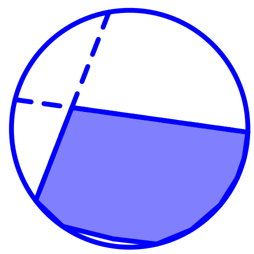
Fractions

Fractions and their Equivalent Forms

Little kids use the concept of a fraction long before we ever formalize their knowledge in school. Watching little kids share a candy bar or a bottle of soda indicates they have a feeling for what will soon be called a fraction.

Can you imagine having a son or daughter come home and slice a cake like this, then eat the piece largest (shaded) piece.



When you got home and saw most of the cake gone, you might have asked; who ate all the cake? If someone were to retort, I only had one piece, the biggest, you may have wanted to make him wear the rest. But, you decided that in the future you would pre-cut the cake into equal pieces. That way you would know how much cake one piece would be, hence you introduced the notion of denominator.

> **A fraction is part of a whole unit or group, it's made up of two parts, a numerator and a denominator. The denominator tells you how many EQUAL parts make one whole unit or group, the numerator tells you how many of those parts you have.**

$$\frac{3}{8} \rightarrow \frac{\text{numerator}}{\text{denominator}}$$

If a pie was cut into eight equal pieces and you ate all eight pieces, you could write that as $^{8}/_{8}$ or one whole pie. One whole day could be described as $^{24}/_{24}$. That allows us to say the $^{24}/_{24} = 1$

There are two types of fractions, proper and improper.

Types of Fractions

1. **Proper fraction** is a fraction less than one. The numerator is less than the denominator. Ex. 5/62.

2. **Improper fraction** is a fraction greater than one. The numerator is greater than the denominator. Ex. 7/5.

Saying and Writing Fractions

To say a fraction like $\frac{3}{4}$ you say the numerator - three, then say the denominator by spraying friends with the suffix *ths.* In other words, you say three fourths. For people that don't know mathematical notation, they might say three slash four. But, since you understand the importance of vocabulary and notation, you would know that was a fraction.

To express a quantity as a fraction, you would need to know how many equal parts make one whole unit. The numerator would be the parts under consideration, the denominator would be how many ***EQUAL*** parts make one full unit.

Example 1 Express 5 hours as a part of a day.

Since you have 5 hours, that will be numerator. There are 24 hours in one full day, so the denominator is 24. Piece of cake! Therefore, 5 hours 5/24 day.

Example 2 Express 7 ounces as part of a pound.

Since there are16 ounces in a pound, the denominator is 16. The numerator is 7. Therefore 7 ounces is 7/16 lb.

If you are not familiar with denominate numbers, then knowing how many parts are in a whole could cause you some difficulty. You should know things like 12 inches is a foot, 16 ounces is a pound, 3 feet is a yard, etc. I hear you, you want to try some problems on our own.

Writing Fractions

Express the following quantities as a fraction of a foot.

1.	1 inch	2.	7 inches	3.	5 inches
4.	9 inches	5.	8 inches	6.	12 inches

Express as a fraction of an hour.

7.	2 min.	8.	23 min.	9.	9 min.

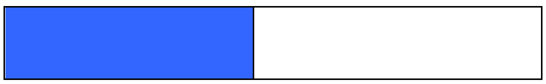
Express as a faction of a day.

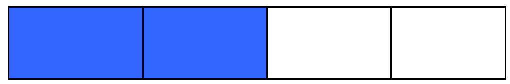
10.	5 hrs.	11.	8 hrs.	12.	20 hrs.

Equivalent Fractions

Have you ever noticed that not everyone describes the same things in the same way. For instance, a mother might say her baby is twelve months old. The father might tell somebody his baby is a year old. Same thing, no big deal. Well, we do the same thing in math. Or in our case, in fractions. Let's look at these two cakes.

One person might notice that 2 out of 4 pieces seem to describe the same thing as 1 out of 2 in the picture above. In other words, $\frac{1}{2} = \frac{2}{4}$





If we continue this process, we would notice we have a number of different ways to express the same thing. When two fractions describe the same thing, we say they are equivalent fractions.

Equivalent Fractions are fractions that have the same value.

Wouldn't it be nice if we could determine if fractions were equivalent without drawing pictures? Well, if we looked at enough equivalent fractions, we would notice a pattern developing. Let's look at some.

$$\frac{1}{2} = \frac{3}{6}, \qquad \frac{3}{4} = \frac{6}{8}, \qquad \frac{2}{3} = \frac{10}{15}, \text{ and } \quad \frac{3}{5} = \frac{30}{50}$$

Do you see any relationship between the numerators and denominators in the first fraction compared to the numerators and denominators in the second?

Hopefully, you might notice we are multiplying both numerator and denominator by the same number to get the 2^{nd} fraction.

Example 3 $\quad \frac{5}{6} = \frac{20}{24}$ if you multiply both the numerator and denominator by 4

Well, you know what that means, when we see a pattern like that, we make a rule, an algorithm or procedure that allows us to show other people simple ways of doing problems. Yes, feel that excitement running through your body.

To generate equivalent fractions, you multiply BOTH numerator and denominator by the SAME number

When you multiply both the numerator and denominator by the same number, we are multiplying by 4/4 or 1. When we multiply by one, that does not change the value of the original fraction.

Example 4 Express $\frac{5}{6}$ as sixtieths.

$$\frac{5}{6} = \frac{?}{60}$$ What did you multiply 6 by to get 60 in the denominator?

By 10, so we multiply the numerator by 10. So $\frac{5}{6} = \frac{50}{60}$. Piece of cake!

Example 5 $\frac{2}{7}$ is how many thirty-fifths?

$$\frac{2}{7} = \frac{?}{35}$$

What did you multiply 7 by to get 35 in the denominator? By 5 you say, we multiply the numerator by 5.

$$\frac{2}{7} = \frac{10}{35}$$

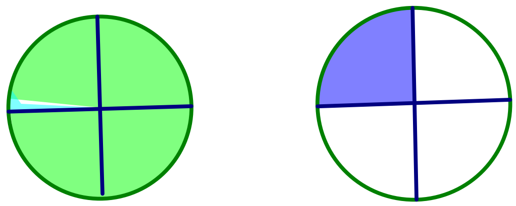
We'll let you try some of these on your own. By the way, do you know what the fraction said to the whole number? Answer, you crack me up! Get it, crack up into parts. Oh yes, math humor, don't you just love it?

Making equivalent fractions.

1. $\frac{2}{3} = \frac{\quad}{6}$
2. $\frac{3}{4} = \frac{\quad}{20}$
3. $\frac{2}{5} = \frac{\quad}{30}$
4. $\frac{5}{8} = \frac{\quad}{40}$
5. $\frac{7}{15} = \frac{21}{\quad}$
6. $\frac{3}{8} = \frac{27}{\quad}$

Converting Mixed Numbers to Improper Fractions

Mixed Number is a whole number and a fraction. A mixed number occurs when you have more than one whole unit. Ex. 1 1/4



Example 6 Let's say Johnny boy ate all the first cake and one piece from the second cake. We could describe that as eating $1\frac{1}{4}$ cakes. His mom might come home and notice Johnny ate 5 pieces of cake. Notice, eating $1\frac{1}{4}$ cakes describes the same thing as eating 5 pieces of cake.

Good news, since we are working with fractions, I can describe eating 5 pieces of cake as a fraction. Again the numerator tells us how many pieces Johnny ate, the denominator tells us how many equal pieces make one whole cake. In our case, that's 4. The fraction then is 5/4.

What we have just seen is $1\frac{1}{4}$ seems to describe the same thing as $\frac{5}{4}$. Therefore, we say they are equivalent. Writing that, we have $1\frac{1}{4} = \frac{5}{4}$

If we looked at some more cakes, we might notice a similar pattern.

Example *7* Describe eating 1 3/8 cakes as eating 11 pieces of cake when 8 pieces make one whole cake.

The whole cake can be described as 1 or $\frac{8}{8}$ and the portion of the other cake is $\frac{3}{8}$. What we have is $1\frac{3}{8} = \frac{8}{8} + \frac{3}{8}$ or 11/8.

If we kept looking at more examples, we might ask ourselves if there is a way we could convert that mixed number to an improper fraction without drawing the picture.

Again, we'll have to look for patterns. Do you see one? It doesn't jump right out at you.

$$1\frac{1}{4} = \frac{5}{4} \text{ and } 1\frac{3}{8} = \frac{11}{8}$$

Well, if we looked long enough and took the time to write the numbers down and play with them, then we might see this little development.

In the first equality, $1\frac{1}{4} = \frac{5}{4}$ If I multiply the denominator by the whole number and add the numerator, that gives me the new numerator 5. The denominator stays the same.

The second equality, $1\frac{3}{8} = \frac{11}{8}$, the pattern of multiplying the denominator by the whole number and adding the numerator also gives me the new numerator of 11. Again the denominator stays the same. Oh yes, you can feel the excitement!

Seeing these developments, we might try looking at a few more examples using cakes, then see if the pattern we discovered still works. If it does, we make a rule.

As you probably guessed, it works. If it didn't, we would not have been discussing it anyway, right?

Converting a Mixed Number to an Improper Fraction

1. ***Multiply the whole number by the denominator***
2. ***Add the numerator to that product***
3. ***Place that result over the original denominator***

Example 8 Convert $2\frac{3}{4}$ to an Improper fraction

$$2\frac{3}{4} = \frac{4\ x\ 2 + 3}{4} = \frac{11}{4}$$

With practice, you should be able to do these in your head. So, if you don't mind, here are some practice problems. No, you don't have to thank me.

Convert to improper fractions

1. $1\frac{3}{4}$ 2. $5\frac{2}{3}$ 3. $10\frac{1}{2}$

4. $8\frac{1}{5}$ 5. $6\frac{5}{7}$ 6. $7\frac{3}{10}$

7. $3\frac{4}{5}$ 8. $4\frac{5}{8}$ 9. $2\frac{1}{2}$

Converting Improper Fractions to Mixed Numbers

Using the pictures of the pie, we noticed that $\frac{5}{4} = 1\frac{1}{4}$. We found a way to convert the mixed number to an improper fraction, do you think you can find a method to convert an improper fraction to a mixed number? How did we get the fraction $\frac{5}{4}$? What we did was count the number of equally sized pieces of pie. All four pieces of the first pie were gobbled down and one piece from the other pie. That gave us a total of 5 pieces. The denominator was 4 because that's how many pieces make one whole pie.

Another way to look at $\frac{5}{4}$ is to look at each pie in fractional form. All four pieces were eaten in the first pie, that's $\frac{4}{4}$. One piece from the second pie, that's $\frac{1}{4}$. That would lead us to believe that $\frac{4}{4} + \frac{1}{4} = \frac{5}{4}$.

In ***Example 8***, converting $1\frac{3}{8}$ to an improper fraction resulted in $\frac{11}{8}$. Now again, the question is can we find a way to convert back without drawing a picture. The fraction $\frac{11}{8}$ clearly indicates more than one pie was eaten. In fact, we can see in the first pie 8 out of 8 pieces were eaten. That's written as $\frac{8}{8}$. The second pie had 3 out of 8 eaten. That would tell us that $\frac{8}{8} + \frac{3}{8} = \frac{11}{8}$.

So the trick to convert an improper fraction to a mixed number seems to be to determine how many whole pies were eaten, then write the fractional part of the pie left.

To convert $\frac{7}{3}$ to a mixed number, how many whole pies were eaten? Well $\frac{7}{3}$ could be written as $\frac{3}{3} + \frac{3}{3} + \frac{1}{3}$. We can see two pies were eaten plus a $\frac{1}{3}$ of another pie.

Could I have done that without breaking apart the fraction?

Sure, to determine how many whole pies are in $\frac{7}{3}$, I could have divided 7 by 3. The quotient is 2, which means I have two whole pies. The remainder is 1, which means I have one piece of the last pie or $\frac{1}{3}$. So $\frac{7}{3} = 2\frac{1}{3}$.

To Convert an Improper Fraction to a Mixed Number

1. ***Divide the numerator by the denominator to determine the whole number***
2. ***Write the remainder over the original denominator***

Example 9 Convert $22/5$ to a Mixed Number

$22 \div 5 = 4$ with a remainder of 2. Therefore $\frac{22}{5} = 4\frac{2}{5}$

Convert to mixed numbers

1. $\frac{7}{4}$ 2. $\frac{16}{7}$ 3. $\frac{23}{6}$ 4. $\frac{81}{80}$

As we continue our study of equivalent forms of fractions, we run into something called reducing fractions. When we reduce fractions, what we are doing is making equivalent fractions. But rather than multiplying both numerator and denominator by the same number, we are going to divide both numerator and denominator by the same number.

To do that, we have to find common factors. That is, we have to find numbers that go into both the numerator and denominator.

Before we reduce fractions, we are going to look for patterns that will make our lives very simple.

Rules of Divisibility

We are going to discuss Rules of Divisibility. To be quite frank, you already know some of them. For instance, if I asked you to determine if a number is divisible by two, would you know the answer. Sure you do, if the number is even, then it's divisible by two.

Can you tell if a number is divisible by 10? How about 5?

Because you are familiar with those numbers, chances are you know if a number is divisible by 2, 5 or 10. We could look at more numbers to see if any other patterns exist that would let me know what they are divisible by, but we don't have that much time or space. So, if you don't mind, I'm just going to share some rules of divisibility with you.

Rules of Divisibility, a number is divisible.

By 2, if it ends in 0, 2, 4, 6 or 8
By 5, if it ends in 0 or 5
By 10, if it ends in 0

By 3, if sum of digits is a multiple of 3
By 9, if sum of digits is a multiple of 9

By 6, if the number is divisible by 2 and by 3
By 4, if the last 2 digits of the number is divisible by 4
By 8, if the last 3 digits of the number is divisible by 8

Example 1 Is 111 divisible by 3?

One way of finding out is by dividing 111 by 3, if there is no remainder, then it goes in evenly. In other words, 3 would be a factor of 111.

Rather than doing that, we can use the rule of divisibility for 3. Does the sum of the digits of 111 add up to a multiple of 3? Yes it does, $1+1+1=3$, so it's divisible by 3.

Example 2 Is 471 divisible by 3?

Do you want to divide or do you want to use the shortcut, the rule of divisibility? Adding 4, 7 and 1, we get 12. Is 12 divisible by 3? If that answer is yes, that means 471 is divisible by 3. If you don't believe it, try dividing 471 by 3.

Let's look at numbers divisible by 4. The rule states if the last 2 digits are divisible by 4, then the number itself is divisible by 4.

Example 3 Is 12,316 divisible by 4?

Using the rule for 4, we look at the last 2 digits, are they divisible by 4. The answer is yes, so 12,316 is divisible by 4.

If you take a few minutes today to learn the ***Rules of Divisibility***, that will make your life a lot easier in the future. Not to mention it will save you time and allow you to do problems very quickly when other students are experiencing difficulty.

Using the rules of divisibility, determine if the following numbers are divisible by 2, 3, 4, 5, 6, 8, 9, or 10.

1.	40	2.	36	3.	72	4.	48
5.	123	6.	111	7.	306	8.	312
9.	1,035	10.	840	11.	7012	12.	3,708

13. Write a 5 digit number that is divisible by 2, 3, 4, 5, 6, 8, 9, and 10.

14. Write a 6 digit number that is divisible by 2, 3, 4, 5, 6, 8, and 10, but not 9.

Now that we have played with rules of divisibility, learning how to reduce fractions will be a much easier task.

Simplifying Fractions

Simplifying fractions is just another form of making equivalent fractions. Instead of multiplying the fraction by one by multiplying the numerator and denominator by the same number, we will divide the numerator and denominator by the same number – which is equivalent to multiplying by one. Now that we know the Rules of Divisibility, reducing fractions is going to be a piece of cake.

> ***To simplify fractions we divide both numerator and denominator by the same number*.**

Example 1 Simplify 18/20

Both the numbers are even, so we can divide both numerator and denominator by 2. Doing that the answer is 9/10.

Example 2 Simplify $\frac{111}{273}$

Notice that the sum of the digits in 111 is 3 and the sum of the digits in 273 is 12. Therefore, both are divisible by 3. $37/91$ is the answer. Don't you just love this stuff?

If you don't know the rules of divisibility, you would have to try and reduce the fractions by trying to find a number that goes into both numerator and denominator. That's too much guessing, so spend a few minutes and commit the rules of divisibility to memory,

Simplify the following fractions.

1. $\frac{4}{8}$

2. $\frac{8}{12}$

3. $\frac{15}{24}$

4. $\frac{25}{30}$

5. $\frac{12}{54}$

6. $\frac{16}{48}$

7. $\frac{111}{213}$

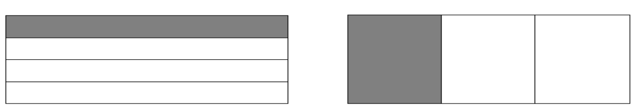
8. $\frac{81}{702}$

9. $\frac{36}{312}$

Common Denominators

Now that we have played with fractions, we know what a fraction is, how to write them, say them and we can make equivalent forms, it's now time to learn how to add and subtract them. But before we do, by definition, we must have equally sized pieces.

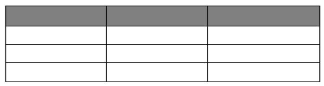
Let's say we have two cakes, one chocolate, the other vanilla. The chocolate was cut into thirds, the vanilla into fourths as shown below. You had one piece of chocolate cake and one piece of vanilla, as shown.



Since you had 2 pieces of cake, can you say you had $\frac{2}{7}$ of a cake?

Let's go back to how we defined a fraction. The numerator tells you how many equal pieces you have, the denominator tells you how many equal pieces make one whole cake. Since your pieces are not equal, we can't say we have $\frac{2}{7}$ of a cake. And clearly, 7 pieces don't make one whole cake. Therefore, trying to add $\frac{1}{4}$ to $\frac{1}{3}$ and coming up with $\frac{2}{7}$ just doesn't fit our definition of a fraction.

The key is we have to cut the cakes into equal pieces. Having one cake cut into thirds and the other in fourths means I have to be innovative – so . . .



Let's get out our knives and do some additional cutting.

By making additional cuts on each cake, both cakes are now made up of 12 equal pieces. That's good news from a sharing standpoint – everyone gets the same size piece. Mathematically, we have introduced the concept of a **common denominator.**

The way I made the additional cuts on each cake was to cut the second cake the same way the first was cut and the first cake the same way the second was cut as shown in the picture. Now, that's a piece of cake!

Clearly, we don't want to make additional cuts in cakes or pies the rest of our lives to make equal pieces from baked goods that have been cut differently. So, what we do is try and find a way that will allow us to determine how to make sure all pieces are the same size. What we do mathematically is find the common denominator.

A **common denominator** *is a denominator that all other denominators will divide into evenly*	**4 Methods of Finding a Common Denominator** ***1. Multiply the denominators*** ***2. Write multiples of each denominator, use a common multiple*** ***3. Use a factor tree and find the Least Common Multiple*** ***4. Use the Reducing Method, especially for larger numbers***

Cake-wise, it's the number of pieces that cakes can be cut so everyone has the same size piece.

Method 1 if I had two fractions like $\frac{1}{3}$ and $\frac{1}{4}$. By multiplying the denominators, I would find a number that is a multiple of 3 and 4. In other words, a number in which both 3 and 4 are factors.   The common denominator would be 3×4 or 12.

Method 2 I would write multiples of each denominator, when I came across a common multiple for each denominator, that would be a common denominator. Again, using $\frac{1}{3}$ and $\frac{1}{4}$, I write multiples of each denominator.

3, 6, 9, 12, 15, 18, …
4, 8, 12,

Since 12 is a multiple of each denominator, 12 would be a common denominator.

Method 3 is a pain in the rear and nobody that I know of uses it to find common denominators, so, we won't either. However, the next method is a direct result of finding LCMs

Method 4 is an especially good way of finding common denominators for fractions that have large denominators or fractions whose denominators are not that familiar to you.

Example 1 Let's say I asked you to find the common denominator for the fractions 1/18 and 5/24.

Using **Method 1**, we'd multiply 18 by 24. The result 432. That's too big of a number for me to know in my head.

Method 2 would have us writing multiples of the two denominators.

18, 36, 54, 72, 90, 108, …
24, 48, 72

72 is a multiple of each, therefore 72 would be a common denominator.

Using **Method 4**, I put the 2 denominators over each other in fractional form as shown and reduce.

$$\frac{18}{24} = \frac{3}{4}$$

Now I cross multiply, either 24 by 3 or 18 by 4. Notice I get 72 no matter which way I go. Therefore 72 is the common denominator.

It does not matter if I put 18/24 and reduce or 24/18, I get the same answer.

Example 2 Find the common denominator for 5/24 and 9/42.

While multiplying will give you a common denominator, it will be a very large number. I'm going to use method 4. Placing the denominators over each other and reducing.

$\frac{24}{42} = \frac{4}{7}$ 4 x 42 gives me a common denominator of 168

Using **Method 4**, reducing the denominators, sure beats multiplying 24 by 42. It's also better than trying to write multiples for both of those denominators and finding a common multiple.

Try reducing the following fractions using the easiest method available to you,

Find the common denominators

1. 2/3, 3/5 2. 1/6, 3/5 3. 2/3, 7/8

4. $\frac{11}{12}, \frac{5}{13}$ 5. $\frac{7}{18}, \frac{5}{24}$ 6. $\frac{7}{24}, \frac{11}{45}$

7. $\frac{3}{4}, \frac{7}{8}$ 8. $\frac{2}{3}, \frac{5}{7}, \frac{2}{5}$ 9. $\frac{7}{36}, \frac{5}{48}$

1. $8 - 5\frac{1}{3}$ 2. $8\frac{1}{4} - 5\frac{1}{3}$ 3. $10\frac{1}{2} - 6\frac{4}{5}$

Comparing Fractions

There are a number of ways to compare (order) fractions; making equivalent fractions with common denominators, converting the fractions to decimals, or by cross multiplying.

Let's look at each method.

Method 1. By equivalent fractions.

1. Find a common denominator
2. Make equivalent fractions
3. Compare the numerators.

Method 2. By converting to a decimal

1. Divide the numerator by the denominator
2. Carry out the division the same number of places
3. Compare the numbers

Method 3. By cross multiplying

1. Using the fractions, a/b and c/d
2. Cross multiply and place the products ABOVE the fractions
3. Compare products

The method you want to use really depends on the fractions with which you are working. Let's do one example all three ways.

Ex. 1. Method 1. Compare by making equivalent fractions

Compare using a > or < sign.

$\frac{2}{3} \quad \frac{4}{5}$

The common denominator is 15, the equivalent fractions are $\frac{10}{15} \quad \frac{12}{15}$

10 < 12, therefore $\frac{2}{3} < \frac{4}{5}$

Ex. 2 Method 2. Compare by converting to a decimal using < o r > signs.

$\frac{2}{3} = .666$ $\frac{4}{5} = .800$ Note both carried out 3 places

666 < 800, therefore $\frac{2}{3} < \frac{4}{5}$

Ex.3 Method 3. Compare by cross multiplying

10 12

$\frac{2}{3} \quad \frac{4}{5}$

Cross multiplying and placing the product above the fraction, 10 < 12.

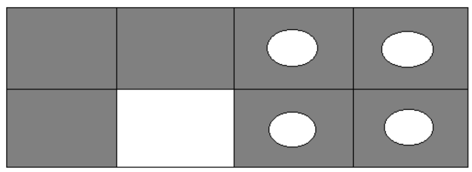
Since 10 < 12, therefore $\frac{2}{3} < \frac{4}{5}$

Compare the following fractions by any method.

1. 2/3, 3/5
2. 1/6, 3/5
3. 2/3, 7/8
4. $\frac{11}{12}, \frac{5}{13}$
5. $\frac{7}{18}, \frac{5}{24}$
6. $\frac{7}{24}, \frac{11}{45}$
7. $\frac{3}{4}, \frac{7}{8}$
8. $\frac{2}{3}, \frac{5}{7}, \frac{2}{5}$
9. $\frac{7}{36}, \frac{5}{48}$

Adding & Subtracting Fractions
With Like Denominators

In order to add or subtract fractions, we have to have <u>equal</u> pieces. If a cake was cut into 8 equal pieces and you had three pieces tonight, then ate four pieces tomorrow, you would have eaten a total of 7 pieces of cake, or 7/8 of one cake.



Mathematically, we would write $\frac{3}{8}+\frac{4}{8}=\frac{7}{8}$

Notice, we added the numerators because that told us how many <u>equal</u> pieces were eaten. Why didn't we add the denominators? Remember how we defined a fraction, the denominator tells us how many equal pieces makes one whole cake. If I added them, we would be indicating that the cake was cut into 16 pieces. But we know it was only cut into 8 equally sized pieces.

Add/subtract

1. $\frac{1}{5}+\frac{2}{5}$

2. $\frac{3}{8}+\frac{5}{8}$

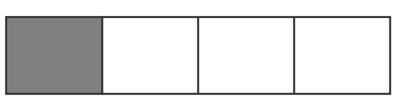
3. $\frac{3}{11}+\frac{2}{11}$

Adding and Subtracting Fractions
With Unlike Denominators

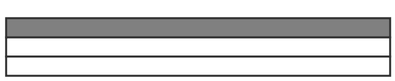
Let's add $\frac{1}{3} to \frac{1}{4}$ Would I get $\frac{2}{7}$? Why not? The $\frac{2}{7}$ would indicate that we have two equal pieces and that 7 equal pieces made one whole unit.

Let's draw a picture to represent this:

$\frac{1}{4}$



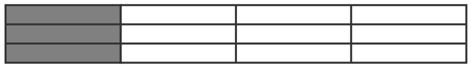
$+\frac{1}{3}$



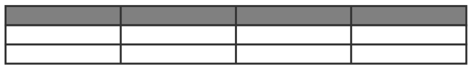
Notice the pieces are not the same size.

Making the same cuts in each cake will result in equally sized pieces. That will allow me to add the pieces together. Each cake now has 12 equally sized pieces. Mathematically, we say that 12 is the common denominator. Now let's add.

$$\frac{1}{4} = \frac{3}{12}$$



$$+\frac{1}{3} = \frac{4}{12}$$



$$\frac{7}{12}$$

From the picture we can see that 1/3 is the same as 4/12 and $1/4$ has the same value as 3/12. Adding the numerators, a total of 7 equally sized pieces are shaded and 12 pieces make one unit.

If I did a number of these problems, I would be able to find a way of adding and subtracting fractions without drawing the picture.

<u>Algorithm for Adding/Subtracting Fractions</u>

1. ***Find a common denominator***
2. ***Make equivalent fractions.***
3. ***Add/Subtract the numerators***
4. ***Bring down the denominator***
5. ***Simplify***

Using the procedure, let's try one.

Example 1

$$\begin{array}{r} \frac{1}{5} \\ +\frac{2}{3} \\ \hline \end{array}$$

Since the denominators are small and relatively prime, I will use multiply the denominators to find the common denominator, $5 \times 3 = 15$. Now I make equivalent fractions and add the numerators.

$$\begin{array}{r} \frac{1}{5} = \frac{3}{15} \\ +\frac{2}{3} = \frac{10}{15} \\ \hline \frac{13}{15} \end{array}$$

Let's try a few. Using the algorithm, first find the common denominator, then make equal fractions. Once you complete that, you add the numerators and place that result over the common denominator.

Remember, the reason you are finding a common denominator is so you have equally sized pictures.

Add or subtract the following problems.

1. $\begin{array}{r} \frac{3}{4} \\ +\frac{1}{5} \\ \hline \end{array}$

2. $\begin{array}{r} \frac{3}{8} \\ +\frac{1}{3} \\ \hline \end{array}$

3. $\begin{array}{r} \frac{5}{7} \\ -\frac{1}{3} \\ \hline \end{array}$

4. $\begin{array}{r} \frac{1}{3} \\ +\frac{2}{5} \\ \hline \end{array}$

5. $\begin{array}{r} \frac{3}{4} \\ -\frac{1}{2} \\ \hline \end{array}$

6. $\begin{array}{r} \frac{5}{9} \\ +\frac{1}{2} \\ \hline \end{array}$

7. $\frac{5}{8} - \frac{2}{5}$

8. $\frac{3}{5} + \frac{3}{4}$

9. $\frac{7}{9} - \frac{1}{2}$

Borrowing From a Whole Number

Subtracting fractions when borrowing is as easy as getting change for your money.

Example 2 Let's say you have 7 one dollar bills and you have to give your friend $3.25. How much money would you have left?

Since you don't have any coins, you would have to change one of the dollars into 4 quarters. Why not ten dimes? Because you have to give your friend a quarter, so you get the change in terms of what you are working with – quarters.

Writing that, this is what it would look like.

```
 7  dollars  ------------>          6 dollars  4 quarters
–3  dollars  1 quarter            – 3 dollars  1 quarter
```

After you get the change, you can subtract

Doing that, we have

```
  6  dollars  4  quarters
– 3  dollars  1  quarter
  3  dollars  3  quarters
```

I'd be left with 3 dollars and 3 quarters. That seems to make sense. You could have done that in your head

Example 3 Let's do the exactly the same problem using fractions. Again, I have 7 one dollar bills and I have to give my friend $3.25. Another way to say that is I have to give my friend three and a quarter.

Let's write it in fractional terms. 7 – 3 ¼

```
   7            6  4/4
– 3 1/4        –3  1/4
                3  3/4
```

Remember, I am borrowing one, $\frac{4}{4}=1$. So what I'm substituting $6\frac{4}{4}$ for 7 so I can subtract.

Example 4 $12 - 9\frac{2}{5}$

```
  12           11  5/5
– 9  2/5       –9  2/5
                2  3/5
```

Remember, when borrowing, when working in fifths, I borrow 5/5. If I was working in thirds, I'll borrow 3/3. If I'm working in twelfths, I'll borrow 12/12. The point being, just like in dealing with money, I get the type of change I'll need to do the problem. Piece of cake, don't you think?

Guess what, it's your turn to do a couple.

1. $12 - 7\frac{2}{3}$

2. $8 - 5\frac{3}{4}$

3. $10 - 7\frac{3}{5}$

4. $9 - 4\frac{5}{12}$

5. $16 - 6\frac{5}{7}$

6. $13 - 8\frac{5}{6}$

I know what you are thinking, these are just too easy. Can I make them more difficult? Unfortunately, the answer is no. All I can do is try to make the problems longer.

In the next section, we do the same thing we did with these problems, we are going to borrow. But, since we already had some change, we are going to take what we borrow and add that to what we already have. Let's go on and see.

Borrowing From Mixed Numbers

Now, what do you think might happen if you had 6 dollars and 1 quarter in your pocket and you had to give your little brother $2.75?

Well, since you only have one quarter, you again would have to get more change.

Using money, let's see what we have.

6 dollars 1 quarter	making change	5 dollars 5 quarters
– 2 dollars 3 quarters		– 2 dollars 3 quarters
		3 dollars 2 quarters

When you got change for the dollar, you received 4 quarters, adding that to the quarter you already had, that gives you 5 quarters.

Now, you can subtract. You'd end up with 3 dollars and 2 quarters.

Let's do the same problem using fractions.

Example 5 $\begin{array}{r} 6\frac{1}{4} \\ \underline{-2\frac{3}{4}} \end{array}$ Notice, you can't take 3 from 1 in the numerators, you must borrow, Just like when we borrowed before, since we are working in fourths, we'll borrow $\frac{4}{4}$. We take what we borrow and add that to what we already have, the result is 5/4. That makes sense.

$$\begin{array}{r} 6\tfrac{1}{4} \\ -2\tfrac{3}{4} \\ \hline \end{array}$$ Adding the 4/4 to the $\frac{1}{4}$ we already had results in $$\begin{array}{r} 5\tfrac{5}{4} \\ -2\tfrac{3}{4} \\ \hline 3\tfrac{2}{4} \end{array}$$

Simplifying, we have $3\frac{1}{2}$

Always check to see if your final answer can be simplified. My guess is you would use the Rules of Divisibility to make that determination.

I can't make these problems difficult if you know the algorithm. So take a few minutes and memorize it.

Borrowing with Mixed Numbers

1. ***Find a common denominator***
2. ***Make equivalent fractions***
3. ***Borrow, if necessary***
4. ***Subtract the numerators***
5. ***Bring down the denominator***

1. $8 - 5\frac{1}{3}$
2. $8\frac{1}{4} - 5\frac{1}{3}$
3. $10 - 6\frac{4}{3}$
4. $10\frac{1}{2} - 6\frac{4}{5}$
5. $8\frac{1}{4} - 6\frac{1}{2}$
6. $9\frac{2}{3} - 3\frac{3}{4}$

Multiplying Fractions

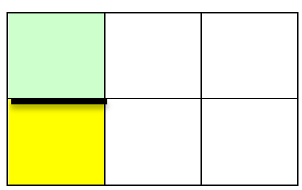
Let’s take a look at what multiplying fractions looks like visually. To begin we will look at 1/3 of a unit, then take ½ of that 1/3.

So here’s what 1/3 of a unit might look like.

Now, I chose a rectangular shaped unit, it could have been triangular, circular, a star or a square.

Now, what I will do is take ½ of that 1/3 and shade that green.

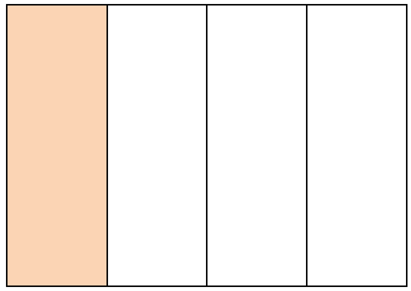
By taking ½ of the 1/3, I end up with smaller pieces. How many of those equally sized pieces would make a whole unit? Remember, the denominator tells you how many equal pieces makes one unit. So, let’s divide the unit into 6 equally sized pieces.



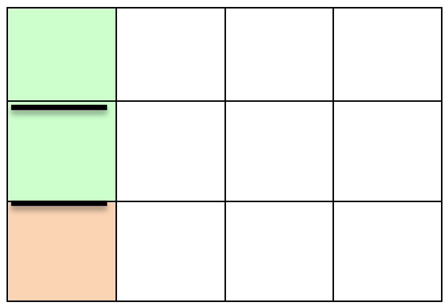
Notice that 1/2 of the 1/3 is in green. So we have $\frac{1}{2} x \frac{1}{3} = \frac{1}{6}$

I know what you are thinking, let’s look at another visual.

Let’s see what 2/3 of 1/4 would look like visually. So we will divide the unit (rectangle) into fourths and shade in ¼.



Now, we want 2/3 of that 1/4. So we will break the ¼ into 3 pieces, and we want two of those three.



So shading those 2 in green, and also creating equally sized pieces, we see that we have 2 pieces out of 12, or 2/12.

Looking at that mathematically, we have 2/3 of 1/4 $$\frac{2}{3} x \frac{1}{4} = \frac{2}{12}$$

Now, if we look at the last two examples and did a few more for good measure, we would see a pattern develop. First, we would realize that the word "of" translates to multiplication. Second, when we performed the multiplication, we multiplied numerator by numerator and denominator by denominator. For illustration purposes, I did not simplify the fraction in the last example.

Multiplying fractions is pretty straight-forward. So, we'll just write the algorithm for it, give an example and move on.

Algorithm for Multiplying Fractions

1. ***Make sure you have proper or improper fractions***
2. ***Cancel, if possible***
3. ***Multiply numerators***
4. ***Multiply denominators***
5. ***Simplify***

Example 1 $3\frac{1}{2}\ x\ \frac{4}{5}$

Since $3\frac{1}{2}$ is not a fraction, we convert it to $\frac{7}{2}$, we rewrite it as follows

$$\frac{7}{2}\ x\ \frac{4}{5}$$

Now what I'm about to say is important and will make your life a lot easier. We know how to simplify fractions, what we want to do now is to cancel with fractions. That's nothing more than simplifying using the commutative and associative properties (changing the order and grouping).

The numerator is $7\ x\ 4$, the denominator is $2\ x\ 5$. Writing that as a single fraction I have $\frac{7\ x\ 4}{2\ x\ 5}$. The Commutative Property of Addition allows me to change the order of the numbers. I will rewrite the numerator; $\frac{4\ x\ 7}{2\ x\ 5}$. Now, I can rewrite them as separate fractions using the Associative Property.

$\frac{4}{2}\ x\ \frac{7}{5}$, I can reduce $\frac{4}{2}$ to $\frac{4}{2}$ to $\frac{2}{1}$ and rewrite the problem as $\frac{2}{1}\ x\ \frac{7}{5}$. The answer is $\frac{14}{5} = 2\ \frac{4}{5}$.

Now rather than going through all those steps, using the commutative and associative properties, we could have taken a shortcut and cancelled.

$$\frac{7}{\cancel{2}_{\,1}}\ x\ \frac{\cancel{4}^{\,2}}{5}$$

To do that, we would look for common factors in the numerator and denominator and divide them out. In our problem, there is a common factor of 2. By dividing out a 2, the problem looks like this

$$\frac{7}{1}\ x\ \frac{2}{5} = \frac{14}{5} \qquad = 2\frac{4}{5}$$

Let's look at another one.

Example 2 $3\frac{3}{5} \; x \; \frac{5}{9}$

Rewriting the mixed number as a fraction, we have $\frac{18}{5} \; x \; \frac{5}{9}$.

We have a common factor of 5 in the numerator and denominator, we also have a factor of 9 in each. Canceling the 5's and the 9's, we have

$$\overset{2}{\cancel{\frac{18}{5}}} \; x \; \underset{1}{\frac{\cancel{5}}{\cancel{9}}}$$

The answer is 2.

\Multiply the following fractions.

1. $\frac{3}{4} \; x \; \frac{2}{5}$
2. $\frac{7}{8} \; x \; \frac{3}{4}$
3. $\frac{8}{9} \; x \; \frac{6}{7}$
4. $5\frac{1}{3} \; x \; \frac{3}{4}$
5. $\frac{1}{2} \; x \; 4\frac{3}{5}$
6. $2\frac{1}{2} \; x \; 3\frac{1}{4}$

Dividing Fractions

Before we learn how to divide fractions, let's revisit the concept of division using whole numbers. When I ask, how many 2's are there in 8. I can write that mathematically three ways.

$2\overline{)8}$ $\qquad \frac{8}{2}$ $\qquad 8 \div 2$

To find out how many 2's there are in 8, I will use the subtraction model:

$$\begin{array}{r} 8 \\ \underline{-2} \\ 6 \\ \underline{-2} \\ 4 \\ \underline{-2} \\ 2 \\ \underline{-2} \\ 0 \end{array}$$

Now, how many times did I subtract 2? Count them, there are 4 subtractions. So there are 4 twos in eight.

Mathematically, we say $8 \div 2 = 4$.

You want some good news, division has been already defined as repeated subtraction. That won't change because we are using a different number set. In other words, to divide fractions, I could also do repeated subtraction.

Example 3

$$1\frac{1}{2} \div \frac{1}{4}$$

Another way to look at this problem is using your experiences with money. How many quarters (1/4) are there in $1.50 (1 ½)? Using repeated subtraction, we have:

$1\frac{1}{2} \div \frac{1}{4}$, rewriting that with a common denominator, we have

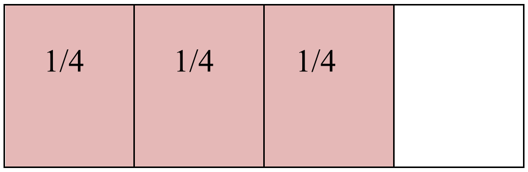
$$1\frac{2}{4} - \frac{1}{4} = 1\frac{1}{4}, \quad 1\frac{1}{4} - \frac{1}{4} = 1, \quad 1 - \frac{1}{4} = \frac{3}{4}, \quad \frac{3}{4} - \frac{1}{4} = \frac{2}{4}, \quad \frac{2}{4} - \frac{1}{4} = \frac{1}{4}, \quad \frac{1}{4} - \frac{1}{4} = 0$$

Note, I subtracted ¼ SIX times. That means there are 6 ¼'s in $1\frac{1}{2}$. Mathematically we write $1\frac{1}{2} \div \frac{1}{4} = 6$

Ready for a shortcut or would you rather subtract your brains out?

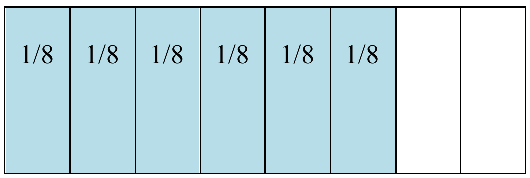
What you are asking when dividing 1 ½ by ¼ is, how many 1/4 (quarters) are there in 1 ½ dollars. You can see there are 6.

Example 4 How many 1/8 are in 3/4?



Dividing each of the ¼ ‘s in half, we get 1/8.

How many 1/8 fit into the pink area representing ¼’s?



Count, there are six $\frac{1}{8}$ ‘s in $\frac{3}{4}$, so $\frac{3}{4} \div \frac{1}{8} = 6$

Now, try, that same thing for how many 1/9 are in 1/3? $\frac{1}{3} \div \frac{1}{9}$

How about how many 1/10 are in 2/5? $\frac{2}{5} \div \frac{1}{10}$

If you do those subtractions or draw pictures and count, then look at the numbers in the division problems, you might see a pattern develop.

Well, because you enjoy playing with numbers, you found a quick way of dividing fractions. You did this by looking at fractions that were to be divided and they noticed a pattern. And here is what they noticed by just looking at the problem and the answer.

$$\frac{3}{4} \div \frac{1}{8} = 6 \qquad \frac{1}{3} \div \frac{1}{9} = 3 \qquad \frac{2}{5} \div \frac{1}{10} = 4$$

3x8 ÷ 4x1= 6 1x9 ÷ 3x1 = 3 2x10 ÷ 5x1 = 4

Notice, the pattern we developed worked for proper and improper fractions. So, our first step is to make sure we have fractions – no mixed numbers.

In this pattern, we are multiplying the numbers diagonally, then dividing. Another way of doing the same operation is to invert the divisor and multiply, we get the same result. And it works faster than if we did repeated subtractions, not to mention it takes less time and less space. Patterns sure do make life a whole lot easier, don’t you think?

Rather than multiplying diagonally and dividing, we will use the second pattern and invert the divisor and then multiply the numerators, then the denominators, we get the same result.

A reciprocal of a number, by definition, is just 1 divided by the number. It’s the multiplicative inverse. The reciprocal of 8 is 1/8. The reciprocal of ¾ is 1÷ 3/4 = 4/3. Many describe a reciprocal as a fraction upside-down. That is, the reciprocal of 3/5 is 5/3.

Algorithm for Dividing Fractions

1. Make sure you have fractions
2. Invert the divisor
3. Cancel, if possible
4. Multiply numerators
5. Multiply denominators
6. Simplify

Use the algorithm derived from the patterns to divide.

Example 5 $\frac{3}{4} \div \frac{1}{5} \longrightarrow \frac{3}{4} \times \frac{5}{1}$ Inverting the divisor and multiply

$= \frac{15}{4}$

$= 3\frac{3}{4}$

Example 6 $2\frac{1}{3} \div \frac{1}{4} \longrightarrow \frac{7}{3} \div \frac{1}{4}$ Converting to an Improper Fraction

$= \frac{7}{3} x \frac{4}{1}$ Inverting the divisor and multiply

$= \frac{28}{3}$

$= 9\frac{1}{3}$

1. $\frac{3}{4} \div \frac{1}{8}$

2. $\frac{3}{5} \div \frac{1}{10}$

3. $6 \div \frac{1}{2}$

4. $3\frac{1}{4} \div \frac{1}{2}$

5. $5\frac{1}{2} \div \frac{1}{4}$

6. $8 \div 2\frac{3}{4}$

Fractions

Name
Date

Definitions

1. ***Fraction

2. *Improper Fraction

3. ***Reciprocal

4. ***In the fraction 4/5, the 4 is called the ______________.

5. ***Write the algorithm for Adding/Subtracting Fractions.

6. ***Write the algorithm for Dividing Fractions.

7. **What method of find a common denominator might be most convenient to use when adding 5/18 and 7/24? Explain why you chose that method.

8. *If the numerator and the denominator of a proper fraction continually increase by 1, what happens to the value of the fraction?

9. *A student argued that any time you multiply numbers, the product is larger than the factors. What is his error and how would you explain the rationale behind the correct answer?

10. *Draw a model to show that 1/5 + 1/4 = 9/20

11-19 are ** questions

11. Simplify the following fractions.

a. 18/24 b. 45/72 c. 111/123

12.

$$\begin{array}{r} \frac{5}{8} \\ + \ \frac{3}{7} \\ \hline \end{array}$$

13. 9 1/5 – 5 2/3

14. $6 \times 3/5$

15. $4 \frac{1}{2} \div \frac{1}{4}$

16. Find the LCM and GCF of 54, 81, and 108.

17. **Order the following numbers from greatest to least.

2/3, 3/5, 5/8

18. Choose values of x from below so that $x < 5$

5, 0, 5/2, 11/2, 21/5

19. Choose values of x such that $4 \frac{3}{4} > x$

6 5/9, 7 1/8 , 3 1/3, – 5 ¼, 4 1/5

20. *Antonio has 4/5 of the money for the trip. His brother Peter has the rest of the money. If Antonio has $24, how much money does Peter have for the trip?

21. *Mike worked 10 ½ hours Monday, 8 ¼ hours on Tuesday, and 3 ¾ hours on Wednesday, how many hours did he work the three days?

22.SBAC *Ted needs 2 ¼ feet of ribbon to wrap each present. He has to wrap 15 presents. If ribbons come in 5 foot rolls, how many rolls of ribbon will Ted need to purchase to wrap all the presents?

23.SBAC *A person has $29\frac{1}{2}$ yards of material available to make uniforms. Each uniform requires $\frac{3}{4}$ yard of material. How many uniforms can be made? How much material will be left over?

24.* Bob owns five ninths of the stock in the family company. His sister Mary owns half as much stock as Bob. What part of the stock is owned by NEITHER Bob nor Mary?

25.* Joel worked $9\frac{1}{2}$ hours one week and $11\frac{2}{3}$ hours the next week. How many more hours and minutes did he work the second week than the first?

26. ***Write a home phone, cell phone, email or home address to contact your parent or guardian. (CHP)