Chapter 17 Logarithms

Sec. 1 Definition of a Logarithm

In the last chapter we solved and graphed "exponential equations." The strategy we used to solve those was to make the bases the same, set the exponents equal, and solve the resulting equation.

You might wonder what would happen if you could not set the bases equal?

Before we do that, I need to show you another way of writing an exponential. We all know $5^2 = 25$. Another way of writing that is: $\log_5 25 = 2$. The way we say that is the ***log of 25 with base 5 is 2*.**

A logarithm (abbreviated log) is an exponent. Say that a couple of times. A log is an exponent. A log is an exponent. So, when I look at the expression $2^3 = 8$, we know that a log is an exponent; log = 3. Now looking at the exponential, what is the base? It's 2. So, filling that information, we have $\log_2 8 = 3$.

Example 1 Rewrite $3^4 = 81$ as a logarithm.

Remember, a log is an exponent. What is the exponent?

So, we can say, log = 4 – an exponent.

The log of what is 4? The 3 is the base in the exponential.

Now we have the $\log_3 n = 4$. That means the log of some number with base 3 is 4. What's the number, 81. Therefore, we have $\log_3 81 = 4$

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Example 2 Rewrite $10^2 = 100$ as a logarithm.

Again, a log is a what? An exponent. Let's identify the exponent; 2 is correct. What's the base? Yes, the base is 10. Therefore, we have:

$$10^2 = 100$$

Let's fill it in, $\log_{10} 100 = 2$

The important thing to remember is: **A LOG IS AN EXPONENT.** Keep that in mind when you are working with logarithms and you'll just think they are peachy!

Let's say the same thing mathematically:

> If $\boldsymbol{a}$ denotes any positive real number and "b" any real number except 1, then there is a unique real number, called the logarithm of $\boldsymbol{a}$ with base "b" ($\log_b a$), which is the exponent in the power of "b" that equals $\boldsymbol{a}$; that is,
>
> $$\log_b a = n \text{ if and only if } a = b^n$$

Example 3 Rewrite $4^3 = 64$ as a log.

Remember a log is an exponent, log = 3. What's the base? 4 you say. OK, let's fill in the numbers.
We have $\log_4 64 = 3$.

Can you go backwards? In other words, if I gave you a logarithm, could you write it as an exponential? Sure, you could. Look at the patterns we used to go from an exponential to a logarithm.

Example: $\log_{10} 100 = 2$; → $10^2 = 100$
Example: $\log_{10} 1000 = 3$; → $10^3 = 1000$
Example: $\log_{10} .001 = -3$; → $10^{-3} = .001$
Example: $\log_5 25 = 2$; → $5^2 = 25$

Example 4 Rewrite $\log_2 8 = 3$ as an exponential.

Now remember, a log is an exponent. Therefore, our exponent is 3. What's the base? Yes, it's 2. We now have $2^3 = 8$.

Example 5 Rewrite $\log_{10} 10000 = 4$ as an exponential

$10^4 = 10,000$

Now, let's put this in perspective. All we are doing is learning to write an exponential as a logarithm and a logarithm as an exponential.

The reason for doing this is so we can solve equations that we might not have been able to solve otherwise.

Let's solve some simple problems involving logarithms.

Example 6 Find the value of x, $\log_x 81 = 4$

Converting to an exponential, we have $x^4 = 81$

$x = \sqrt[4]{81}$

$x = 3$

Example 7 Find the value of x, $\log_{16} 8 = x$

Converting to an exponential, we have $16^x = 8$

Changing the base $(2^4)^x = 2^3$

Simplifying $2^{4x} = 2^3$

Set the exponents equal $4x = 3$

$x = \frac{3}{4}$

***** If a base is not written, it is understood to be 10.**

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Example 8 Solve for x; $\log_9 27 = x$

$$9^x = 27$$

$$(3^2)^x = 3^3$$

$$3^{2x} = 3^3$$

$$2x = 3$$

the answer is $x = 3/2$

Problems involving logarithms can easily be converted to exponentials. We then solve those problems the way we did in the last chapter.

As usual, I can't make the problems more difficult, I can only make them longer. If you think about some of the rules you have learned previously about exponentials, then you'll have a good idea of what the rules will be concerning logarithms.

Since $\mathbf{b^{y1} = b^{y2}}$ **iff** $\mathbf{y_1 = y_2}$. That implies that $\mathbf{\log_b x_1 = \log_b x_2}$ **iff** $\mathbf{x_1 = x_2}$

Sec. 2 Inverse of an Exponential

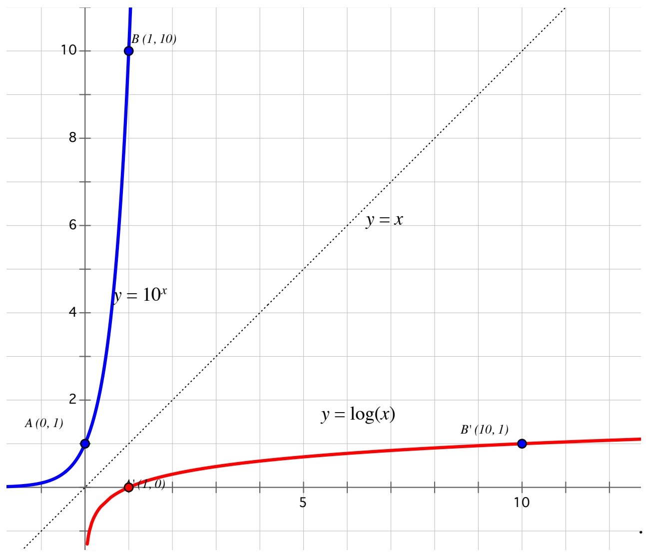
The inverse of the exponential equation, $\mathbf{y = b^x}$ is found by interchanging the domain and range, the x and y. So, the inverse of $\mathbf{y = b^x}$ is $\mathbf{x = b^y}$ which we now know can written as $\mathbf{f^{-1}(x) = y = \log_b x}$.

So, if we look at the graph of $y = 10^x$ and interchange the domain and range, x and y coordinates, we have the inverse $x = 10^y$ or $y = \log_{10}x$ as shown below.

Since the logarithm is the inverse of the exponential, we know the domain and range are interchanged and the graph is reflected over the graph of the line y = x. So, if you can graph $y = 10^x$, the graph of the logarithm will consist of the interchanged points of $y = 10^x$

What that means is this; if the graph $y = 10^x$ results in ordered pairs (0, 1), (1, 10), (2, 100) etc., then the graph of the inverse, the log will have (1, 0), (10,1), (100, 2) etc. as ordered pairs.

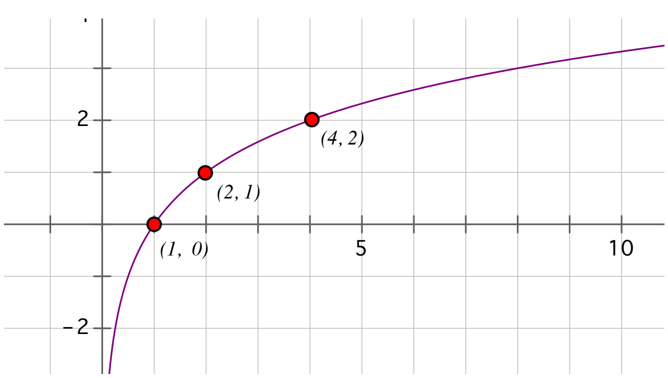
Let's look at these graphs. Notice, I ca graph the y = $\log_b x$ by just interchanging the x and y coordinates as mentioned earlier.
Notice A and A' and B and B'.



Sec. 3 Graphing Logarithms

As we did with exponentials, I am able to move the graph of logarithms around the coordinate system by graphing the parent function first**.**

For convenience of numbers and space, I'm going to use base 2 in the following log graphs. Here's what we know, the graph of a log will pass through (1, 0) and (1, 2)



$y = \log_2 x$

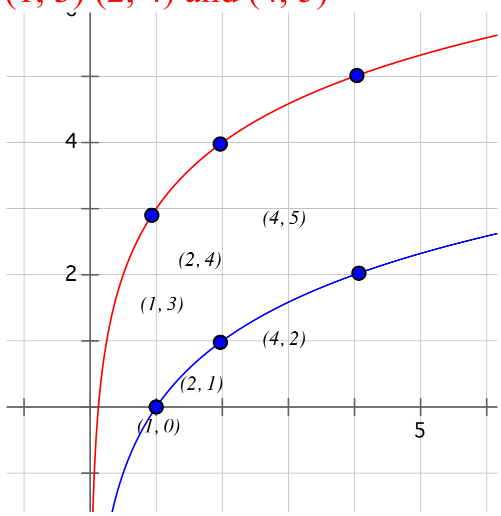
Now, those ordered pairs came from the inverse of $y = 2^{x}$, which is the log with base 2

(0, 1), (1, 2) (2, 4)
(1, 0), (2, 1), (4, 2)

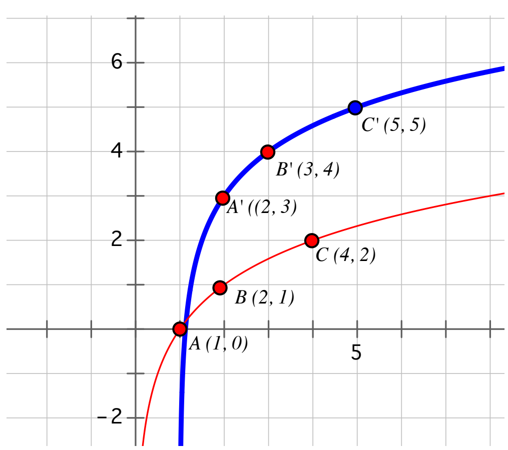
Now, if I use (1, 0), (2, 1), (4, 2) as the ordered pairs in my parent function with base 2, then I move the graph around just like I did with exponentials.

Example 1 Graph $y = \log_2 (x) + 3$

Based on our previous work, the graph of the parent function should all be moved up 3 units. That should result in ordered pairs
(1, 3) (2, 4) and (4, 5)



Example 2 **Graph $y = \log_2(x - 1) + 3$**



Geometrically, I moved each point of the parent graph over 1 to the right and up 3. From A to A', B ot B', and C to C'.

Algebraically, I added 1 to each x-coordinate and 3 to each y-coordinate.

Sec. 4 Rules for Logs

Now, let's leave graphing and go back and look at logarithms from an algebraic point of view. And remember, since logarithms are exponents, all the rules for exponents should apply to logarithms.

Rewriting the exponential and logarithmic in functional notation, we have

$$f(x) = b^x \text{ and } f^{-1}(x) = \log_b x.$$

We also know by the definition of inverse functions from a previous chapter that

$$f[f^{-1}(x)] = b^{\log_b x} = x.$$

Let's use that information and prove that:

Statements

Reasons

1.	$y = b^x = f(x)$	Given
2.	$x = b^y$	Interchange x and y
3.	$y' = \log_b x = g(x)$	Def of log
4.	$f(g(x)) = b^{\log_b x}$	Composition of fcts.
5.	Let $b^{\log_b x} = n = b^t$	Substitution
6.	$\log_b x = t$	Bases =, exponents =
7.	$\therefore\ b^t = x$	Def of log
8.	$b^{\log_b x} = x$	Substitution (5)

Turns out this is an important identity.

Using that identity, we can see the following if the base of the logs is 10 – called common logarithms:

$10^{\log a} = a$ $\quad\quad$ $10^{\log b} = b$ $\quad\quad$ $10^{\log ab} = ab$

So, let's use the above three equalities that are a direct application of

$b^{\log_b x} = x$ because they are inverse functions.

Using this relationship, we can derive rules for working with logarithms. Let's take a look at $10^{\log ab} = ab$ and see what we can develop for multiplication.

$10^{\log ab} = ab$	- Given
$= (10^{\log a})(10^{\log b})$	- Substitution
$= 10^{\log a + \log b}$	- Mult Rule Exp.
$= 10^{\log a + \log b}$	- Transitive Prop.
$\therefore \rightarrow \quad \log ab = \log a + \log b$	- Exp Equation

So, we can see the **$\log ab = \log a + \log b$.**

Therefore, we can say, *to find the logarithm of a product of positive numbers, you add the logarithms of the numbers.*

That follows our rules of exponents, when you multiply numbers with the same base, you add the exponents.

We can use a similar derivation to find the $\log \frac{a}{b}$.

Again, as a result these being inverse functions, we know that

$10^{\log a} = a$ $\quad\quad$ $10^{\log b} = b$ $\quad\quad$ $10^{\log a/b} = a/b$

Again, using the three equalities that are a direct application of

$b^{\log_b x} = x$, let's look at what we can develop for division.

$$10^{\log a/b} = a/b \qquad \text{- Given}$$

$$= \frac{10^{\log a}}{10^{\log b}} \qquad \text{- Substitution}$$

$$= 10^{\log a - \log b} \qquad \text{- Div Rule Exp.}$$

$$10^{\log a/b} = 10^{\log a - \log b} \qquad \text{- Transitive Prop.}$$

$$\log a/b = \log a - \log b \qquad \text{- Exp Equation}$$

So, we can see $\mathbf{\log \frac{a}{b} = \log a - \log b}$

Therefore, we can say, *to find the logarithm of a quotient of positive numbers, you subtract the logarithms of the numbers.*

That follows our rules of exponents, when you divide numbers with the same base, you subtract the exponents.

Another helpful rule in logarithms can be seen by raising them to a power.

Again, we know that $a = 10^{\log a}$. If each side is raised to the power of n, we have

$$a = 10^{\log a} \qquad \textbf{Given}$$

$$a^n = (10^{\log a})^n \qquad \textbf{Exponent Power Rule}$$

$$= 10^{n \log a} \qquad \textbf{Exp. Raise Power to Power}$$

$$(10^{\log a})^n = 10^{n \log a}$$

So, we can see $\mathbf{\log a^n = n \log a}$

Therefore, we can say, *to find the logarithm of a power, you multiply the logarithm by the exponent.*

That follows our rules of exponents, when you raise a power to a power, you multiply the exponents.

Sometimes it is helpful to change the base of a logarithm such as $\log_b n$ to a logarithm in another base.

Let $x = \log_b n$	
$b^x = n$	**- Def of log**
$\log_a b^x = \log_a n$	**- log of both sides**
$x\log_a b = \log_a n$	**- Power rule – logs**
$x = \dfrac{\log_a n}{\log_a b}$	**- Div Prop. Equality**
$\log_b n = \dfrac{\log_a n}{\log_a b}$	**- Substitution**

So, we can see to change the base of a logarithm, we have

$$\log_b n = \frac{\log_a n}{\log_a b}$$

Now, we can use these rules for logarithms to help us solve logarithmic equations.

1. $\log_b a = n$ **iff** $b^n = a$ 2. $\log_b x_1 = \log_b x_2$ **iff** $x_1 = x_2$
3. $10^{\log x} = x$
4. $\log ab = \log a + \log b$
5. $\log a/b = \log a - \log b$ 6. $\log a^n = n\log a$
7. $\log_b n = \log_a n / \log_a b$

Hopefully you see the rules for logarithms as extensions of the rules of exponents. And, when a base is not written, it is understood to be 10 – a common logarithm.

We can use these rules to expand or condense logarithm expressions.

Sec. 5 Rewriting Logarithmic Expressions

Using the above rules, specifically the product, quotient and power rules, we can rewrite the logarithms. For instance, the logarithm of a product can be written as a sum of logarithms.

Expand the following:

Example 1 $\log_{10} 100x = \log_{10} 100 + \log_{10} x$, by the product rule

by definition $\log_{10} 100 = 2$

substituting 2, $2 + \log_{10} x$

Example 2 $\log_{10} (1000/y) = \log_{10}1000 - \log_{10}y$

$= 3 - \log_{10}y$

Example 3 $\log_{10} xy^2/Z = \log x + \log y^2 - \log Z$

$= \log x + 2 \log y - \log Z$

Notice I did not write the base 10, it is understood to be 10. Also, notice this problem has multiplication, power, and quotient rules.

Now the question becomes, can I condense problems using the same rules. The answer is yes.

Rewrite as a single logarithm

Example 4 $\log n + \log 5 =$ This is an example of the product rule

$=$ $\log n5$ or $\log 5n$

Example 5 $3\log x - \log y =$ This appears to be the power and quotient

$= \log x^3 - \log y$

$= \log x^3/y$

Like everything else in math, I can't make these more difficult – only longer. You do need to know the rules!

Sec. 6 Solving Equations Containing Logarithms

There are essentially two types of logarithmic equations, equations that have logs on both sides and equations where there are logs only on one side of the equation. Remember, when no base is written, the base is understood to be 10.

Remember, if $b^{y1} = b^{y2}$ iff $y_1 = y_2$, that implies that $\log_b x_1 = \log_b x_2$ iff $x_1 = x_2$

2 Types of Log Equations

Type I. $\log_b x = \log_b y$, **then** $x = y$

Type II. $\log_b x = y$, **then** $b^y = x$

Using these 2 rules allows us to eliminates logs from equations, then we solve the resulting equations the way we have done in the past.

Strategy for Solving Logarithmic Equations

The strategy for solving logarithmic equations is to rewrite the equations as a single log on one or both sides of the equality.

A) If you have logs on both sides of the equation, then you use the rule under Type I above and eliminate the logs.

B) If you have logs on only one side of the equation, then you use the rule under Type II, to get rid of the log.

And, finally you solve the resulting equation.

Example 1. Solve for x: $\log(3x - 2) = \log(x + 6)$

Is this Type I or II? Since you have logs on **BOTH** sides of the equation, it's Type I. That results in the logs being eliminated in the equation. Base understood to be 10.

$$\begin{aligned} \text{So, } 3x - 2 &= x + 6 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

Example 2. Solve for x: $\log_4(x^2 - 9) = 2$

Is this a Type I or II problem? Since there a log ONLY on one side, then it is Type 2. Using the definition of log (rule 1 in the previous section), results in the log being eliminated.

$$\begin{aligned} \text{So, } x^2 - 9 &= 4^2 \\ x^2 - 9 &= 16 \\ x^2 &= 25 \\ x = 5 \quad &\text{or} \quad x = -5 \end{aligned}$$

Now, we know how to solve equations containing logarithms. There are only two methods, when we use them, the logs are no longer part of the equation.

As in all of math, I can not make these problems more difficult, but I can make them longer.

Looking at the next example, I have the sum of logs equal to a number. In order to solve the equation, I have to write the sum of the logs as a single log. We derived the $\log A + \log B = \log AB$ – the product rule. So, let's use it. Remember, when there is no base written, it is understood to be 10.

When a base of a logarithm is not explicitly written, it is understood to be base 10 – it's called the common logarithm. A logarithm with base e is called the natural logarithms and is written as $\ln x$ rather than $\log_e x$.

Example 3: Solve for n; $\log n + \log 5 = 1$

$\log n + \log 5 = 1$	Given
$\log 5n = 1$	Mult Rule – logs
$10^1 = 5n$	Def of log
$2 = n$	Div Prop =

Example 4: Solve for x; $3\log_5 x - \log_5 x = 2$

$3 \log_5 x - \log_5 x = 2$	Given
$\log_5 x^3 - \log_5 x = 2$	Power rule
$\log_5 (x^3/x) = 2$	Div Rule – logs
$\log_5 x^2 = 2$	Div – Exp
$5^2 = x^2$	Def of log
$\pm 5 = x$	Sq Root

We cannot take the log of a negative number, so $x = 5$

In both the preceding examples, there were logs on one side of the equation. This next one appears to be a little different. Please note, the logs on the left side of the equation have no base written, so the base is understood to be 10. On the right side the base of the log is written and it is 6.

Example 5: Solve for x; $\log (x - 1) + \log (x + 2) = \log_6 6$

$\log (x - 1) + \log (x + 2) = \log_6 6$	Given
$\log (x - 1)(x + 2) = 1$	Mult Rule/ Def of log

Why is $\log_6 6 = 1$?

$10^1 = (x - 1)(x + 2)$	Def of log
$10 = x^2 + x - 2$	Mult polynomials
$0 = x^2 + x - 12$	Sub Prop =
$0 = (x + 4)(x - 3)$	Factoring
$x = -4$ or $x = 3$	Solving

x cannot be equal to -4, so $x = 3$ — Domain of logs

Example 6: Solve; $\log_8 (m + 1) - \log_8 m = \log_8 4$

To solve, I need to write the difference of the logs as a single logarithm. Fill in the reasons.

$\log_8 (m + 1) - \log_8 m = \log_8 4$	Given
$\log_8 \frac{m+1}{m} = \log_8 4$	_______________
$\frac{m+1}{m} = 4$	_______________
$m + 1 = 4m$	_______________
$1 = 3m$	_______________
$1/3 = m$	_______________

So far, we have looked at equations that had a log on one or both sides. So, the question we posed in the last chapter on exponentials, what happens if you can't change the base to set the exponents equal?

The Natural Logarithm is a logarithm with base ***e***. As we have previously stated, the common logarithm is a logarithm with base 10. The 10 is typically not written. With natural logs, logs with base ***e***, they could be written as $\log_e x$. But, the preferred notation is ln x, where ln stands for the natural log.

Let's take a look at this exponential equation.

Example 7 Solve for x; $7^x = 9$

I clearly cannot make the bases (7 & 9) the same. So, I will take the log of each side. The base does not matter.

$\ln 7^x = \ln 9$	
$x \ln 7 = \ln 9$	Power Rule
$x = \frac{\ln 9}{\ln 7}$	Div. Prop =

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$$x = \frac{\ln 9}{\ln 7} = \frac{2.197}{1.945} = 1.129$$ Look upthose values

So now we are able to solve an exponential when we could not change the base.
As usual, I cannot make math more difficult, but I can make the problem longer. Let's look at the next example. In the last example, I indicated the base you choose does not matter. In this example, again the base you choose does not matter, but….

Since I have a number with a base of 10, I will use a log in base 10, a common log because that will make the problem easier.

Example 8 Solve for x, $10^{5-x} = 8$

$10^{5-x} = 8$	Given
$\log 10^{5-x} = \log 8$	Take log both sides
$(5-x) \log 10 = \log 8$	Power Rule
$(5-x)(1) = \log 8$	$\log_a a = 1$
$5 - x = \log 8$	
$5 - \log 8 = x$	Subtract Prop =
$5 - .9030 \approx x$	Look up the log 8
$4.0969 \approx x$	

In the last problem, I noticed I had an exponential with base 10. To use our rules to make the problem easier, I decided to use logs with base 10. In this next problem, I see have have an exponential with base ***e***, I'll use **ln** (the natural log) to make the problem simpler using the sam logic.

Example 9 Solve for t; $e^{t+6} = 2$

$e^{t+6} = 2$	Given
$\ln e^{t+6} = \ln 2$	Take log
$(t + 6) \ln e = \ln 2$	Power Rule
$(t + 6)\, 1 = \ln 2$	$\ln e = 1$
$t + 6 = \ln 2$	Dist Prop
$t = \ln(2) - 6$	Sub Prop of =
$t = .0931 - 6$	Look up value
$t \approx -5.3068$	Arithmetic

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Usually, we work with logs with base 10. However, as we have seen from the last two examples, choosing the base of a log should be determned by the problem. It just makes math easier.

In this next problem, I have 2 exponentials with different bases. So choosing a log isn't going to have much impact on my work. I'll use the natural log, ln, for no other reason than we just introduced it.

Example 10 Solve for y; $2^{4y+1} - 3^y = 0$

$2^{4y+1} - 3^y = 0$	Given -
$2^{4y+1} = 3^y$	Add Prop =
$\ln 2^{4y+1} = \ln 3^y$	log both sides
$(4y+1)\ln 2 = y\ln 3$	Power Rule
$4y\ln 2 + \ln 2 = y \ln 3$	Distribitive Prop
$4y\ln 2 - y\ln 3 = -\ln 2$	Variables one side; Sub Prop =
$y(4\ln 2 - \ln 3) = -\ln 2$	Factor; Didtributive Prop
$y = \dfrac{-\ln 2}{4\ln 2 - \ln 3}$	Div Prop =

$$y = \frac{-\ln 2}{4\ln 2 - \ln 3} = \frac{-.6931}{4(.6931) - 1.0986} = .-.41407$$

I chose to take the ln, could have used a log with a different base.

Example 11 Solve for x; $5e^{2x+4} = 8$ in terms of ln

$5e^{2x+4} = 8$	Given
$e^{2x+4} = \frac{8}{5}$	Div Prop =
$\ln e^{2x+4} = \ln \frac{8}{5}$	ln both sides
$2x + 4 = \ln(8/5)$	$\log_e e$
$2x = \ln(8/5) - 4$	Sub Prop =
$x \approx \frac{1}{2}(.4700 - 4)$	Div Prop =
$x \approx -1.7649$	

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Let's use log equations to solve problems.

Example 1. Acidity is defined by the formula pH = –log[H+], where [H+] is the hydrogen ion concentration measured in moles per liter. If the pH is less than 7, then it's considered acidic. If greater than 7, its basic and if it measures 7, then its neutral. If a solution was tested and the hydrogen ion concentration given by [H+] = .0003, find the pH value and determine if it is basic or acidic.

Evaluating pH when [H+] is .003, we have pH = – log[H+]

$$pH = -\log(.0003)$$
$$pH \approx 3.523$$

That's less than 7, so the solution is acidic.

Example 2 Using acidity as defined by the formula pH = –log[H+], where [H+] is the hydrogen ion concentration measured in moles per liter, testing a solution of ammonia with [H+] =1.3 x 10^{-9},determine if it's acidic or basic.

$$\begin{aligned} pH &= -\log(1.3 \text{ x } 10^{-9}) \\ &= -\log 1.3 + \log 10^{-9} \\ &= -(\log 1.3 - 9\log 10) \\ &\approx \ +8.\ 8861 \end{aligned}$$

Since that pH is greater than 7, it's basic.

Example 3 The formula for loudness is given by dB = 10log (I/I_o). dB is decibels, Io is sound that can be barely heard and I is how more times intense than the initial sound being barely heard. If a cat's purr is 316 times as intense as a threshold sound, find the decibel rating.

$$\begin{aligned} dB &= 10 \log (I/I_0) \\ &= 10 \log (316 \text{ Io} \div \text{Io}) \\ &= 10 \log (316) \\ &\approx 24.9969 \end{aligned}$$