.

Example
2 3
1 1
2 3; 2 + 3 = 5, so the answer is 253

```
    2  3
    1  1
    ----
    2  3
 2  3
 -----
 2  5  3
```

Example
5 4
1 1
5 4; 5 + 4 = 9, so the answer is 594

In both of these cases, there was no need to regroup when the numbers were added.

Example 6 8
1 1
6 8; 6 + 8 = 14, put down the 4 and add 1

7 4 8

We can show why that works using a little algebra. The number 11 can be written as 1(10) + 1 and any two digit number can be written as a(10) + b. Let's multiply those numbers using the standard algorithm.

$$\begin{array}{r} a(10) + b \\ \underline{1(10) + 1} \\ a(10) + b \\ \underline{a(100) + b(10)} \\ a(100) + (a+b)10 + b \end{array}$$

Notice the middle term as we saw from the pattern is a + b/

Do you think you can find a pattern for multiplying a 3-digit number by 11?

Multiply by 15

When you multiply by 15, we use the Distributive Property and change 15 to (10 + 5).

Example 18 x 15 = 18(15 + 10)
= 18(10 + 5)
= 180 + 90 or 270

Notice the product of 5n is half the product of 10n

Multiply using the Distributive Property

Example 23(99) = 23(100 – 1) = 2300 – 23 = 2477

Example 18 x 25 = 25 x 18 = 25(10 + 8) = 250 + 200 = 450

Multiplying by 25

When you multiply by 25, that's the same as dividing by 4 and multiplying by 100.

Example 32 x 25, divide 32 by 4, that's 8 and multiply by 100, answer is 800

Example 28 x 25, 28 ÷ 4 = 7, multiply by 100, answer is 700

Example 44 x 25, 44 ÷ 4 = 11, multiply by 100, answer is 1100

Those three examples used multiples of 4. Let's look at numbers that are not.

Example 33 x 25, 33 ÷ 4 = 8 ¼ or 8.25, multiply by 100, answer is 825

Example 30 x 25, 30 ÷ 4 = 7 2/4 or 7.50, multiply by 100, answer is 750

Example 47 x 25, 47 ÷ 4 = 11 ¾ or 11.75, multiply by 100, answer is 1175

Let's see how that works mathematically.

Link to decimals

	3	2
	2	5
1	6	0
6	4	
8	0	0

÷ 100→

	3	2
	.2	5
1	6	0
6	4	
8 .	0	0

Reasoning

Multiplying by 25 looks the same as multiplying by .25

Multiplying by .25 is the same as multiplying by ¼.

Multiplying by ¼ is the same as dividing by 4.

Since we originally divided by 100, we multiply by 100 - 800

Multiply When the Tens Digits are the Same and Units Digits Sum is 10.

Examples 25 x 25 = 625, 75 x 75 = 5625 45 x 45 = 2025

Do you see where the 25 is coming from in each of those products? What about the hundreds column written in blue?

Examples

53	42	36
57	48	34

1. Multiply the unit digits and write the product

53	42	36
57	48	34
21	16	24

2. Mentally add ONE to the tens digit in the multiplicand, then multiply the tens digits

^{6}53	^{5}42	^{4}36
57	48	34
3021	2016	1224

In this proof, we see the numbers in the tens place are the same, the units column sum is 10.

Proof

$$a(10) + b$$
$$\underline{a(10) + (10-b)}$$
$$10a(10-b) + b(10-b)$$
$$\underline{100a^2 + 10ab}$$
$$100a^2 + 10ab + 100a - 10ab + b(10-b)$$

Tens digits the same, units sum is 10

Subtract out 10ab's and factor out 100a

$$100a^2 + 100a + b(10-b)$$

Factoring

$$100a(a + 1) + b(10-b)$$

You can see to get the 100's column, I'm multiplying a(a+1).

Difference of 2 Squares

In algebra, a special product occurs when you multiply $(a + b)(a - b) = a^2 - b^2$

Let's look at that again using the standard algorithm.

$$\begin{array}{r} a + b \\ \underline{a - b} \\ -ab - b^2 \\ \underline{a^2 + ab } \\ a^2 - b^2 \end{array}$$

Notice +ab and –ab add out.

When I multiply numbers that have a nice midpoint (middle #), I can use this to mentally multiply the numbers using the Difference of 2 Squares.

Example $16 \times 24 = (20 - 4)(20 + 4) = 20^2 - 4^2 = 400 - 16$ or 384

Example $32 \times 28 = (30 + 2)(30 - 2) = 30^2 - 2^2 = 900 - 4$ or 896

Example $53 \times 47 = (50 + 3)(50 - 3) = 2500 - 9 = 2491$

Squaring Numbers Close to 50

Look at the following squares, can you see a pattern?

$50^2 = 2500$

$49^2 = 2401$

$48^2 = 2304$

$47^2 = 2209$

$46^2 = 2116$

That pattern doesn't jump out at you. Let me expand that.

$50^2 = 2500$

$49^2 = (50 - 1)^2 = 2401$ Where does the 01 come from?

$48^2 = (50 - 2)^2 = 2304$ Where does the 04 come from?

$47^2 = (50 - 3)^2 = 2209$ Where does the 09 come from?

$46^2 = (50 - 4)2 = 2116$ Where does the 16 come from?

$43^2 = (50 - 7)^2$ Will that end in 49? Now ask, where do the 24, 23, 22, and 21 come from in the above problems. If you said those numbers came from subtracting from 25, you'd be right!

So $43^2 = (50 - 7)^2$, subtracting 7 from 25, we get 18, squaring 7 we get 49, therefore 432 = 1849

Let's see where that comes from: **$(a \pm b)^2 = a^2 \pm 2ab + b^2$**

Using numbers around 50, we have $(50 \pm x)^2 = 50^2 \pm 2(50)x + x^2$

$= 2500 \pm 100x + x^2$

That gives us our pattern, when I rewrite the number around 50 as a binomial, I subtract that number from 2500 to get the 100's column and square that number to get the tens or ones.

Example $54^2 = (50 + 4)^2 = 2500 +400 + 4^2 = 2916$

Example $58^2 = (50 + 8)^2 = 2500 + 800 + 8^2 = 3364$

Using the same reasoning, we can easily square numbers close to 100.

Squaring Numbers Close to 100

$100^2 = 10{,}000$

$(100 \pm x)^2 = 10{,}000 \pm 2(100)x + x^2$

$= 10{,}000 \pm 200x + x^2$

The tens and ones column come from squaring the x just as before. But the hundreds column comes from doubling the x and subtracting from 10,000.

Example $97^2 = (100 - 3)^2 = 10{,}000 - 600 + 09 = 9409$

Example $106^2 = (100 + 6)2 = 10{,}000 + 1200 + 36 = 11{,}236$