Ch. 8 COUNTING METHODS

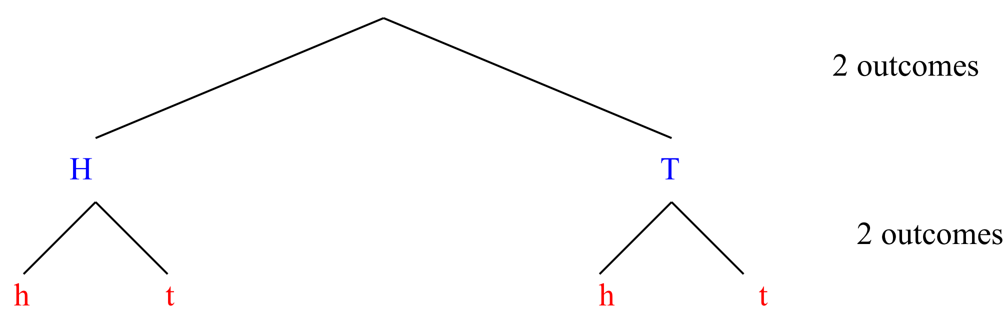
From our preliminary work in probability, we often found ourselves wondering how many different scenarios there were in a given situation. In the beginning of that chapter, we merely tried to list all the possible outcomes, hoping we didn't miss any. When we determined that was not good enough, we began to use a tree diagram to lend some order to the listing. While that worked out well for smaller sample spaces, we quickly saw its limitations when there were a great number of outcomes. For that reason, we will look at some counting methods that should make our work a lot easier.

Methods Used for Counting

1. Listing
2. Cartesian product
3. Tree Diagram
4. Fundamental Counting Principle
5. Permutation
6. Combination

If we looked at the number of outcomes in a sample space being described using a tree diagram, we might notice a pattern that would suggest a counting method. For instance, if I drew the tree diagram for tossing 2 coins, I would see there would be four possible outcomes – Hh, Ht, Th, and Tt.

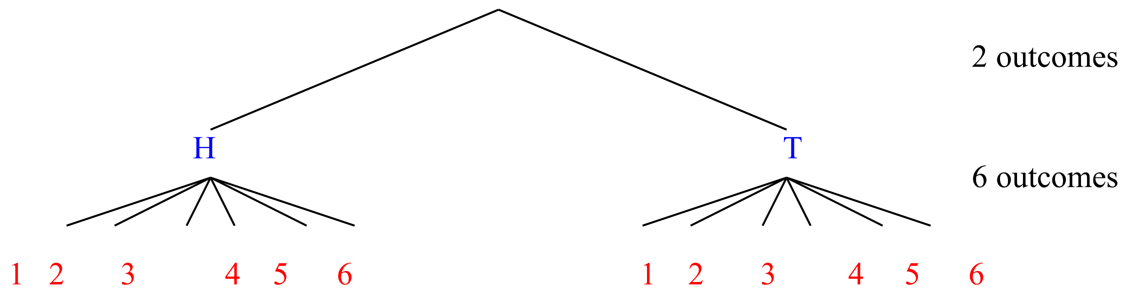
Example 1 Draw a tree diagram for tossing two coins.



With a little investigation, I might also notice there were two possible outcomes throwing the first coin, the H and the T. If I got an H on the first throw, then I could get an h or t or the second toss. If I got a T on the first toss, then I could still get and h or a t on the second toss. That results in four outcomes; Hh, Ht, Th and Tt.

Example 2 Draw a tree diagram for flipping a coin and tossing a die to determine the possible outcomes.

If I looked at another example, say throwing a coin and rolling a die. Drawing a tree diagram I would quickly see there are twelve possible outcomes – H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, and T6. Again, you might notice there are two possible outcomes when tossing the coin and six outcomes when rolling the die.



To draw that tree diagram that will tell me all the possible outcomes, we see we have two events; flipping a coin, then rolling a die.

When I flip the coin, two things can happen, a heads (H) or a tails (T). So I draw two branches for each outcome; H and T.

Now for the second event, rolling a die. There are six possible outcomes; 1, 2, 3, 4, 5 or 6, when we roll a die.

If I flipped a coin and got a heads (H), then rolled the die, I could get 1, 2, 3, 4, 5 or 6. That is reflected on the left side of the tree diagram. However, I could have gotten a tails (T) when I tossed the coin, I would still roll the die with the possible outcomes 1, 2, 3, 4, 5 or 6.

That leaves me with the following outcomes by reading down the branches:

H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, and T6.

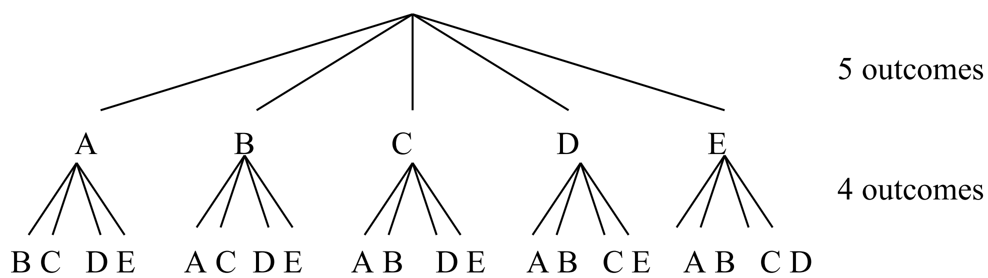
I can't make tree diagram more difficult, but they can become very cumbersome by having larger numbers or increased number of stages to an experiment.

By doing a few more of these problems, I might begin to see a pattern that would suggest a way of determining the total number of outcomes without listing and without using a tree diagram.

In the first example, we tossed two coins and discovered there were four possible outcomes. Each stage of the experiment having two outcomes, we notice that $2x2=4$.

In the second example, tossing a coin and rolling a die, we discovered using a tree diagram there were twelve possible outcomes. By looking at each stage of the experiment, we see that there are two outcomes possible for stage one, and rolling the die resulted in six possible outcomes. Notice that by multiplying 2 X 6 we end up with 12 possible outcomes. Getting excited?

Drawing a tree diagram for selecting a president and vice president from a group of 5 people if the first person chosen is the president and the second person is the vice president.



Notice, to choose the president there are 5 choices. If A was chosen as president, there would be only 4 choices for vice president - only B, C, D or E could be chosen for vice president. If B was chosen president, then A, C, D or E could be chosen vice president. That's 5x4 or 20 possible outcomes.

Those observations lead us to a pretty important way of determining the number of ways something can happen, it's called the Fundamental Counting Principle.

Fundamental Counting Principle

Fundamental Counting Principle – if an event **M** can happen in ***m*** ways, and after that occurs another event **N** can happen in ***n*** ways, then event **M** followed by event **N** can happen in ***mxn*** ways.

This counting principle will allow me to determine how many different outcomes exist quickly in my head that could be verified using tree diagrams.

In the coin tossing example, since there were 2 things that could happen on the first toss, followed by two things that could happen on the second toss, the Fundamental Counting Principle states that there will be 2x2 or 4 possible outcomes

Example 3 **Suppose Jennifer has three blouses, two pairs of slacks, and four pairs of shoes. Assuming no matter what she wears, they all match, how many outfits does she have altogether?**

She has three choices for a blouse, two choices for her slacks, and four choices for her shows. Using the Fundamental Counting Principle, she has 3 x 2 x 4 or 24 different outfits.

Now, doesn't that beat drawing a tree diagram?

Example 4 **Abe, Ben, and Carl are running a race, in how many ways can they finish?**

I could draw a tree diagram to see all the possible outcomes or I could use the Fundamental Counting Principle. There are three ways I could choose the winner, and after that occurs, there are two ways to pick second place, and one way to pick the third place finisher. Therefore, there are 3 x 2 x 1 or 6 different way these three boys could finish the race.

Example 5 **How many five-digit numbers can you make if the number in the ten thousands column cannot be zero and the number in the units column must be even?**

There are 10 digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. We have 5 events, picking a number for each column.

1) Since the ten thousands column cannot be 0, we have 9 choices.
2) There are 10 choices for the thousands column
3) There are 10 choices for the hundreds column
4) There are 10 choices for the tens column
5) For the units column, to make the digit even, we have 0, 2, 4, 6, or 8; 5 choices

Using the FCP, we have 9x10x10x10x5 = 45,000 numbers

Example 6 **How many different license plates can be made with 6 spaces if the first two spaces have to be letters that cannot be repeated and the next 4 spaces are numbers?**

We have six events, 2 picking letters and picking 4 numbers. We know there are 26 letters in the alphabet and 10 digits to create numbers.

1) There are 26 ways to pick the first letter
2) After we pick the first letter, there are 25 ways to pick the second letter
3) There are 10 ways to pick the first number
4) There are 10 ways to pick he second number

5) There are 10 ways to pick the third number
6) There are 10 ways to pick the last number

Using the FCP, we have 26x25x10x10x10x10 = 6,500,000 different license plates.

Example 7 **In a 5-person committee, how many ways can the president and vice president be chosen if the first person picked is the president and the second person is the vice president.**

We have two events, picking the president, then the vice president.

1) There are 5 options (choices) for president
2) After the president is chosen, there are 4 choices for vice president

Using FCP, there are 5x4, 20 ways of choosing the president and vice president.

Example 8 **How many ways can a 5 question True False test be answered?**

We have 5 questions, that means 5 events.

Each question can be answered two ways, true or false.

Using FCP, we have 2x2x2x2x2, 32 ways to answer those questions.

Doesn't that beat using a tree diagram? Before we go on, we need to learn a little notation that will make our lives easier in the long run.

Factorials

In **Example 4**, we saw that we had to multiply 3 x 2 x 1. As it turns out, we will have a number of opportunities to multiply numbers like 5 x 4 x 3 x 2 x 1. An issue with this is that if I have to multiply 30 x 29 x 28 x …x 3 x 2 x 1, that is a lot of writing and a lot of arithmetic.

So, we are going to introduce notation to shorten that up. We are going to abbreviate products that start at one number and work their way back to one by using an exclamation point (!). In math, however, the "!" point won't mean the number is excited. And we won't call it an exclamation point, we'll call it a factorial. So "5!" is read as five factorial and means 5 x 4 x 3 x 2 x 1 which equals 120.

Now don't you wish there was a key on your calculator to do that? Oh, there is! Yeah!!!

Examples

4! = 4 x 3 x 2 x 1 = 24

6! = 6 x 5 x 4 x 3 x 2 x 1 = 720

3!= 3 x 2 x 1 = 6

2! = 2 x 1

1! =1

*** Later on, we will find it to our advantage to **define 0! = 1.**

Now that we have learned factorials, we can more efficiently determine the size of sample spaces.

One of the best strategies to use when trying to find out how many different outcomes are in a sample space is using the Fundamental Counting Principle. Find out how many ways the first event can occur, then multiply that by the number of ways events can occur in subsequent events.

Example 9 How many outcomes are there if two dice were thrown*?*

There are 6 outcomes on the first die, and after that occurs, there are 6 outcomes on the second die.

Using the **Fundamental Counting Principle**, there are 6 x 6 or 36 possible outcomes.

Example 10 How many different ways can the letters in the word "ACT" be arranged?

There are six different ways to write those letters. We can see that in the following list.

ACT
CTA
TAC
CAT
ATC
TCA

We could have determined there were six by using the **Fundamental Counting Principle**, three ways to pick the first letter, 2 ways to pick the second, then one way to pick the third. That could also have been described using 3! = 3x2x1 = 6.

In **Example 10**, the question was how many different ways could I arrange the letters; ACT. We used the FCP and said, 3x2x1 = 6. Since the order of the arrangement worked and we were NOT using repeats, we could have said 3! How many ways do you think you could rearrange the letters in the word "POST"? Four letters, no letters repeated – 4!

Let's look at another example.

Example 11 If a different person must be selected for each position, in how many ways can we choose the president, vice president, and secretary from a group of seven members if the first person chosen is the president, the second the vice president, and the third is the secretary?

We have a total of 7 people taken three at a time. Using the **Fundamental Counting Principle**, the first person can be chosen 7 ways, the next 6, and the third 5, we have

7 x 6 x 5 or 210 ways of choosing the officers.

Another way of doing the same problem is by developing a formula using factorials. Let's see what that might look like and define it.

Let me make a minor change **Example 11**, for the next example.

Example 12 How many different ways can a 7-person race finish?

In Example 11, we had to choose 3 people from a group of 7 and the order mattered without any repetition, meaning the president could not be the vice president too. In this problem, we again have a group of 7, but we will use all of all the runners.

So, instead of writing 7x6x5x4x3x2x1, we will just write 7!

Permutations

Permutation is an arrangement of objects in which the order matters – without repetition.

Order typically matters when there is position or awards. Like first place, second place, or when someone is named president, or vice president. Various notations are used to represent the number of permutations of a set of **n** objects taken **r** at a time nPr and $P(n,r)$ are the most popular.

$_n\mathbf{P}_r$ is read as a permutation of ***n*** things taken ***r*** at a time.

$_7\mathbf{P}_3$ is read as a permutation of 7 things taken 3 at a time.

In the example choosing the president, vice president and secretary, we would use the notation $_7P_3$ to represent picking three people out of the seven. We could then use the Fundamental Counting principle to determine the number of permutations.

$$
\begin{aligned}
{}_7P_3 &= 7x6x5 \\
&= 210
\end{aligned}
$$

Generalizing this algebraically, we could develop the following formula for a permutation using factorials.

$$n\mathrm{Pr} = \frac{n!}{(n-r)!}$$

Using that formula for the last example would give us

$$
\begin{aligned}
{}_7P_3 &= \frac{7!}{(7-3)!} \\
&= \frac{7!}{4!} \\
&= \frac{7x6x5x\cancel{4}x\cancel{3}x\cancel{2}x\cancel{1}}{\cancel{4}x\cancel{3}x\cancel{2}x\cancel{1}} \\
&= 7x6x5 \text{ or } 210
\end{aligned}
$$

By dividing by (n – r)!, I'm simply dividing out the numbers, (4x3x2x1) so I'm just left with 7 x 6 x 5, the first three choices.

Example 13 Three friends buy an all day pass to ride a two-seated bike, if only two of them can ride at a time, how many possible seating arrangements are there?

Since order matters (sitting in front or back), using the **Fundamental Counting Principle**, the number of permutations of 3 friends taken 2 at a time is 3 x 2 or 6.

The front seat can be chosen three ways, and after that occurs, the person in the back seat can be chosen two ways. We could have also used the Permutation Formula.

$$_3P_2 = \frac{3!}{(3-2)!} = 6$$

The question that might come to your mind is why use the formula when we can use the Fundamental Counting Principle so easily? The answer is simple, right now I'm using smaller numbers so you can see the relationship between the FCP and a permutation. When the numbers get larger, you will want to use the Permutation Formula.

Example 14 There are 5 runners in a race. How many different ways can the racers finish?

By formula, we have a permutation of 5 runners being taken 5 at a time.

$$_5P_5 = \frac{5!}{(5-5)!}$$

$$= \frac{5!}{0!}$$

$$= 5! \quad \text{Or} \quad 5 \times 4 \times 3 \times 2 \times 1$$

Notice, we could have just as easily used the Fundamental Counting Principle to solve this problem.

Using a permutation or the Fundamental Counting Principle, order matters. A permutation does not allow repetition, the FCP does. For instance, in finding the number of arrangements of license plates, the digits can be re-used. In other words, someone might have the license plate 333 333. To determine the possible number of license plates, I could not use the permutation formula because of the repetitions, I would have to use the Fundamental Counting Principle.

Since there are 10 ways to chose each digit on the license plate, the number of plates would be determined by – 10 x 10 x 10 x 10 x 10 x 10 or 1,000,000.

The permutation formula is nothing more than a special formula of the FCP. With larger numbers, it will be of great benefit to use. Having said that, any problem can be solved by the Fundamental Counting Principal – the permutation formula is just a shortcut.

Example 15 How many distinguishable ways could we arrange the letters in the word "LOOK"?

At first glance, we might say 4 letters, so 4! = 4x3x2x1 = 24

But, that's not quite true, what if I interchanged the "O's", LOOK and "LOOK" are not distinguishable. Since there are 2 "O's", I would need to divide the 4! By 2! To get rid of the repeats.

To ensure you are seeing that, list the possible 24 arrangements by using subscripts on the O's, the first O call O_1, the second O call O_2. You can see LO_1O_2K and LO_2O_1K are different, but without the subscripts they are Not distinguishable.

Example 16 How many distinguishable arrangements of letter are there in the word "cheese"?

There are 6 letters, so 6!, but the letter "e" repeats three times. So the number of distinguishable arrangements is 6!/3!

You try some,

1. You have 4 posters to hang in the game room. You want to put one poster on each wall. How many ways can you arrange the posters?
2. Judges at the art fair are awarding prizes to the first, second and third place finalists. The art fair has 10 contestants. How many different ways can first, second and third place be awarded?
3. The cable network has 15 comedy shows it plans to broadcast. They want to choose 4 of them for Tuesday evenings. How many possible line ups are there for Tuesday?
4. You have 8 pairs of pants, and plan to wear a different each day from Monday to Friday. How many possible arrangements of pants are there during the five days?
5. You are planning a trip to Boston for 3 days. You want to see Paul Revere's house, The Old North Church and Plymouth Rock. How many ways can you arrange your schedule so you complete one activity per day?
6. The dining hall has 12 tables. One afternoon the dining hall accepts 6 reservations. How many choices does the dining hall have for seating the six groups?

7. 3! 8. 0! 9. 10! 10. 9!

11. How many distinguishable arrangements are there in "Massachusetts"?

Combinations

Sometimes "order" just does not matter. For instance, let's say we had 8 slots open for a trip. Twenty people wanted to go on that trip. How many ways could that occur?

Would using the FCP work? 20x19x18x17x16x15x14x13

How about $_{20}P_8$? 20!/(20–8)!

Both would give you the same answer. But the answer would be wrong. Why?

With the FCP and Permutation formula, order matters. That is being picked first or second makes a difference. When you just want to go on the trip, it doesn't matter if you were picked first or eighth, as long as your name is on the list.

So, this problem is just a little different than the ones we have been doing; here order does not matter. We have to determine a way to divide out repetitions. Ahh, yes, by dividing.

> ***Combination*** is an arrangement of objects in which the order does not matter – without repetition.

This is different from a permutation because the order does **not** matter. If you change the order, you don't change the group, you do not make a new combination. So, a dime, nickel, and penny is the same combination of coins as a penny, dime, and nickel.

Example 15 Bob has four golf shirts. He wants to take two of them on his golf outing. How many different **combinations** of two shirts can he take?

Using the Fundamental Counting Principle, Bob could choose his first shirt four ways, his second shirt three ways – 12 ways.

But, hold on a minute. Let's say those shirts each had a different color, by using the Fundamental Counting Principle, that would suggest picking the blue shirt, then the yellow is different from picking the yellow first, then the blue.

We don't want that to happen since the order does not matter. To find the number of combinations, in other words, eliminating the order of the two shirts, we would divide the 12 permutations by 2! or 2 x 1

There would be six different shirt combinations Bob could take on his outing.

In essence, a combination is nothing more than a permutation that is being divided by the different orderings of that permutation.

The notation we will use will follow that of a permutation, either nCr or $C(n,r)$, meaning a combination on n things being taken r at a time.

So, the combination formula is permutation formula divided by the number of repetitions.

$$nCr = \frac{n\,\text{Pr}}{r!}$$

Simplifying that algebraically, we have

$$nCr = \frac{n!}{(n-r)!r!}$$

In a permutation, A,B is different from B,A because order is important. In a combination, you would either have A,B or B,A – not both, they are the same grouping.

Example 16 From among 12 students trying out for the basketball team, how many ways can 7 students be selected?

Does the order matter? Is this a permutation or combination? Well, if you were going out for the team and a list was printed, would it matter if you were listed first or last? All you would care about is that your name is on the list. The order is not important, therefore this would be a combination problem of 12 students take 7 at a time.

$${}_{12}C_7 = \frac{12!}{(12-7)!7!}$$

$$= \frac{12!}{5!7!}$$

$$= \frac{12\ 11\ 10\ 9\ 8\ \cancel{7}\ \cancel{6}\ \cancel{5}\ \cancel{4}\ \cancel{3}\ \cancel{2}\ \cancel{1}}{5\ 4\ 3\ 2\ 1\ \cancel{7}\ \cancel{6}\ \cancel{5}\ \cancel{3}\ \cancel{2}\ \cancel{1}}$$

$$= \frac{12\ 11\ 10\ 9\ 8}{5\ 4\ 3\ 2\ 1}$$

$$= 792$$

There would be 792 different teams that can be chosen.

Example 17 Ted has 6 employees, three of them must be on duty during the night shift, how many ways can he choose who will work?

Does order matter? Since it does not matter, this problem can be solved by using the Fundamental Counting Principle, then dividing out the same grouping or you could use the formula for combination of 6 people being taken three at a time.

There are 6 ways to choose the first person, 5 ways to choose the second, and 4 ways to choose the third, that's 120 permutations. Each group of three employees can be ordered 3! Or 6 ways.

So, we divide the number of permutations by the different ordering of the three employees.

$$\frac{6x5x4}{3!} = \frac{120}{6} \rightarrow 20 \text{ ways to pick the shifts}$$

By formula, we'd have $_6C_3 = \frac{6!}{(6-3)!x3!}$. Working that out, there would be 20 ways.

Doing these problems by hand can be very distracting, you would be able to concentrate on the problem more if you had a calculator that had permutations and combination on it. That way, when you had $_{12}C_7$, all you would do is plug those numbers in, press the appropriate buttons, and wala, you would have gotten 792. Don't you just love technology?

In summary, when order matters and there is no repetition, use a permutation. If order matters and there is repetition, then use the Fundamental Counting Principle. If order does not matter, use a combination.

In just about all cases, you can still use the Fundamental Counting Principle to determine the size of the same space.

The formulas for permutation and combination just allow us to compute the answers quickly. However, if you read a problem and have trouble determining if it's a permutation or combination, then do it by the Fundamental Counting Principle and dividing out the repeats.

Remember, in a permutation or combination, repetition is not allowed. If you are having repetition, you will have to use the Fundamental Counting Principle.

In **Example 10** we asked how many different ways can the letters in the word "ACT" be written. We determined by listing and the Fundamental Counting Principle there are 6 ways to arrange those letters – 3x2x1 or 3!.

Example 18 How many different ways can we arrange the letters in the word "CHOOSES"?

Using the Fundamental Counting Principle, it appears simple enough; 7 ways to choose the first letter, 6 ways for the second letter, 5 ways for the third letter, etc.. That's 7!

7! = 720 ways to arrange the letters.

Here's the question, is "CHOOSES" different from "CHOOSES"?

Those are not distinguishable from each other. Let me place subscripts with the O's so we can see those are actually different.

$$C\ H\ O_1\ O_2\ S\ E\ S \quad \text{and} \quad C\ H\ O_2\ O_1\ S\ E\ S$$

Since they are not distinguishable, we will divide out the repeating O's that we cannot tell apart. The same applies to the "S's"

So there are seven letters, that is 7!, now we divide the 2 "O's" and the 2 "S's" that are not distinguishable. The result $\frac{7!}{2!\,2!}$

Example 19 How many distinguishable ways can we rearrange the letters in the word "MISSISSIPPI"

There are 11 letters, that translates to 11! Now, because we have repeating letters that are not distinguishable, we need to divide those out.

We have 4 "I's" that need to be divided out by 4!
We have 4 "S's" that have to be divided out by 4!
We have 2 "P's" that have to be divided out by 2!

So, our answer is

$$\frac{11!}{4!\,4!\,2!}$$

In the beginning of this chapter, we indicated we had a number of methods to determine how many ways something might happen. We could try and list everything, that's too easy to miss some. We could draw a tree diagram, that gets messy quickly. But from the tree diagram, we saw a pattern develop that led to the Fundamental Counting Principle. That recognition made finding the number of outcomes a lot easier.

From the Fundamental Counting Principal, we developed factorial notation to do some of the same problems. From there, we developed a formula that worked when order matter and there were no repetitions – the Permutation Formula. That provided us choices, use the FCP or the Permutation Formula to solve problems.

From there, we came across problems where order was not important and like permutations, repetitions did not occur. So, we either took the FCP and divided out the repeats or we took the Permutation Formula and divided out the repeats – hence the Combination Formula.

1. You want to rent 4 different Red Box movies. You can afford to rent only 3 of them. How many combinations of DVDs can you rent?
2. You need to write four essays for your Biology class. Your professor gives the class a list 7 articles to choose from. How many different groups of 4 articles can you choose from that list?
3. A social studies class names 5 presidents and each student is to choose two to compare in an essay. How many different pairings are possible?
4. A poll asks people to rank tennis, hockey, soccer, baseball and basketball according to how much they love to watch them. How many possible responses are there?
5. How many ways can you choose three of the numbers 1, 2, 5, 8, and 9 so that at least one of the numbers is greater than 6?
6. Maddy has two cats to take to be groomed. She has to take them on separate days. Appointments are available only Monday thru Friday. How many possible combinations of the days she can take the cats to be groomed?
7. You buy six movies. In how many different ways can you watch all six movies?
8. Write the number of combinations of 15 objects taken at 5 at a time.

Let's compare these methods essentially using one problem.

A) Select 3 people out of seven where the first person chosen gets $100, the second person gets $50 and the third person receives $25. How many ways can this occur?

First ting to notice is order matters, I'd rather be picked first and get $100. And, you can't be picked twice, there are no repetitions.

1) Using the FCP, we have 7 x 6 x 5, 210 ways to distribute that money between 7 people.

or

2) Using the permutation formula, ${}_7P_3$, choosing 3 people out of 7, 7!/(7–3)! = 7 x 6 x5, 210 ways to distribute the money.

Now, making a minor change to the problem, eliminating the importance of ordering.

B. Select 3 people out of seven, each person receives $100. How many ways can this occur?

1) Again, I could use FCP, 7x6x5, but the order of the 3 people does not matter, so I divide out the repeats 3x2x1 – 6. So, I have 35 ways of handing out the money.

or

2) I could have used the permutation formula $\frac{n!}{(n-r)!}$, then divide out the repetition of the 3 people resulting in 7!/[(7–3)!3!]. But wait, when you divide out the repeats, that's the Combination Formula. Don't you just love how this is all connected!

$$nCr = \frac{n\Pr}{r!} \quad = \quad nCr = \frac{n!}{(n-r)!r!} = \quad 7!/[(7-3)!3!] = 35$$

Probability and the Counting Methods

A lot of probability problems can be solved by using the definition of probability, the rules we developed to this point and the counting principle. A more efficient method of solving some of these problems involves using permutations and combinations.

Example 1 Two cards are drawn at random from a deck of 52, without replacement. What is the probability that both cards are fives?

Solve using Counting Principle
The first card can be drawn 4/52 ways, if the first card was a 5, then the second card can be drawn 3/51 since a first card was removed.

P(Both fives) = P(first 5) x P(second 5)
= 4/52 x 3/51 = 1/221

Solve using Permutations
P(Both cards are 5's), we have to know the number of ways to pick two cards from a deck of 52. That $_{52}P_2$, that's the total number of things that can happen, my sample space – denominator. The numerator is picking 2 of the four fives, that's $_4P_2$.

P(Both cards are 5's) = $_4P_2/_{52}P_2$ = 4x3/52x51 = 1/221

Example 2 Two cards are drawn at random from a standard deck of 52, without replacement. What the probability of drawing a five on the first card , then an eight on the second card.

Solve using Counting Principle
There are 4 fives in a deck of 52, 4/52. Then there are 4 eights with 51 cards left. P(5, then 8) = P(5) x P(8)
= 4/52 x 4/51 = 4/663

Solve using Permutations
Two cards from a deck of 52 is $_{52}P_2$ is the sample space.
There are 4 ways to draw one five, $_4P_1$. And after that there are 4 ways to draw one eight, that's $_4P_1$.

$$P(5,\text{ then }8) = \frac{n(E)}{n(S)}$$
$$= \frac{_4P_1 \cdot {_4P_1}}{_{52}P_2}$$
$$= \frac{4\cdot 4}{52\cdot 51} = \frac{4}{663}$$

That's the probability of getting a 5, then an 8 IN THAT ORDER. If we were asked to find the probability of getting a 5 and 8, in other words the order does not matter. Then that's the P(5, 8) + P(8, 5) or double our previous probability.

$$\frac{4}{663}+\frac{4}{663}=\frac{8}{663}$$

Example 3 Bob has homework in four subjects: math, science, English and social studies. If Bob does his homework in random order, what is the probability that he does his math homework first?

Solve by definition
There are 4 subjects, so P(math first) = ¼

Solve using Permutations
The sample space, the number of ways to do his homework is ${}_4P_4$.
There is one way to do math first, then there are ${}_3P_3$ ways to do the other assignments.

$$P(Math\ first)=\frac{1\cdot({}_3P_3)}{{}_4P_4}$$
$$=\frac{1\cdot 6}{24}=\frac{1}{4}$$

So, you may be wondering why do the problem using a permutation if using the definition was so much easier. Well, you can see that depends on the problem and the numbers in the problem. Math is first and foremost about decision-making – logic

Probability and Combinations

As we have seen, the probability of some event E is given by

$$P(E)=\frac{n(E)}{n(S)}$$

We have also seen that *n*(E) and *n*(S) are found by one of the counting methods, the tree diagrams, fundamental counting principle as well as permutations and combinations. Remember, a combination is used when order does ***not*** matter.

Example 4 There are 4 juniors and 5 sophomores on the school's golf team. If the coach has to create a foursome with this group, what is the probability that 2 will be juniors and 2 will be sophomores?

The total number of foursomes from the group of 9 people, the denominator, is ${}_9C_4$. We will be selecting 4 people from a group of 9. Notice, order does not matter.

There are ${}_4C_2$ ways of picking a junior and ${}_5C_2$ ways of picking a sophomore. Since picking a sophomore and junior are two independent events, we can apply the fundamental counting principle. That gives us the number of ways of being successful.

$$C(E) = \frac{n(E)}{n(S)} = \frac{{}_4C_2 \cdot {}_5C_2}{{}_9C_4}$$
$$= \frac{6(10)}{126} = 10/21$$

Example 5 A bag contains equally sized marbles, 4 are white and 5 are blue. If three marbles are drawn with no replacement, what's the probability there are **at least** 2 blue marbles drawn?

Again, order does not matter, so we can use combinations. The total number in the sample space, we will draw 3 marbles from a bag containing 9 is ${}_9C_3 = 84$

Now, to be successful I have to draw at least 2 blue, which means 2 blue and 1 white or 3 blue and 0 white.

n(2 blue, 1 white) = ${}_5C_2\ {}_4C_1 = 40$
n(3 blue, 0 white) = ${}_5C_3\ {}_4C_0 = 10$

Getting that math out of the way,
P(at least 2 blue) = P(2 blue) + P(3blue)
Since these 2 sets are disjoint, we have

$$P(atleast 2blue) = P(2blue) + P(3blue)$$
$$= \frac{40}{84} + \frac{10}{84} = \frac{50}{84} = \frac{25}{42}$$

Name

Date

Counting Problems

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1. Probability

2. Odds

3. Fundamental Counting Principle

4. Permutation & Formula

5. Combination & Formula

6. How was the Combination Formula derived?

7. How many five digit numbers can be obtained given that the ten thousands place cannot be zero?

8. Ten cars are in a race. How many ways can we have first, second, and third place?

9. How many ways can a True-False test be answered if there are 4 questions?

10. How many distinguishable words are in the word "grammar"?

11. In how many ways can six items be chosen from nine things?

12. License plates consist of two letters followed by five digits. How many different license plates can be made?

13. A shipment is received containing ten items, two of which are defective. Determine how many ways two items can be chosen so that a defective item is not chosen? One defective item is selected?

14. A man has a nickel, dime, quarter, and a half dollar. How many ways can the waitress be tipped if he gives her two coins?

15. There are five rotten apples in a crate of 25 apples: How many samples of three could be chosen if all three are rotten?

16. How many two card hands can be dealt from a deck of 52 cards?

17. A city council is composed of 5 liberals and 4 conservatives. A delegation of three is to be selected to attend a convention. How many could have 2 liberals and one conservative?

18. How many ways can a basketball team of 5 players be selected from a squad of 12 players if the first player picked will be the captain?

19. A man's wardrobe consists of 5 sport coats, 3 dress slacks, and 2 pairs of shoes. Assuming they all match, in how many ways can he select an outfit?

20. The school board consists of seven members, if the first person selected is the president, the second is the vice president and the third is the treasurer, how many ways can the officers of the board be chosen?

21. Provide parent/guardian contact information: email, home or cell phone, etc. (CHP)

