Ch. 5 Sequences & Functions

Sec. 1 Skip Counting to Arithmetic Sequences

When you skipped counted as a child, you were introduced to arithmetic sequences.

Example 1 2, 4, 6, 8, … → adding 2
Example 2 10, 20, 30, 40, … → adding 10
Example 3 5, 10, 15, 20, … → adding 5

are all very recognizable patterns. In grade school, you skip counted by fives by using five as the first term. Another way of saying that is when you skip counted in early grades the first number you used was the number you added to find subsequent numbers.

These next examples represent a slight variation to the first three examples.

Example 4 **2, 12, 22, 32, …** → **adding 10**
Example 5 10, 13, 16, 19, … → **adding 3**

This skip counting is a slight variation because the number I am adding to find subsequent terms is not necessarily the first number – not necessarily a multiple.

All of these examples fit the definition of an Arithmetic Sequence.
Arithmetic sequence is a sequence in which every term after the first term is found by adding a constant – called the common difference (d).

If I asked you to find the 100th term of any of the arithmetic sequences using examples 1, 2 or 3, my guess is you'd know those answers by inspection, the answers are 800, 4000, and 2000 respectively. You would have gotten those answers by just multiplying.

Now, look at examples 4 and 5. If asked to find the 100th term of those arithmetic sequences, the answers are not so obvious.

So, let's look at example 4 and ask ourselves what we are doing to find the next term of a sequence. Hopefully, you would respond by saying I am adding 10. If you knew that, we are off to a good start.

Let's write down what we say we are doing and see if we can see a pattern develop. What is the next term in the following sequence? What is the 6th term.

Example 6 Find the 6th term of the sequence.

2, 12, 22, 32, 42, ____

+10 +10 +10 +10 +10

Example 7 Find the 10th term of the sequence

3, 8, 13, 18, 23, …

Writing that out we have:

3, 8, 13, 18, 23, 28, 33, 38, 43, ____

How'd we get from one term to the next in example 7? You added 5. Write that down.

3, 8, 13, 18, 23, 28, 33, 38, 43, ____

+5 +5 +5 + 5 +5 +5 +5 +5 +5

Example 8 Find the 4th term of the following arithmetic sequence

7, 15, 23, ____

This one is easy enough, you can do in your head. But let's write down how we got from one number to the next again.

7, 15, 23, ____

+ 8 + 8 + 8

Let's put this together:

In example 6, to find the 6th term, how many times did I add 10? – 5 times

In example 7, to find the 10th term, how many times did I add 5? – 9 times

In example 8, to find the 4th term, how many times did I add 8? – 3 times

In Example 6 to find the 6th term of the sequence, we found we added 10 five times to the first term – which is 2. So, we added 5 tens or 50 to the first term. The answer was 52.

In Example 7, to find the 10th term of the sequence we added 5 nine times to the first term – which is 3. So, we added 9 fives or 45 to the first term.

Generally, what we found was we added the common difference to the first term one less time than the term we were looking. So, looking for the 6^{th} term, the common difference, we added 10 five times to the first number in the sequence.

Mathematically, we have $a_6 = 2 + (6–1)10$

A_6 – sixth term of the sequence,
2 – is the first term of the sequence

The 6^{th} term of the sequence 52. Generalizing, what would be the 101^{st} term?

$$\mathbf{a_n = a_1 + (n – 1)d}$$

a_n represents the n^{th} term of the sequence
a_1 represents the 1^{st} term of the sequence
d represents the common difference (what we are adding)
n–1 represents we are multiplying by one less than the n^{th} term

so

$$a_{101} = 2 + (101 –1)10$$
$$a_{101} = 2 + (100)10 = 1002$$

So, what is nice about this is I have a quick way of finding out what a specific term is by using the formula we just developed and don't have to write it out the long way.

Example 9 Find the 21^{st} term of the sequence 3, 7, 11, 15, …

Since I am looking for the 21^{st} term, n = 21
The common difference is 4
The first term, a_1, is 3
Since $a_n = a_1 + (n–1)d$

$$a_{21} = 3 + (21–1)4$$
$$a_{21} = 3 + (20)\ 4$$
$$a_{21} = 83$$

Example 10 Find the 51st term of the sequence 12, 7, 2, –3, –8,…

In this case, $n = 51$, $a_1 = 12$, and $d = -5$

$$a_n = a_1 + (n-1)d$$

$$a_{51} = 12 + (51-1)(-5)$$
$$a_{51} = 12 + (50)(-5)$$
$$a_{51} = -238$$

In example 10, you can see the numbers in the sequence were getting smaller so we were adding a (–5).

Sec 2. Arithmetic Sequences as Functions

If we think about arithmetic sequences a little bit in terms of our previous study, of functions, we might realize when we add the same number over again to find the next value, we are adding a constant. That suggests the common difference in an arithmetic sequence could be viewed as a slope, a rate of change from one value to the next in a linear function.

I can write the formula for the n^{th} term using function notation by substituting values for a_1, d, and rewriting a_n as f(n), then simplifying.

Example 1 Given $a_1 = 4$ and $d = 5$, write a rule to find the terms of the sequence.

$\mathbf{a_n = a_1 + (n-1)d}$	**Given**
$\mathbf{a_n = 4 + (n-1)\,5}$	**Substitution**
$\mathbf{a_n = 4 + 5n - 5}$	**Distributive Prop**
$\mathbf{a_n = -1 + 5n}$	**Combine like terms**
$\mathbf{a_n = 5n - 1}$	**Comm. Property**

Now, rather than saying a_n, I will write that in functional notation f(n).

$$\mathbf{f(n) = 5n - 1}$$

or $\mathbf{f(x) = 5x - 1}$

Since I am adding a constant, we should clearly recognize the pattern as a linear function. Using the function rule instead of the formula makes working with arithmetic sequences even easier. Now, if I want to find the 20^{th} term of the sequence defined by the function, I merely substitute that into the function rule.

f(n) = 5n – 1 describes the sequence
f(20) = 5(20) – 1
f(20) = 99.

It gets better, since I know arithmetic progressions are linear functions, if I know two terms of a sequence, I can find a formula for all the terms. The terms between two given terms of an arithmetic sequence are called the arithmetic means between the two terms.

Example 2 **Given the sequence with the missing terms (arithmetic means) 4, __, __, 22, …, find the missing terms and describe the sequence as a function.**

Let me rewrite the sequence using a chart.

1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	5^{th} term	6^{th} term
4			**22**		

Treat this like an x-y chart and find the slope. Slope is the $\frac{\Delta y}{\Delta x} = \frac{22-4}{4-1} = \frac{18}{3} = 6$

If we know the slope is 6, we just add 6 to each preceding term.

1^{st} term	2^{nd} term	3^{rd} term	4^{th} term	5^{th} term	6^{th} term
4	**10**	**16**	**22**	**28**	**34**

Substituting m = 6 into the Point Slope Form of a Line (finding an equation of a line), we can write a rule, an equation, a function. Using (1, 4) and (4, 22) as ordered pairs.

$y - y_1 = m(x - x_1)$	**Given Pt Slope Form of a Line**
$y - 4 = 6(x - 1)$	**Substite (1, 4)**
$y - 4 = 6x - 6$	**Distributive Property**
$y = 6x - 2$	**Addition Prop of Equality**
$f(x) = 6x - 2$	**Substitution**

Please take note that the common difference in an arithmetic sequence is the rate of change, the slope. **f(x) = 6x – 2** is in the Slope Intercept Form of a Line and the slope is 6 - the same as the common difference.

By using this formula, to find the 101^{st} term, we merely substitute 101 in to the function.

$$\mathbf{f(x) = 6x - 2}$$
$$\mathbf{f(101) = 6(101) - 2}$$
$$\mathbf{f(101) = 604}$$

Just as we got before**.**

Example 2 **Insert four terms in an arithmetic sequence between 2 and 37 and write the rule using functional notation.**

The first number in the sequence is 2, that corresponds to the ordered pair (1, 2). Since 2 is the first term and I'm looking for 4 terms, that means 37 is the 6^{th} term of the sequence. So that is represented by (6, 37).

To determine what is being added to each term, I find the slope.

$$m = \frac{37-2}{6-1} = \frac{35}{5} = 7$$

Since the slope is 7, my sequence becomes 2, 9, 16, 23, 30, 37

Writing the function, I use the Point-Slope Form of a Line

$y - y_1 = m(x - x_1)$ and pick the easiest ordered pair to substitute, which in my opinion is (1, 2)

y – 2 = 7(x– 1), when simplified is y = 7x – 5. Writing that using functional notation, we have f(x) = 7x – 5

Sec 3. Recursive Functions

Now we see a linear function can describe an arithmetic sequence by substituting values of x beginning with one and finding the corresponding values of y.

Example 1 Find the arithmetic sequence defined by **f(x) = 2x + 3**

x	1	2	**3**	4	5
y	5	7	**9**	11	13

The resulting sequence is 5, 7, 9, 11, 13, …

Notice the slope is 2 – the common difference of each term of the sequence is 2.

To find the third term of the sequence, we can see from the chart it is 9. Or, we could find the value of f at 3, written mathematically as f(**3**) = **9 from the rule f(x) = 2x + 3**.

Look at the chart, look at the sequence, and look how f is defined. Now look at the next three statements.

Note the 3rd term is found by adding 2 to the 2nd term. In other words

$$f(3) = f(2) + 2$$

Note the 4th term is found by adding 2 to the 3rd term. In other words

$$f(4) = f(3) + 2$$

Note the 5th term is found by adding 2 to the 4th term. In other words

$$f(5) = f(4) + 2$$

I'm adding 2 because that is the common difference, the rate of change, the slope. That's how I am getting from one term to the next. Also note, I am adding 2 to the preceding function. That is, f(5) is being described in terms of f(4). So f(10) would be f(9) + 2; f(10) = f(9) + 2

Generalizing, to find the function recursively, I'm merely adding the common difference (which is the slope) to the preceding term in the sequence which is described in functional form.

Now, to ensure we understand the notation, let's look at **f(5) = f(4) + 2**.
Another way to write that is **f(5) = f(5 – 1) + 2**

Example 2 **Write f(7) in terms of the preceding term**

$$\begin{aligned} f(7) &= f(7-1) + 2 \\ &= f(6) + 2 \end{aligned}$$

So, to write the function **recursively**, we write the n^{th} term in term in terms of the preceding term; $(n - 1)$.

Mathematically, we say that $f(n) = f(n-1) + d$, where d is the common difference or slope.

Or we could manipulate **$f(n) = f(n-1) + d$, and solve for $f(n-1)$**

$$f(n) - d = f(n-1)$$

Example 2 If $g(x) = 5x + 3$, write $g(x)$ in terms of $g(x - 1)$

What's the slope? It is 5, so **$g(x) = g(x - 1) + 5$**

Example 3 If $h(x) = -2x + 10$, write $h(x)$ in terms of $h(x - 1)$

What's the slope? It is –2, so **$h(x) = h(x - 1) - 2$**

Example 4 If $p(x) = 3x + 7$, write $p(x)$ in terms of $p(x - 1)$

The slope is 3, therefore $p(x) = p(x - 1) + 3$

So, $p(x)$ can be defined in terms of the preceding term $p(x - 1)$ plus the slope.

1. Find the 11^{th} term in the arithmetic sequence if $a_1 = 1$ and $d = 2$.
2. Find the 21^{st} term in the arithmetic sequence if $a_1 = 2$ and $d = 5$
3. Find the 51^{st} term in the arithmetic sequence if $a_1 = 200$ and $d = -4$

4. Find the 101^{st} term of the arithmetic sequence if a1 = 80 and d = –1/2
5. Find the 31^{st} term of 4, 9, 14, 19, …
6. Find the 41^{st} term of 8, 11, 14, …
7. Find the 61^{st} term of 12, 7, 2, –3, …
8. Insert three terms in the arithmetic sequence between 16 and –12
9. Insert 6 terms in the arithmetic sequence between – 6 and 6
10. Find the missing terms in the arithmetic sequence; 7, _, 17, _ ,
11. Find the missing terms in the arithmetic sequence _, 49, _ , _, 28
12. Which term in the arithmetic sequence –2, 5, 12, … is 138?
13. A teacher earns $40,000 per year. If the teacher receives an annual increase of $2500, what will his salary be in his 15^{th} year?
14. A lay-a-way plan requires customers to deposit $15 the first week and to increase the deposit by $4.00 weekly for the nest 9 weeks. How much money will be deposited in the 10^{th} week?

Sec 4. Arithmetic Series

A series is an indicated sum of the terms of a sequence represented by S_n.

Example 1 Find the sum of the first 10 terms of the arithmetic sequence 1, 2, 3, 4, …, 8, 9

In other words, add the numbers one through 9.

To add those numbers, I could add them sequentially from left to right using the Order of Operations. That would give me the correct sum. Another way to add them is to add the first term to the last term, the second term to the second from last, the third term to the third from last, etc.

1 + 9, 2 + 8, 3 + 7, 4 + 6, 5 + 0; I'm adding 10 four times then adding 5 = 45

To come up with a nice formula for finding the sum of an arithmetic sequence, I'm going to write out the sums in algebraic notation from the first term to the last term. Then I'm going to write out the sum again in reverse. After writing out those sums, I will add those two equations together, the result will be interesting.

S_n – sum of the sequence, a_1 is the first term, d is the common difference, and a_n is the n^{th} term – just as we have identified before.

Now remember, to get from one term to the next, we add the common difference to the preceding term. But, essentially what we are doing is adding the sum of the common differences to the first term.

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_n - 2d) + (a_n - d) + a_n$$

a_n represents the last term, $(a_n - d)$, represents the second to last term, $(a_n - 2d)$ represents the third to last term of the sequence. Writing that formula in reverse, we have:

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_1 + 2d) + (a_1 + d) + a_1$$

Now, let's line these two equations vertically and add them together. Notice all the d's add out.

$$\begin{aligned} S_n &= a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_n - 2d) + (a_n - d) + a_n \\ S_n &= a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_1 + 2d) + (a_1 + d) + a_1 \\ \hline 2S_n &= (a_1+a_n)+(a_1+a_n)+(a_1+a_n) + \ldots + (a_1+a_n) + (a_1+a_n) + (a_1+a_n) \end{aligned}$$

I'm adding $(a_1 + a_n)$ n times or $n(a_1 + a_n)$

So $2S_n = n(a_1 + a_n)$

$$S_n = \frac{n(a_1 + a_n)}{2}$$

Example 2 Find the sum of the numbers between 1 and 100.

$$S_n = \frac{n(a_1 + a_n)}{2};\ a_1 = 1,\ a_{100} = 100,\ n = 100$$

$$S_{100} = \frac{100(1+100)}{2} = 50(101) = 5050$$

Example 3 Find the sum of the first 21 terms of an arithmetic whose first term is 8 and common difference is 4.

To use the formula we just developed, we need to know the first and last term - $S_n = \frac{n(a_1 + a_n)}{2}$ or $S_n = \frac{n}{2}(a_1 + a_n)$

The last term, the 21[th] term was not given to us. So we will use the n[th] term formula to find a_n.

$$a_n = a_1 + (n - 1)d; \qquad a_{21} = 8 + (21 - 1)4$$
$$a_{21} = 88$$

Now to find the sum of the first 21 terms, we substitute our values into that equation.

$$S_n = \frac{n(a_1 + a_n)}{2}; \qquad S_{21} = \frac{21(8 + 88)}{2}$$

$$S_{21} = \frac{21(96)}{2} = 1008$$

1. Find the sum of the arithmetic progression is $a_1 = 5$, $d = 3$ and $n = 12$
2. Find the sum of the arithmetic progression if $a_1 = -1$, $d = 4$, and $n = 7$
3. Find the sum of the arithmetic progression if $d = -2$, $n = 12$ and $a_n = 5$
4. Find the first 3 terms of an arithmetic sequence if $a_1 = 8$, $a_n = 408$, and $S_n = 2288$
5. The third term of an arithmetic series is 9 and the 7[th] term is 21, find the sum of the first 10 terms. (Hint: Use the a_n formula to find d and a_1)
6. How much will an engineer earn in ten years if his starting salary was $80,000 and he received annual increases of $5000.00?
7. In an auditorium, there are 25 seats in the first two and two seats more in each successive row, how many seats are there in ten rows?
8. If the cab fare is $2.50 for the first mile and $0.80 for each additional mile, what will be the cost of the cab traveling 21 miles?
9. A test consisting of ten questions is administered to the class and told that each question after the first is worth 2 more credits than the preceding question. If the third question is worth 5 credits, what is the maximum score on the test?

Sec. 5 Geometric Sequences

A geometric sequence is a sequence in which every term after the first is obtained by multiplying the preceding term by a constant, called the common ratio (r). This definition parallels the definition of an arithmetic sequence, the difference between an arithmetic sequence and geometric sequence is adding vs. multiplying.

Example 1 Is 3, 6, 12, 24, … is a geometric sequence?

Yes, when dividing I get a ratio of 2 for each successive term.

Example 2 Find the next term in the geometric sequence and find the common ratio (r)

4, 20, 100, 500, ___

To find r, divide the next term by the preceding term, 20/4 = 5

The next term in the sequence is 2500

As we did with arithmetic sequences, let's list the first few terms of a few and see if we see a pattern to develop.

a_1 – is the first term
a_1r – is the second term
a_1r^2 – is the third term
a_1r^3 – is the fourth term

Do you see a pattern to help you determine the exponent?

Example 3 What would be the 11th term of that geometric sequence?

It appears that exponent is one less than the term, n. So the 11th term would be a_1r^{10}.

Generalizing what we see from Example 3, we have the nth term of a geometric sequence be represented as $\mathbf{a_n = a_1 r^{n-1}}$

Example 4 Find the 7th term of the sequence using the formula
3, 15, 75, 375, …

$a_1 = 3$, $r = 15/3 = 5$

Substituting, $a_7 = 3\,(5)^6 = 15{,}625$

Example 5 Find the 17th term in the sequence
1, ½, ¼, …

$a_1 = 1$, $r = ½$, $n = 17$

Substituting, $a_{17} = 1\,(1/2)^{16} = 1/65{,}536$

Geometric sequence problems are set up the same way arithmetic sequence problems are set up. The difference is the formulas. Arithmetic sequence formulas were developed by adding a constant called the common difference (d); geometric sequence formulas were developed by multiplying by a constant called the common ratio (r).

Example 6 Insert two geometric means between 3 and 24.

$a_1 = 3$, 24 would be a_4, why?

Using our formula, $a_n = a_1 r^{n-1}$, we have

$$24 = 3r^3$$
$$8 = r^3$$
$$2 = r$$

So the terms are 3, 6, 12, 24

1. Insert the 2 geometric means between 1 and 729.
2. Find the geometric mean between 6 and 24.
3. Find *r* in the geometric sequence if $a_1 = 5$ and $a_4 = 135$

Sec. 6 Geometric Series

Geometric series is the sum of the terms of a geometric sequence.

You might recall, to find the sum of an arithmetic series, 1) I wrote each term algebraically and added them together, 2) Rewrote them in reverse order, and 3) added the two sums together.

To find the sum of a geometric series, I will do something similar. I will write the sum of the terms algebraically as I did with the arithmetic series. Next, I will multiply S_n, by (–r), then add them together.

$$\begin{aligned} S_n &= a_1 + a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-2} + a_1r^{n-1} \\ -rS_n &= \quad - a_1r - a_1r^2 - a_1r^3 - \quad \ldots \quad - a_1r^{n-1} - a_1r^n \\ \hline S_n - rS_n &= a_1 \qquad\qquad\qquad\qquad\qquad\qquad - a_1r^n \end{aligned}$$

All the terms in the middle add out. Then factor S_n using the Distributive Property on the left side, we have

$$S_n(1 - r) = a_1 - a_1r^n$$

$$S_n = \frac{a_1 - a_1r^n}{1-r} = \frac{a_1(1-r^n)}{1-r}$$

Note: We already know that $a_n = a_1r^{n-1}$. If I multiply both sides of that relationship by r, that results in $ra_n = a_1r^n$. *Substituting ra_n for a_1r^n in the above equation results in the following formula.*

$$S_n = \frac{a_1 - ra_n}{1-r}$$

