Strategies for

Learning the Basic Arithmetic Facts

Easier

By Bill & Tammy Hanlon

Memorizing the arithmetic facts is important for students' future success in math and science. Let's be very clear, students who are not required to memorize the basic arithmetic facts are being set up for failure, frustration and will have a much more difficult time as they continue their study in mathematics.

Having said that, memorizing facts without strategies and understanding makes memorization of those facts more difficult and often times more frustrating for not only students, but parents and teachers as well.

To address this, we'll ask students to identify different patterns, connect that pattern to a strategy, then use that specific strategy to find the correct answer. That takes a couple of seconds when students are first learning the facts, but with practice, automaticity develops.

Example A: If I asked most students to find "12 – 12", they would almost immediately know the answer is zero. They learned to identify the pattern, when they subtract the same number, the answer is zero.

Example B: If I asked the same students to find "12 – 3", many would go to their fingers to determine an answer. They would not identify a pattern as they did above, they would not connect it to a rule, and arithmetic begins to be cumbersome and time consuming.

As you will find out later in this booklet under subtraction strategies, when you subtract numbers with "consecutive" digits in the units' column, the answer is either 9 or ends in 9. Recognizing that pattern would make the following subtractions a lot easier. 12 – 3; 13 – 4; 14 – 5; 15 – 6, 16 – 7; 17 – 8 and 18 – 9. All the answers are 9

In this booklet, we will cover basic fact strategies that won't be taught sequentially as many parents and grandparents were taught, the facts will be grouped by strategies so students can see patterns that help them learn the facts. After they have learned the strategies so they can find the answers, then students will need a lot of practice so they know the facts with automaticity.

The more sophisticated mental operations in mathematics of analysis, synthesis, and evaluation are impossible without rapid and accurate recall of specific knowledge.

Before any memorization takes place, the concept for each operation should be explained fully, with visuals, so the students are comfortable in their understanding of the operation.

I also recommend that we should teach these strategies in a way that builds success on success. That is, use the facts they know and praise them as they can recall them. While we need students to learn and memorize all the facts, we can use their knowledge to boost their confidence.

Addition Fact Strategies

Thinking Strategies for Learning the Addition Facts

There are 100 basic addition facts, zero through nine. That's a bunch. But if we use more effective strategies to help students learn, then memorizing these facts will become easier for the students.

There are 5 combinations that don't have a strategy that just have to be memorized, we will leave those 5 facts till the end.

These strategies have been listed in the order we suggest teachers use to teach their own students. Before we begin with these strategies, an over-riding strategy for addition will be the Commutative Property.

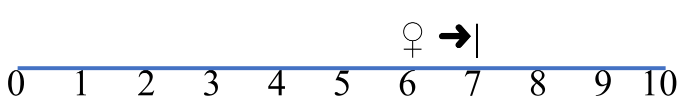
*** **Commutativity:** By changing the order, 4 + 1 to 1 + 4, we get the same result. This property should be stressed with each of the strategies presented.

Commutative Property Addition $\forall a,b \in \mathbb{R}, a + b = b + a$
For all *a* and *b* that belong to the Real numbers, *a* + *b* = *b* + *a*

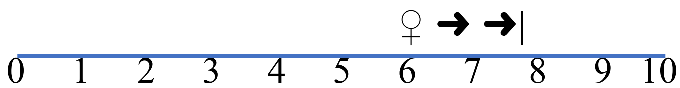
While the commutative property just makes sense to us as adults, be aware that for our little ones just learning, adding 1 + 4 looks different than 4 + 1 and for little ones counting on by one is a lot easier than counting on by four.

1. **Adding zero:** Students can quickly grasp the rule for adding zero; the sum is always the other number. 8 + 0 = 8, 0 + 8 = 8, 5 + 0 = 5.

2. **Counting on by 1:** Students can find sums like 5 + 1 or 6 + 1 by simply ***counting on*** to the next number. For beginning students, 7 + 1 = 8 is easy to teach it's the next number and we can see it on the number line. But, 1 + 7 is not so easy for the beginner. Stressing the commutative property, we strongly suggest we tell students to count on from the largest number. So, 1 + 6 becomes 6 + 1 = 7.



3. **Counting on by 2**: Students can find sums like 5 + 2 or 6 + 2 by simply counting on to the next number, then the next again. For beginning students, 7 + 2 = 9 is easy to teach because the students will count on twice. In other words, 7 + 2 is 8, 9 = 9. Students should also be able to see this on a number line. But, 2 + 7 is not so easy for the beginner. Stressing the commutative property, we strongly suggest you tell students to count on from the largest number again. So, 2 + 6 becomes 6 + 2 = 8.



4. **Sums to 10:** Sums to 10 facts have to be memorized. But the good news is we just learned three strategies; 10 + 0, 9 + 1, and 8 + 2 from the Counting on Strategies and most students come in knowing 5 + 5 = 10. So, when teaching the sums to 10, we are actually reviewing 4 facts and introducing two more, 6 + 4 = 10 and 7 + 3 = 10. The students, after initial practice, should also be able to use the commutative property; 4 + 6 = 10

Before moving on to the next set of addition strategies, it is real important that students recognize these first 4 patterns, identify the rule associated with the pattern, then use the rule to find the answer. With daily practice, students should be able to answer these as quickly as they could identify their own name.

5. **Doubles:** For whatever reason, students seem to be able to remember the sums of doubles. That might be a consequence of skip counting in earlier grades. But the doubles the students don't know need to be memorized. My guess is most students come in already knowing 1 + 1, 2 + 2, 3 + 3, 5 + 5, and 10 + 10. So while those will need to be reinforced, most of the time learning the doubles will be spent on the other 5 doubles; 4 + 4, 6 + 6, 7 + 7, 8 + 8, and 9 + 9.

6. **Doubles plus one:** This strategy should be taught when the addends are **consecutive numbers**; 6, 7 or 4, 5. The students double the smaller number and count on by 1. So 4 + 5 becomes 4 + (4 + 1) = (4 + 4) + 1 = 9. The strategy requires students to recognize the pattern of consecutive numbers, double the smaller number and count on by 1.

It's clear to learn to add with consecutive numbers in the units' column, the students need to know their doubles first.

6 + 7	**recognize consecutive numbers**	
= 6 + (6 + 1)	**rewrite 7 as 6 + 1**	**n + (n + 1)**
= (6 + 6) + 1	**regroup**	**(n + n) + 1**
= 12 + 1	**double and add 1**	**2n + 1**
= 13		

Don't forget the commutative property, 8 + 7, 7 is the smaller and is doubled, then add 1. So, 8 + 7 = 7 + 8 = 14 + 1 = 15

Students should be shown this with visuals as well, 3 balls + 4 balls, circling 3 balls, then circling or underlining 3 out of the 4 balls and having one more.

3 + 4 3 + (3 + 1)

We will use this same concept for adding consecutive EVEN or ODD numbers in the next strategy.

7. **Doubles plus two:** This method works when the addends differ by two, consecutive even or consecutive odd numbers. This strategy requires students to recognize the pattern, then double the smaller number and count on by 2. So, 6 + 8 becomes 6 + (6 + 2) = 14. Like the doubles plus one strategy for consecutive numbers, we will decompose the numbers so we have a double.

6 + 8	**recognize the pattern**	
6 + (6 + 2)	**rewrite 8 as 6 + 2**	**n + (n + 2)**
(6 + 6) + 2	**regroup**	**n + n + 2**
12 + 2 = 14	**double and add 2**	**2n + 2**

Students should recognize the commutative property works, that they double the smaller number. So, 7 + 5, would be doubling 5 and adding 2.

Students should be shown this with visuals as well, 3 triangles + 5 triangles, circling or underlining 3 triangles, then underlining 3 out of the 5 triangles and having 2 more.

$$\begin{array}{ccccccc} 3 & + & 5 & & 3 & + & (3 + 2) \\ \underline{\Delta\Delta\Delta} & + & \underline{\Delta\Delta}\Delta\Delta\Delta & \Longrightarrow & \Delta\Delta\Delta & + & (\Delta\Delta\Delta + \Delta\Delta) \end{array}$$

Before moving on to the next two strategies, daily drill and practice is required for students to attain automaticity. What's nice so far is we can see how these strategies are related and build upon one another. That makes learning them easier and less cumbersome than counting on fingers, but it does take practice and continual reinforcement. And, recognizing the community property for addition just cuts our work pretty much in half as long as we are stressing it as we teach new strategies.

8. **Adding 10's;** Adding tens is an easy pattern to recognize, learn and apply for most students. But it still needs to be taught! 10 + 5 = 15. The sum always ends in the number being added to 10.

9 **Nines:** It should be pointed out to students that when adding nine, the ones digit in the sum is always one less than the number added to 9. For example, 7 + 9 = 16, the 6 is one less than 7. Another example, 5 + 9 = 14.

This can be demonstrated by rewriting 9 as (10 – 1)

6 + 9	**recognize the pattern**
= 6 + (10 – 1)	**rewrite 9 as 10 – 1**
= (6 + 10) – 1	**regroup**
= 16 – 1	**add 10**
= 15	**subtract 1**

If you teach these strategies in this order, building on previously learned strategies and building success on success, we find there are only 5 more facts to learn. And there is no way around these 5 facts, there are NO strategies to help students learn them. That means students don't have an aid in memorizing them, they just memorize them by rote – by repetition. The last 5 addition facts are:

3 + 6, **3 + 8,** **4 + 7,** **4 + 8,** and **5 + 8**

Using these strategies helps the students find the answers to addition facts – to help the students learn them. After they learn them, they now have to do enough practice and drill so they can answer them as quickly as they can say their name when asked. And one final word, once they learn them, they have to be reviewed regularly in order for the facts to remain automatic. In other words, the job is not done.

It is expected that students respond automatically when asked a basic addition fact.

Teach/Review these strategies in this order, building on previously learned strategies and the communitive property, building success on success.

With initial learning students have to 1) recognize the number pattern, 2) connect that to a strategy, 3) use that strategy to find the sum. When first learning the strategies, students need a few seconds to process that information. With **repeated purposeful practice**, it is expected that students respond automatically when asked a basic addition fact. Here's a realistic schedule to review addition facts for those students who have not mastered them with automaticity,

LTMR

Day 1	Strategies for adding zero, one and two; Counting on
Day 2	Review strategies for 0, 1, and 2, introduce sums to 10
Day 3	Review strategies 0,1, 2, sums to 10
Day 4	Review 0, 1, 2, sums to 10. Introduce Doubles
Day 5	Review 0, 1, 2, sums to 10, and Doubles, Introduce Consecutive Numbers, Doubles + 1
Day 6	Review 0, 1, 2, sums to 10 and Doubles + 1
Day 7	Review all strategies, Introduce Consecutive Even or Odd, Doubles + 2
Day 8	Review all strategies, heavy emphasis on D+1 and D+2 strategies
Day 9	Review all strategies, Introduce +10

Day 10 Review all strategies, Introduce +9

Day 11 Review all strategies

Day 12 Introduce facts with no strategies; **3 + 6, 3 + 8, 4 + 7, 4 + 8,** and **5 + 8**

Day 13 Review all strategies, emphasize no strategy facts

When teaching these to middle or high school students, remember to strongly suggest you are showing them these strategies so they can help their little brothers, sisters, nieces and nephews. Pretend like they already know them so they are not embarrassed or subject to ridicule.

Subtraction Fact Strategies

Thinking Strategies for Learning the Subtraction Facts

A strategy we used to cut our work in half for learning the addition facts was using the Commutative Property. The commutative property does not work for subtraction, so a strategy we will use to assist us in learning the subtraction facts are Fact Families. Connecting what we have already learned to what we are now doing.

1. **Fact families:** This strategy is the most commonly used and works when students understand the relationship between addition and subtraction. When students see 6 – 2 and think 2 + ? =6. However, if this strategy is used with the following strategies, students will find greater success in a shorter period of time.

2. **Counting backwards:** This method is similar to Counting on used in addition. It isn't quite as easy. Some might think if you can count forward, then you can automatically count backward. This is not true –try saying the alphabet backwards. Students should only be allowed to count back **at most** three.

   8 – 1

   |←♀

   0 1 2 3 4 5 6 7 8 9 10

   6 – 2

   |←←♀

   0 1 2 3 4 5 6 7 8 9 10

3. **Zeros:** The pattern for subtracting zero is readily recognizable. 5 – 0 = 5

4. **Sames:** This method is used when a number is subtracted from itself; this is another generalization that students can quickly identify. 7 – 7 = 0**.**

5. **Recognizing Doubles:** Recognizing the fact families associated with adding doubles.

   If 6 + 6 = 12, then 12 – 6 = 6
   If 7 + 7 = 14, then 14 – 7 = 7

   That's really using special case of fact families/

6. **Subtracting tens:** This is a pattern that students can pick up on very quickly, seeing that the ones digit remains the same.

   16 – 10 = 6; 18 – 10 = 8 13 – 10 = 3

7. **Subtracting from ten:** Recognizing the fact families for Sums to 10.

8. **Subtracting nines:** Again, the pattern that develops for subtracting 9 can be easily identified by most students. They can quickly subtract 9 from a minuend by adding 1 to the ones digit in the minuend. 17 – 9 = 8, 16 – 9 = 7.

   $$\begin{aligned} \mathbf{15 - 9} &= \mathbf{15 - (10 - 1)} \\ &= \mathbf{15 - 10 + 1} \\ &= \mathbf{5 + 1} \\ &= \mathbf{6} \end{aligned}$$

   $$\begin{aligned} \mathbf{13 - 9} &= \mathbf{13 - (10 - 1)} \\ &= \mathbf{13 - 10 + 1} \\ &= \mathbf{3 + 1} \\ &= \mathbf{4} \end{aligned}$$

9. **Subtracting numbers with consecutive ones digits:** This pattern will always result in a difference of 9, 16 – 7 = 9, 13 – 4,= 9 15 – 6 = 9 all have ones digits that are consecutive and the result is always 9.

   $$\begin{aligned} \mathbf{16 - 7} \rightarrow &\ \mathbf{(10 + 6) - (6 + 1)} \\ = &\ \mathbf{10 + 6 - 6 - 1} \\ = &\ \mathbf{10 - 1} \\ = &\ \mathbf{9} \end{aligned}$$

   $$\begin{aligned} &\mathbf{(10 + n) - (n + 1)} \\ &= \mathbf{10 + n - n - 1} \\ &= \mathbf{10 - 1} \\ &= \mathbf{9} \end{aligned}$$

10. **Subtracting numbers with consecutive even or consecutive odd ones digits:** This pattern will always result in a difference of 8. 14 – 6 = 8, 13 – 5 = 8, 12 – 4 = 8.

   $$\begin{aligned} \mathbf{13 - 5} \rightarrow &\ \mathbf{(10 + 3) - (3 + 2)} \\ = &\ \mathbf{10 + 3 - 3 - 2} \\ = &\ \mathbf{10 - 2} \\ = &\ \mathbf{8} \end{aligned}$$

   $$\begin{aligned} &\mathbf{(10 + n) - (n + 2)} \\ &= \mathbf{10 + n - n - 2} \\ &= \mathbf{10 - 2} \\ &= \mathbf{8} \end{aligned}$$

These strategies clearly help students to subtract quickly. How you teach these strategies, allowing the students see the patterns develop, will make students more comfortable using these “shortcuts” and get them off their fingers.

Having said that, as with many of the concepts and skills in math, students need to compare and contrast problems to make them more recognizable to them. Without being able to identify the proper strategy by examining the problem, memorizing these strategies may become more burdensome and cause greater confusion than just rote memorization.

So, while you might teach one strategy at a time, as you add to the number of strategies students can use for a specific numbers, you will need to review previous strategies and, this is important, combine strategies on the same work sheets asking students to only identify the strategy they would use for each problem and why they are using it. Being able to compare and contrast will lead to increased student understanding, comfort, and achievement using these strategies.

For example,
16 – 9, students are subtracting 9, they add one to the units digit answer is 6+1 = 7.
15 – 7, students are subtracting numbers with consecutive odd units digits, the difference is 8.
17 – 8, students are subtracting numbers with consecutive units digits, the difference is 9

Knowing these basic subtraction facts will make math so much easier for students. And they will certainly help later on in math.

Multiplication Fact Strategies

Thinking Strategies for Learning the Multiplication Facts

******Commutativity:*** As with learning the addition facts, order can be changed when learning the multiplication facts. The importance of the following strategies is the order in which they are taught. My experiences suggest teachers have the students memorize the facts sequentially. That works well for the ones, two, and threes, but hits a bump at the fours. The fives go well, then the students die on the vine with the 6, 7, and 8s. So please use this sequencing and it will become apparent why it takes the frustration out of learning the multiplication facts.

1. **Multiplication by zero:** Students can easily grasp that 0 times any number is zero.
2. **Multiplication by one:** Again, the generalization is easy for students to see that 1 times any number is the number.
3. **Multiplication by two:** Students should be taught that multiplying by two is the Doubling strategy used in addition. So 2 x 8 is the same as 8 + 8.
4. **Multiplication by three:** Use the fact that students learned to skip count by 3 in previous grades. many students will already know how to multiply 3 by 1, 2, 3, 5, and 10 from previous knowledge and outside experiences. So we are left with multiply 3 by 4, 6, 7, 8, and 9.

   ***** N.B. I'm leaving multiplying by 4, then 6, 7 and 8s for later.**

5. **Multiplication by five:** Students can often be taught the fives by referring to the minute hand on a clock and skip counting. Having said that, the students need to be cautioned the products always end in zero or five. Helping them with doubling and halving also helps. That is, if they know 5 x 4 = 20, then 5 x 8 must be 40, because the 4 was doubled.

6. **Squaring:** As with the addition facts, students seem to learn square numbers faster than other facts. Students come in knowing how to square 1, 2, 3, 5, and 10 from the strategies, we must concentrate on 4, 6, 7, 8, and 9.
7. **Multiplication by ten:** This pattern is very easy for students to see.

8. **Multiplication by nine:** Patterns emerge when multiplying by 9. One pattern is the sum of the digits in the product is always equal to 9. The other pattern is the ten's digit is always one less than the factor multiplied by 9. $9 \cdot 6 = 54$. Notice $5 + 4 = 9$ and the 5 in the product is one less 6, the number being multiplied by 9. Another example, $8 \times 9 = 72$, the sum is 9 and the tens digit is one less than the tens digit. Only 10 facts to go**.**

9. **Multiplication by 4:** By using the order suggested and the commutative property, the students already know four times 1, 2, 3, 4, 5, 10, and 9 – what is left is 4 times three factors 6, 7, and 8.

10. **Multiplication by six:** By using the suggested order and the commutative property, the students know 6 times 1, 2, 3, 4, 5, 6, 10, and 9 – what is left is 6 times 7 & 8.

11. **Multiplication by seven:** Again, by using the suggested order and the commutative property, the students know 7 times 1, 2, 3, 4, 5, 6, 7, 10, and 9 – what is left is 7 x 8.

12. **Multiplication by 8: Think about it – all the facts have been covered using the suggested sequencing.**

13. **Distributive property:** Students should feel comfortable breaking numbers apart and using previously learned information. For instance, $7 \cdot 6$ might be rewritten as $7(5 + 1)$. This would allow a student to use the $7 \cdot 5$ fact that he knows and add that to the $7 \cdot 1$ fact to get 42.

14. **Finger math:** Facing your palms of your hands to your face, let the baby finger become the 6 finger, the ring finger the 7, the middle finger the 8, the index finger be the nine finger. To multiply the $7 \cdot 7$, place the two ring fingers together. The number of fingers touching and below represent the tens digit. There are four fingers, multiply the fingers above those on each hand to determine the ones digit. That would be 3 x 3. The product is 49. Try $7 \cdot 9$ by touching the index finger on one hand with the index finger on the other hand. There are 6 fingers touching or below so the answer is sixty something and you

now multiply the fingers on top to get the ones digit. That's 1 x 3, the final answer is 63.

There are other strategies for learning the multiplication facts. There are two things I want to point out to you before we go any further.

First, you don't have students memorize the multiplication facts until you have taught the concept – repeated addition. Make sure the kids feel comfortable in what they are learning.

Second, notice we have not taught the facts sequentially. We taught them in an order to help students learn so students experience a sense of accomplishment. Students need to continually practice so these facts become automatic. With respect to the math facts, we have given students ways of learning them more effectively and effectively, but they have to know them with automaticity.

Division Fact Strategies

Memorization of the division facts is based on the mastery of multiplication facts. Using fact families, that is, 6x4 = 24 so, 24÷4 = 6 and 24÷ 4 = 6 is the most important strategy presented.

This strategy works when students understand the relationship between multiplication and division. When students see 24 ÷ 6, they have to relate that to $6 \cdot ? = 24$.

Division by Zero Undefined.

The reason that division by zero is not allowed is because of how division is defined: $\boldsymbol{\frac{a}{b} = c}$ if and only if a = bc. Let's look at an example of that definition.

$\frac{8}{2} = 4$ if and only if 8 = 2 x 4. (cross multiplying) That's true. Now let's try dividing by zero, $\frac{8}{0} = n$, where n represents any number. That's true by definition only if 8 = 0 x $\boldsymbol{n}$.

Well zero times any number *n* will never result in 8, therefore since this is not true, it does not fit the definition of division. Therefore, we are not allowed to divide by zero. It's that simple. Knowing how math is defined is very important to understanding rules.