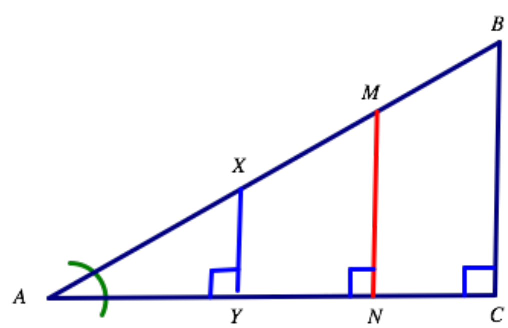
Ch. Y Intro Right Triangle Trig

In our work with similar polygons, we learned that, by definition, the angles of similar polygons were congruent and their sides were in proportion - which means their ratios are the same. We also learned the Angle-Angle Postulate. That is, if two angles of one triangle were congruent to two angles of another triangle, the triangles were similar.

As we look at the ratios of the sides of similar triangles, we are going to use that information and name some of those ratios - an introduction to right triangle trigonometry. Look at the three triangles below - ΔABC, ΔAMN and ΔAXY. They are similar because of the AA Postulate. Each triangle has a right angle and each triangle contains $\angle A$.



That means BC:MN have the same ratio as MN:XY. In fact, I could continue the relationships by saying BC:AB as MN:AM as XY:AX.

Let me write some of those relationships in words: the side opposite $\angle A$ to the hypotenuse of each of the triangles, ΔABC , ΔAMN and ΔAXY have the same ratio.

$$\frac{BC}{AB} = \frac{MN}{AM} = \frac{XY}{AX}$$

This is kind of important, so we are going to name that ratio. The side opposite an angle over the hypotenuse will be called the sine ratio, abbreviated sin, of that angle.

So, sin A is equal to any of those ratios; $\frac{BC}{AB} = \frac{MN}{AM} = \frac{XY}{AX}$

If we were to look at other ratios based on the above triangles, we would also realize because of similar triangles, these ratios are also equal; $\frac{AC}{AB} = \frac{AN}{AM} = \frac{AY}{AX}$

Putting that ratio in words, the side adjacent to the angle over the hypotenuse will be called the cosine, abbreviated cos, of that angle.

So, the cos A is equal to any of those ratios; $\frac{AC}{AB} = \frac{AN}{AM} = \frac{AY}{AX}$

And one more set of ratios that would be equal are the ratio of the side opposite that angle over that adjacent side. That ratio is called the tangent, abbreviated tan.

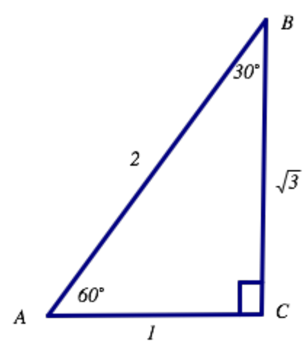
Summarizing those names, we have

$\sin A = \frac{opposite}{hypotenuse}$ $\qquad$ $\cos A = \frac{adjacent}{hupotenuse}$ $\qquad$ $\tan A = \frac{opposite}{adjacent}$

A popular way to remember those ratios is by memorizing **SOHCAHTOA**.

The S stands for sine, O for opposite side, H for hypotenuse, C for cosine, A for adjacent side and T for tangent.

Example 1 Using a 30-60-90˚ special right triangle, find the sine, cosine and tangent of 30˚



Using SOHCAHTOA

$$\sin 30^\circ = \frac{opposite}{hypotenuse} = \frac{1}{2}$$
$$\cos 30^\circ = \frac{adjacent}{hypotenuse} = \frac{\sqrt{3}}{2}$$
$$\tan 30^\circ = \frac{opposite}{adjacent} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Example 2 Using the same 30-60-90˚ special right triangle, find the tan 60˚ and the cos 60˚.

Using SOHCAHTOA

tan 60˚ = $\frac{opposite}{adjacent} = \frac{\sqrt{3}}{1} = \sqrt{3}$

cos 60˚ = $\frac{adjacent}{hypotenuse} = \frac{1}{2}$

Using your knowledge of special right triangles, you should be able to find the sine, cosine and tangent ratios for 30˚, 45˚ and 60˚ angles. The ratios for other angles would have to be looked up in your book or found on your calculator.

We just said, based on our knowledge of special right triangles, the cos 60˚ = 1/2. If you were to use a table or your calculator, you would find the cos 60˚ = .500.

To find the tan 42˚, use either your calculator or book and you will find that ratio of the opposite side to the adjacent side is always 0.9004. Because of similar triangles and the sides being in proportion, the ratio of the opposite side : adjacent side is called the tangent and the tangent of 42˚ will always be equal to 0.9004.

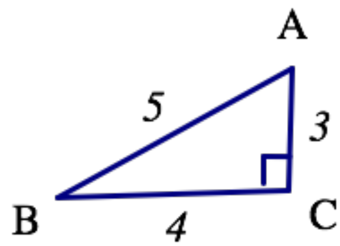
Example 3 Use the tables in your calculator or in your book to find the sine, cosine and tangent of the following angles.

1. 30˚
2. 60˚
3. 40˚
4. 63˚

Find the measure of the angle with the following values (ratios).

5. sin A = .5150
6. cos B = .7071
7, tan C = 1.3270

Example 4 Let's find the sin B, cos B, and the tan B



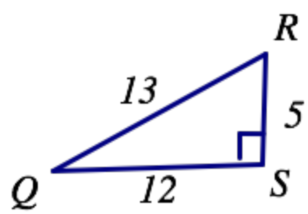
Remember SOHCAHTOA.

sin B = 3/5 cos B = 4/5 tan B = 3/4

Was that hard? Of course not. Using the same triangle, find the sin A, cos A, and tan A.

sin A = 4/5 cos A = 3/5 tan A = 4/3

Example 5 Find the sin, cos, and tan for angles Q and R.



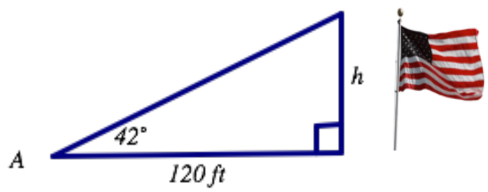
Using SOHCAHTOA

sin Q = 5/13 sin R = 12/13
cos Q = 12/13 cos R = 5/13
tan Q = 5/12 tan R = 12/5

In these examples, we have just asked you to find the ratios given a specific angle. To solve problems, you will have to determine which trig ratio is most appropriate based on the information provided.

Example 6 A boy standing on level ground notices (or can measure) that the angle he has to look up (angle of elevation) to see the top of a flagpole is 42 degrees. He can also measure the distance he is from the pole and finds it to be 120 feet. How high is the flagpole?

We'll call the height of the flagpole "*h*." Let's look at a picture and fill in any information we have



Now, how can we determine the height? Well, since we are studying trig, let's hope we use one of the trig ratios. Which one do we use? That's the question.

The sine is the opposite over the hypotenuse; we don't know the opposite or the hypotenuse. So, we won't use the sine. Cosine is the adjacent over the hypotenuse; that does not give the height. So, I won't use that. Tangent is the opposite, which is *h*, over the adjacent. There, I know the adjacent is 120 and I'm looking for the opposite (height). Therefore, I will use the tangent ratio.

I will have to look up the tan 42 on my trig table.
The tan 42˚ = .9004

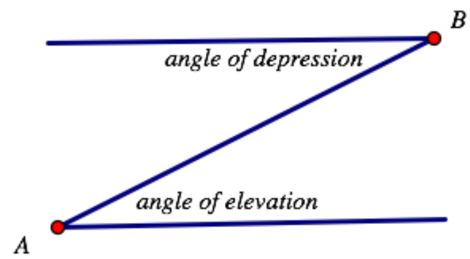
$$\tan 42^\circ = \frac{opposite}{adjacent} = \frac{h}{120}$$

$$.9004 = \frac{h}{120}$$

$$108 \approx h$$

So, the height of the flagpole is approximately 108 feet.

In a number of applications, professionals use terms like "angle of elevation" or "angle of depression"

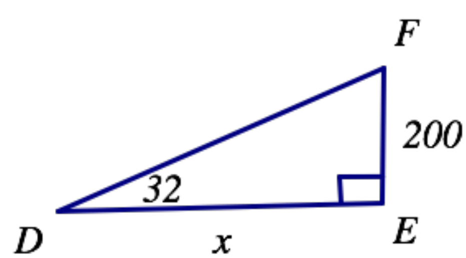


Line segment AB is the line of sight, the other two lines are horizontals. Notice how those two angles are defined with the horizontals. If you are looking up, that's an angle of elevation. Looking down is an angle of depression.

To solve problems involving the trigonometric ratios, it's always wise to think first, then set up a ratio. As we have said before, the choices we make can either make math easier or more cumbersome.

So, we have to ask, given certain information, am I going to use the sine cosine or tangent ratios and from what angle? So, thinking first is important.

Example 7 Find the measure of DE in the diagram.



Using the trig ratios, we are working with 2 sides and an angle. To solve problems, we need to know 2 of those 3 and find the unknown.

In this problem, we have 2 sides and an angle. Notice that the hypotenuse is NOT one of the sides given. For me, that eliminates using the sine or cosine ratios.

So, that leaves me with the tangent ratio, do I want to use angle D or F? Remember, if $\angle D = 32^\circ$, then $\angle F = 58^\circ$. The sum must be 90°.

So now decide, which is an easier equation to solve, A or B?

$$A. \quad \tan 32^\circ = \frac{200}{x} \qquad or \qquad B. \quad \tan 58^\circ = \frac{x}{200}$$

In either case, I have to use a table or calculator to find the respective tangent values. However, since x is in the numerator of the B, I would choose to multiply, rather than divide.

$$\tan 58^\circ = \frac{x}{200}$$

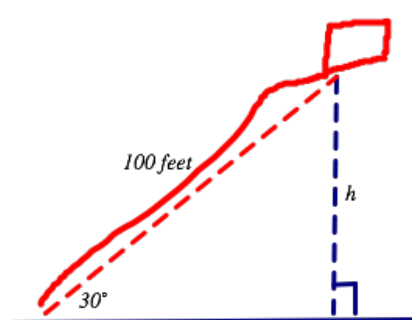
$\tan 58^\circ \approx 1.6003$ (using a calculator)

$$1.6003 = \frac{x}{200}$$

$$x \approx 200(1.6003) \approx 320$$

Example 8 The length of a kite string is 100 feet long. How high is the kite if the kite make a 30˚ angle with the round?

The first best thing to do is draw a picture and label what we can.



Drawing the picture, you can see I straightened out the string and the height of anything is always denoted by perpendicular lines to form a right triangle.

Now, the question is; which trig ratio is best to use? To solve a problem with a trig ratio, I need to know 2 of 3 things; sides and angles.

In this case, the sin 30˚ is something I know from applications of special triangles or I could look it up.

$$\sin 30^\circ = \frac{opposite}{hypotenuse}$$

Substituting sin 30˚ = 1/2 and the hypotenuse is 100 ft.

$$\frac{1}{2} = \frac{h}{100}$$

$$2h = 100$$
$$h = 50$$

The tip of the kite string is 50 feet above the ground.

Let's examine what we know. All trig is in the study of the ratios of the sides of the right triangle. Those ratios are given names; sine, cosine, tangent, cosecant, secant, and cotangent. To know which ratio we are talking about, we stand on the vertex of the angle and use the acronym SOHCAHTOA. By looking at different right triangles and knowing the ratios are the same for equal angles (using SIMILAR TRIANGLES) we are able to create trig tables for all the angles that will give us the ratio of those sides. Those ratios are written in decimal form.

Don't you just love this stuff? You are probably thinking "trig is my life." Try these next two problems. In the first problem, the angle of depression is given. You need to know how to draw that - see page 6. We should not need tables because we are working with a 60˚ angle - use the special right triangle relationships.

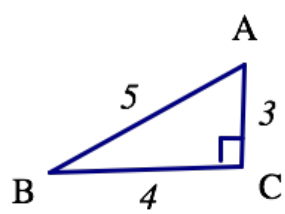
1. From the top of a building 300 feet high, you see a parked car on a bridge, if the angle of depression is 60˚, how far is the car from the building?

In this problem, we don't know the ratio of sides for an angle measuring 20˚, s we will have to use the book or calculator to determine that value based on the trig ratio you chose.

2. A pilot reads the angle of depression from his position 5000 feet in the air is 20˚, how far is the plane from the airport?

Sec. 2 Angle Relationships

In the previous section, we identified the trig values of angles when solving problems. In example 4, we had a right triangle shown below with the following relationships.



sin B = 3/5	cos B = 4/5	tan B = 3/4
sin A = 4/5	cos A = 3/5	tanA = 4/3

Notice the color coding. The sin A = 4/5 and the cos B = 4/5 and the sin B = 3/5 and the cos A also equals 3/5. Interesting?

Also notice, since ΔACB is a right triangle, then ∠A and ∠B are complementary – their sum is 90˚.

That means if we know one of the acute angles is 50˚, then the other angle must be 90 – 50 or 40˚.

Now, let's combine those two ideas. sin B = 3/5 and the cos A = 3/5. Using substitution and letting ∠A = 90 – B, we have sin B = cos (90 – B).

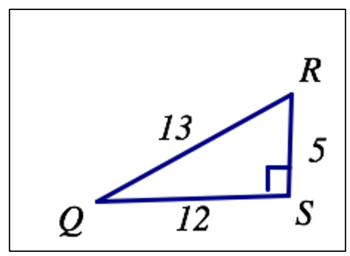
Using the same reasoning, we have sin A = cos (90 – A).

If we looked further at these relationships, we would also see if we placed the value of the sine over the value of the cosine, we get the value of the tangent. Let's look.

$\tan B = \frac{3}{4}$, placing the $\frac{\sin B}{\cos B} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$, the same as the tan B. Therefore, we can have the identity:

$$\tan B = \frac{\sin B}{\cos B}$$

From Example 5, we had the following relationships:



sin Q = 5/13	sin R = 12/13
cos Q = 12/13	cos R = 5/13
tan Q = 5/12	tan R = 12/5

We can see the sin Q = cos R, another way to write R is (90 – Q) or

sin Q = cos (90 – Q).

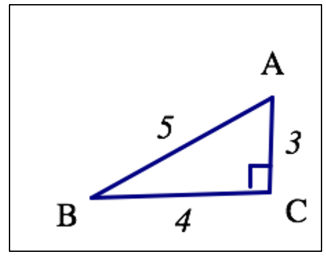
We can also see the cos Q = sin R, another way to write R is (90 – Q) or

cos Q = sin (90 – Q).

And the $\tan Q = \frac{\sin Q}{\cos Q}$

If we continued to play with these angles, we will continue to see more and more relationships.

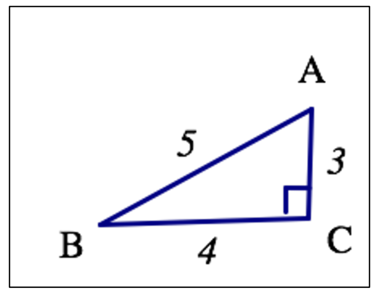
Previously using the triangle on the right, we said the sin B = $\frac{3}{5}$ and the cos B = $\frac{4}{5}$. If I find the sum of those trig ratios squared, we have $\sin^2 B + \cos^2 B = (\frac{3}{4})^2 + (\frac{4}{5})^2 = 1$. If we continued looking at the other ratios from previous examples, we would see a pattern develop. That is



$$\mathbf{\sin^2 x + \cos^2 x = 1}$$

Later, we will see, using a unit circle how this important identity was derived.

And, using the same triangle we have been using, we can see there are a total of 6 ratios that can be made formed by the sides. The second set of three ratios are the reciprocals of the first three.



$\frac{3}{5}$ $\frac{3}{5}$ $\frac{3}{4}$

$\frac{5}{3}$ $\frac{5}{3}$ $\frac{4}{3}$

As we defined the sine, cosine and tangent ratios, we will define the reciprocals as the cosecant, secant and cotangent.

The cosecant x, abbreviated csc x, is equal to the reciprocal of the sine x.

The secant x, abbreviated sec x, is equal to the reciprocal of the cosine x.

The cotangent x, abbreviated cot x, is equal to the reciprocal of the tangent.

Mathematically, we write

$\csc x = \frac{1}{\sin x}$ $\sec x = \frac{1}{\cos x}$ $\cot x = \frac{1}{\tan x}$

So, if the sin 30˚ = ½, then the csc 30˚ = 2/1 or 2. Piece of cake, right?

Ch. Y1 Solutions of Triangles

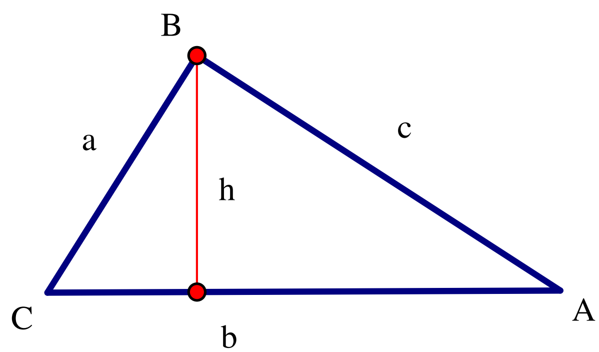
We have studied right triangle trigonometry and learned how to find sides and angles of right triangles. Well, a question that one might ask is what happens when we want to find sides or angles of triangles that are not right triangles. An oblique triangle is a triangle that does not contain a right triangle.

Good news, we can always divide an oblique triangle into right triangles. But, that can be a little cumbersome. So, we will use our knowledge of right triangles and develop some relationships that will work for oblique triangles.

Law of Sines

If two angles and a side of a triangle are known, or if two sides and an angle opposite one of them is known, then the remaining parts can be found.

Given ΔABC, construct an altitude (h), from $\angle$B to $\overline{AC}$



Drawing the altitude forms 2 right triangles. Using sin A = $\frac{opp}{hyp}$,

we have

$$\sin C = \frac{h}{a} \quad \text{and the} \quad \sin A = \frac{h}{c}$$

Solving both equations for h, $h = a \sin C$ and $h = c \sin A$

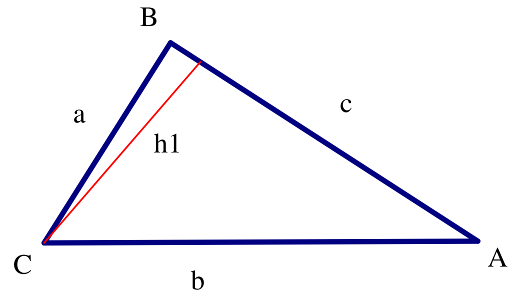
Using substitution $a \sin C = c \sin A$

Divide both sides by (sin C)(sin A)

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Using the same ΔABC, construct an altitude h_1 from $\angle C$ to $\overline{AB}$

Go through the same process for the right triangles formed. Note $h \neq h_1$



$\sin A = \dfrac{h_1}{b}$ $\quad\quad$ $\sin B = \dfrac{h_1}{a}$

Solving for h_1 $\quad$ $\boldsymbol{h_1 = b \sin A}$ **and** $\boldsymbol{h_1 = a \sin B}$

Using substitution $\quad$ **b sin A = a sin B**

Divide both sides by (sin A)(sin B)

We have already shown that $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$, so using the Transitive Property we have

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

That gives us the **Law of Sines**

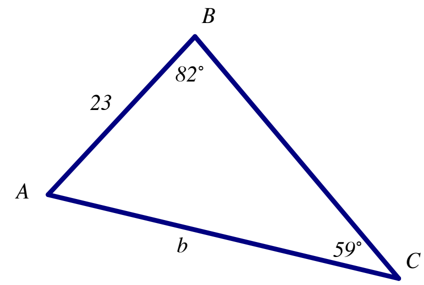
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The ratio of any side of a triangle to the sine of the opposite angle is a constant.

As we said earlier, we use the Law of Sines under a a couple of conditions

A) If two angles and a side of a triangle are known
B) If two sides and an angle opposite one of them are known

Example 1 Given the ΔABC, find *b*.



Using the Law of Sines

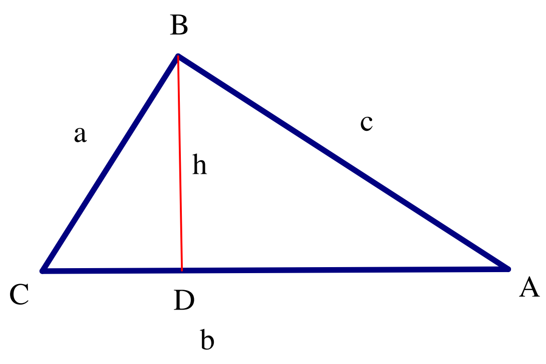
$$\frac{23}{\sin 59^\circ} = \frac{b}{\sin 82^\circ}$$

$$b\sin 59^\circ = 23\sin 82^\circ \quad \text{(look up values for sine)}$$
$$b\,(.86) \approx 23\,(.99)$$
$$b \approx 26.5$$

Law of Cosines

If two sides and the included angle or if three sides of a triangle are given, the Law of Sines can not be applied directly.

Given ΔABC, construct an altitude (h), from $\angle$B to $\overline{AC}$. That altitude forms two right triangles that allows us to use the trig.

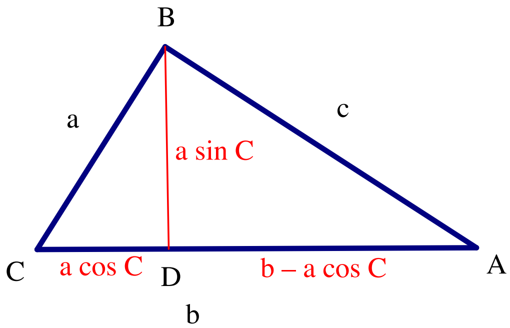


$$\sin C = \frac{h}{a} \quad \text{or} \quad h = a\sin C$$
$$\cos C = \frac{adj}{hyp} \rightarrow \cos C = \frac{CD}{a}$$
$$\therefore \text{ CD} = a\cos C$$

By the diagram AC = b and we just found that CD = a cos C, then

AD = b – *a cos C*

We will use that information to relabel our triangle ΔABC 14



ΔBDA is a right triangle, we can use the Pythagorean Theorem to write an equation

$$(a \sin C)^2 + \underline{(b - a \cos C)^2} = c^2$$

Squaring the binomial - underlined

$$a^2 \sin^2 C + \underline{b^2 - 2ab \cos C + a^2 \cos^2 C} = c^2$$

Rewriting the equation using the Commutative and associative properties that will result in a trig identity

$$a^2 \sin^2 C + a^2 \cos^2 C + b^2 - 2ab \cos C = c^2$$

Factoring a^2 out of the first two terms

$$a^2 (\sin^2 C + \cos^2 C) + b^2 - 2ab \cos C = c^2$$

Substituting 1 for $\sin^2 C + \cos^2 C$

$$a^2 (1) + b^2 - 2ab \cos C = c^2$$

$$a^2 + b^2 - 2ab \cos C = c^2$$

OR

Law of Cosines

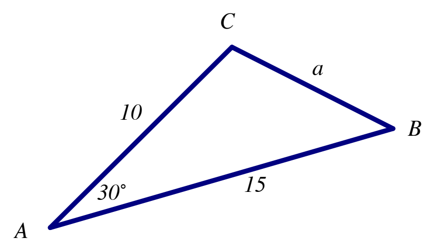
$$\mathbf{c^2 = a^2 + b^2 - 2ab \cos C}$$

In any triangle the square of a side equals the sum of the squares of the other two sides decreased by twice the product of those sides and the cosine of the included angle.

$$\mathbf{a^2 = b^2 + c^2 - 2bc \cos A}$$

$$\mathbf{b^2 = a^2 + c^2 - 2ac \cos B}$$

Example 2 Given ΔABC, find the value of *a*.



Using the Law of Cosines

$a^2 = b^2 + c^2 - 2bc \cos A$

$a^2 = 10^2 + 15^2 - 2(10)(15)\frac{\sqrt{3}}{2}$

$a^2 \approx 100 + 225 - 300(.86)$

$a^2 \approx 325 - 258$

$a^2 \approx 67$

$a \approx 8.2$

Example 3 Given three sides of a Δ, $a = 15$, $b = 17$, and $c = 19$, find $m\angle C$, the side opposite side *c*.

$c^2 = a^2 + b^2 - 2ab \cos C$

$19^2 = 15^2 + 17^2 - 2(15)(17) \cos C$

$361 = 225 + 289 - 510 \cos C$

$361 = 514 - 510 \cos C$

$-153 = -510 \cos C$

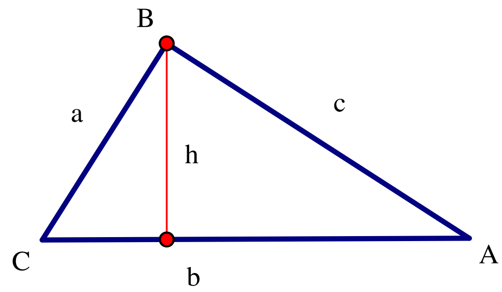
$-153/-510 = \cos C$

$0.3 \approx \cos C$

$73^\circ \approx \angle C$

Area of a Triangle

Let's use ΔABC, and the area formula for a triangle; $A = \frac{1}{2}bh$



$$\sin C = \frac{h}{a} \quad \rightarrow \quad h = a \sin C$$

$$\text{Area}_\Delta = \frac{1}{2}(AC)h$$

$$\text{Area}_\Delta = \frac{1}{2}(b)a\sin C$$

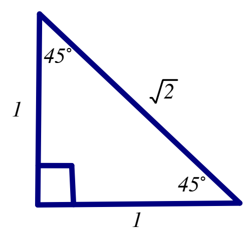
$$\text{Area}_\Delta = \frac{1}{2}ab\sin C$$

Angular Rotations

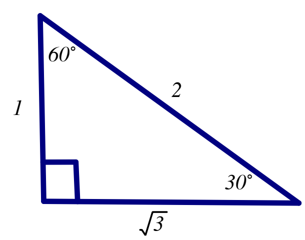
This unit is built upon your knowledge and understanding of the right triangle trigonometric ratios. A memory aid that is often used was SOHCAHTOA.

$$\sin x = \frac{opposite}{hypotenuse} \quad \cos x = \frac{adjacent}{hypotenuse} \quad \tan x = \frac{opposite}{adjacent}$$

In addition, we then learned about special right triangles. Those special right triangles allowed us to find the sides of triangles without using the Pythagorean Theorem. The two special right triangles were the 45-45-90 and the 30-60-90 degree triangles shown below.



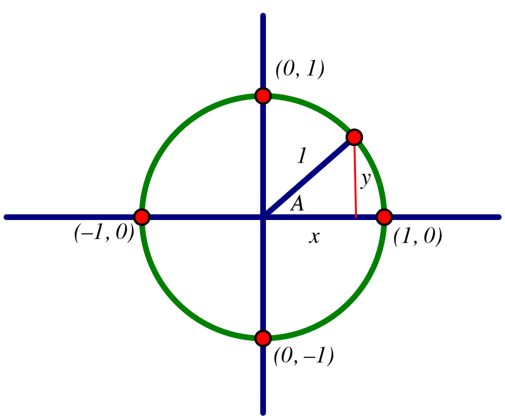
45-45-90° Δ
Hypotenuse = √2 leg



30-60-90° Δ
Hypotenuse = 2 short leg,
long leg = √3 short leg

Using SOHCAHTOA, we see the sin 30° = 1/2, sin 60° = √3/2 and sin 45° = 1/√2

Now, if I extend the trig ratios to the rectangular coordinate system and draw a circle of radius 1 around the origin, we will see how those coordinates are related to the special right triangles.



Using SOHCAHTOA on the unit circle

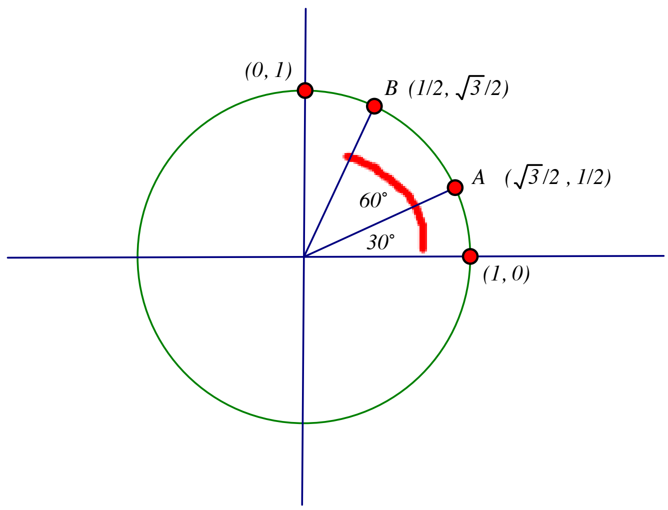
sin A = y/1 or sin A = y

cos A = x/1 or cos A = x

Putting this altogether, we have the trig ratios from the two special right triangles, the cosine and sine being defined on the unit circle (r =1) as the x and y coordinates respectively, and understanding that because the triangles that make up the trig ratios are similar, the ratios are equal.

(x, y) —-≥ (cosine, sine)

Using the 30-60-90˚ triangles above, we know the sin 30˚ = 1/2 and the cos 30˚ = √3/2. We also see the sin 60˚ = √3/2 and the cos 60˚ = 1/2.

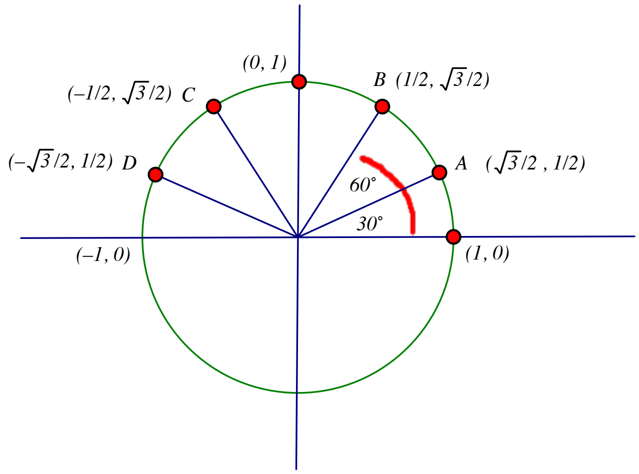


The terminal side of the 30˚ angle intersects the circle at A. How was I able to label the ordered pair (√3/2, 1/2)? Well, I looked at the special right triangle to come up with the ratios and based on the previous drawing, we **defined the cosine as the x-coordinate and the sine as the y-coordinate** using SOHCAHTOA.

The same is true for the 60˚ angle whose terminal side interests the unit circle at B. Using the special right triangle, the cos 60˚ = 1/2, the sin 60˚ = √3/2 and I write them as an ordered pair. To help you remember the ordered pairs, my suggestion is the “c” in cosine comes before the “s” in sine as the “x” comes before the “y”.

Notice the graph of the unit circle intersects the y-axis at (0, 1). Therefore, the cos 90˚ (the x-coordinate) is 0, the sin 90˚ (y-coordinate) is 1.

Either using “reflections” across the y-axis and our understanding of similar triangles, we can come up with more coordinates in the second quadrant.

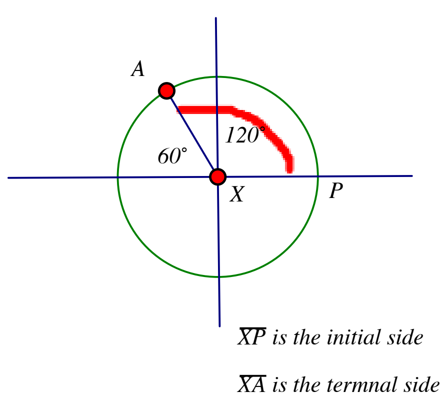


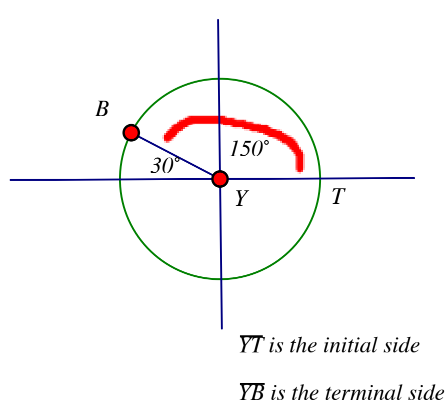
Notice the ordered pairs in the second quadrant, the only thing that seems to change is the sign of the x-coordinate. Point C would represent 120˚ and point D would represent 150˚. So, through symmetry we can find those ordered pairs.

Reference Angles

Unfortunately, we don't have special triangles that have 120 or 150˚ angles. But, if we use our knowledge of similar triangles, we can see an angle of 120˚ corresponds to a triangle of 60˚. We can also see the 150˚ angle would correspond to a 30˚ angle in the similar triangles.

Another way of saying this is the terminal side of a 150˚ angle is 30˚ above the x axis. The terminal side of the 120˚ angle is 60˚ above the x-axis. That would suggest that the coordinates of a 150˚ and 30˚ angle would be the same - except for the sign of the x-coordinate.





We can continue this process into the third and fourth quadrants and see the pattern of the coordinates with only sign changes. That means we can look at a 210˚ angle and realize that is 30˚ below the x-axis, a 240˚ angle is 60˚ below the x-axis, a 300˚ angle is 60˚ below the x-axis in the 4th quadrant and a 330˚ angle is 30˚ below the x-axis. Since all these coordinates can be found using angles in the first quadrant and similar triangles or reflections, we call them reference angles. You just need to remember the signs.

In the above problems, we used the special right triangle; 30-60-90˚. We could also use the 45-45-90˚ special triangles to find angles of 45˚, 90˚, 135˚, 225˚, 270˚ and 315˚.

If you are not using the special triangle relationships, you will use a table or calculator and have to remember, as before, to assign the correct sign depending upon the quadrant that terminal side of the angle is in.

Example 1 Find the reference angle that corresponds to an angle that measures 170˚.

I can draw a picture to make that determination, but I might be able to visualize that 170˚ is 10˚ above the x-axis. So the reference angle is 10˚.

Example 2 Find the reference angle that corresponds to an angle that measures 140˚.

The terminal side is 40˚ above the x-axis, therefore the reference angle is 40˚.

Example 3 Find the reference angle that corresponds to an angle that measures 200˚.

The terminal side lies in the third quadrant, 20˚ below the x-axis, therefore the reference angle is 20˚.

Example 4 Find the reference angle that corresponds to an angle whose measure is 300˚.

Since the terminal side of that angle lies in the fourth quadrant, 60˚ below the x-xis, the reference angle is 60˚

Example 5 Find the reference angle that corresponds to an angle whose measure of –200˚.

The terminal side of an angle of –200˚ lies in the second quadrant, it is 20˚ above the x-axis, so the reference angle is 20˚.

Example 6 Find the reference angle that corresponds to an angle whose measure is 400˚.

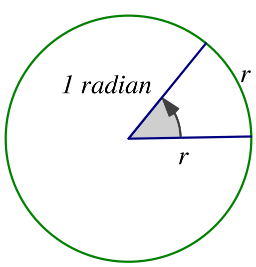
A 400˚ angle does one complete rotation (360˚) and goes another 40˚. Therefore the reference angle is 40˚.

Example 7 Find the reference angle that corresponds to an angle whose measure is 510˚.

A 510˚ angle does one couple rotation (360˚) and results in an angle whose terminal side lies in the second quadrant at 150˚. An angle of 150˚ is 30˚ above the x-axis, so the reference angle is 30˚.

Radian Measure

When a central angle intercepts an arc that has the same length of the radius of the circle, the measure of the angle is defined to be one radian.



Because the circumference of a circle is 2πr, there are 2π radians in any circle. Therefore, 2π radians = 360˚ or π radians = 180˚.

That leads us to the following proportion: $\frac{d^\circ}{180^\circ} = \frac{r\ radians}{\pi\ radians}$

Example 1 Find the radian measure of an angle of 60˚.

Using the proportion and filling in the degree measure, 60˚

$$\frac{60^\circ}{180^\circ} = \frac{r \ radians}{\pi \ radians}$$

$$60\pi = 180r$$

$$\frac{60\pi}{180} = r$$

$$\frac{\pi}{3} = r$$

Example 2 Find the degree measure of –3π/4 radians.

$$\frac{d^\circ}{180^\circ} = \frac{\frac{-3\pi}{4} \ radians}{\pi \ radians}$$

$$d\pi = 180\frac{-3\pi}{4}$$

$$d = 45(-3)$$

$$d = -135^\circ$$

Rather than substituting values into the proportion, we can convert a set of units to another set by multiplying by 1 in terms of what is given and what we want to convert to.

For instance, to convert 3 miles to feet, we start with 3 miles and multiply by 1 in terms of feet and miles: $\frac{5{,}280 \ ft}{1 \ mile}$

$$3 \text{ miles x } \frac{5{,}280 \ ft}{1 \ mile} = 15{,}840 \text{ ft}$$

To convert degrees to radians, multiply by $\frac{\pi \text{ radians}}{180^\circ}$

To convert radians to degrees, multiply by $\frac{180^\circ}{\pi \text{ radians}}$

Example 3 Convert 45° to radians.

Multiply 45° by 1; $45^\circ \frac{\pi \text{ radians}}{180^\circ}$

Simplifying, we have $\pi/4$ radians

To help you remember this, multiply by 1; if you want radians, then radians should be in the numerator. If you want degrees, then degrees should be in the numerator.

Example 4 Convert $5\pi/6$ to degrees.

Multiply $5\pi/6$ by 1
I want to convert to degrees, so I use the
factor that has degrees in the numerator - $\frac{180^\circ}{\pi \text{ radians}}$

$$\frac{5\pi}{6}\frac{180^\circ}{\pi \text{ radians}} = 150^\circ$$

Now, let's look at our reference angles in terms of radian measure. We want to know our special triangles. In essence, from our formula, we multiply the degree measure by π and divide by 180.

Example 5 Convert 30°, 45° and 60° to radians.

$30[\pi/180] = \pi/6$

$45[\pi/180] = \pi/4$

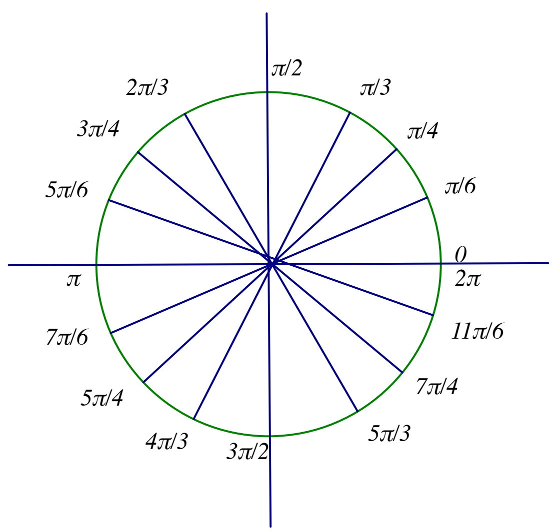
$60[\pi/180] = \pi/3$

It's that easy and you should be able to do those in your head.

The simplest way to convert radians to degrees is let $\pi = 180^\circ$

$\pi/6 = 30^\circ$
$2\pi/6 = 60^\circ$ — $2\pi/6 = \pi/3 = 60^\circ$
$3\pi/6 = 90^\circ$ — $3\pi/6 = \pi/2 = 90^\circ$
$4\pi/6 = 120^\circ$ — $4\pi/6 = 2\pi/3 = 120^\circ$
$5\pi/6 = 150^\circ$
$6\pi/6 = 180^\circ$ — $6\pi/6 = \pi = 180^\circ$

Using this information, we can draw the unit circle and label the angles in terms of radians.



By inspection, we can see

The reference angle of $5\pi/6$ is $\pi/6$

The reference angle of $2\pi/3$ is $\pi/3$

The reference angle of $3\pi/4$ is $\pi/4$

Example 6 Find the reference angle that corresponds to $11\pi/6$.

$11\pi/6$ is in the 4th quadrant and $\pi/6$ below the x-axis, the reference angle is $\pi/6$.

You should also know that $\pi/6 = 30^\circ$. So $11\pi/6 = 330^\circ$

Know these angles and their multiples!

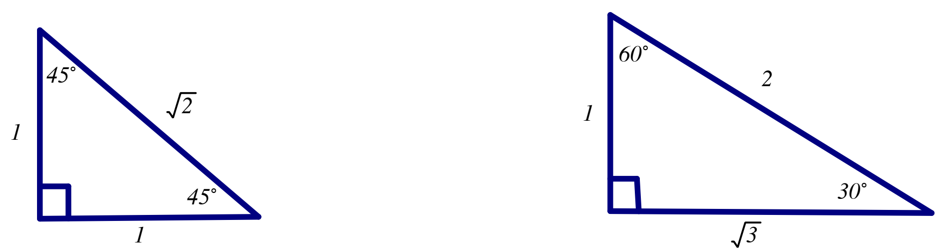
$\pi/3 = 60^\circ$ $\pi/4 = 45^\circ$ $2\pi/3 = 60^\circ$ $\pi/2 = 90^\circ$ $\pi = 180^\circ$ $2\pi = 360^\circ$

Graphing Trig Functions - Sine & Cosine

Sec. 1 Graphing the Sine and Cosine Functions

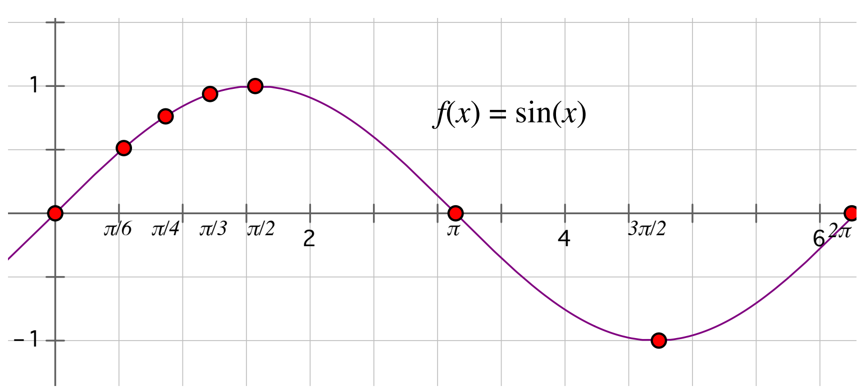
Up to this point, we have learned how the trigonometric ratios have been defined in right triangles. Most of us learned SOHCAHTOA as a memory aid. We then used that information to find the those ratios on a unit circle by using reference angles. Now, we will use that same information to graph trigonometric functions on the Cartesian Coordinate System.

You have to love how all this information ties together. So, let's use the angles in special triangles to help construct my graph and remembering the the x-coordinate is the cosine and the y-coordinate is the sine.



From that information, I know these values in degrees and radians.
sin 0˚ = 0, sin 30˚ = 1/2, sin 45˚ = √2/2, sin 60˚ = √3/2, sin 90˚ = 1
sin 0 = 0, sin π/6 = 1/2, sin π/4 = √2/2, sin π/3 = √3/2, sin π/2 = 1

Example 1 Graph y = sin x for all x such that $0 \leq x \leq 2\pi$



Notice, I labeled my x-axis in terms of radian measure ~ π. Once I relabeled in terms of π, I graphed the first 5 points in the first quadrant ($0 \le x \le \pi/2$). Since the y-coordinates are symmetric in the second quadrant, I can use symmetry to find the graph for $\pi/2 \le x \le \pi$. That gives me the graph in the $0 \le x \le \pi$.

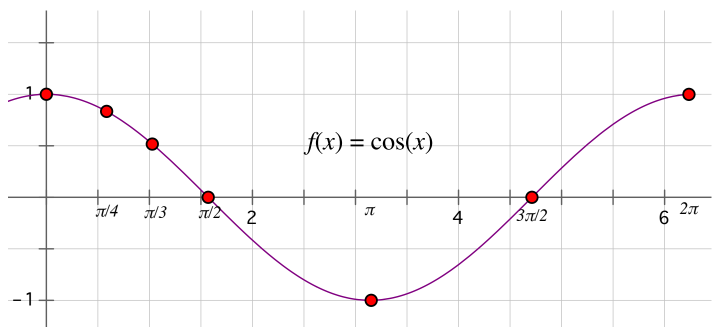
Angles greater than 180 and less than 360˚ fall in the third and fourth quadrants. In the third and fourth quadrants, the ratios are the same but the y-coordinates are negative. In the graph, we can see those points for the special angles fall below the x-axis.

So my point is this, once I label the first few points in the first quadrant, $0 \le x \le \pi/2$, the rest is easy because the only difference in the y- coordinates is the sign.

Using my special right triangles, lets find the values of the cosine, just like we did for the sine.

$\cos 0 = 1$, $\cos \pi/6 = \sqrt{3}/2$, $\cos \pi/4 = \sqrt{2}/2$, $\cos \pi/3 = 1/2$, $\cos \pi/2 = 0$

Example 2 Graph $y = \cos x$, $0 \le x \le 2\pi$

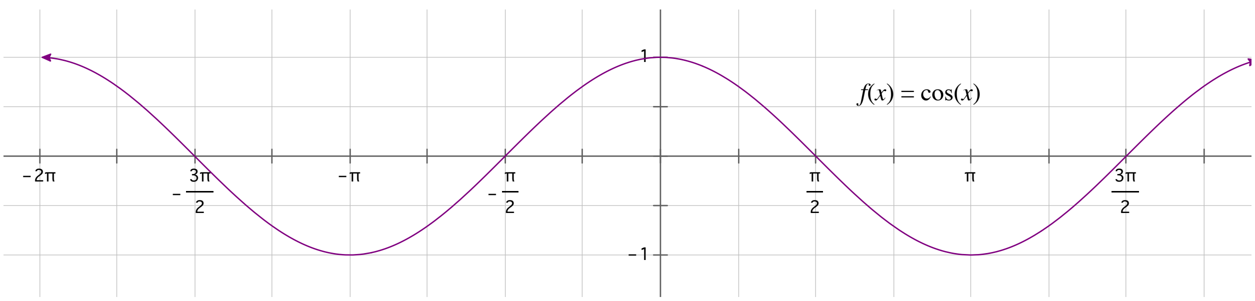


I labeled my x-axis in terms of π and used my special right triangles to find their corresponding values. Again, I only used values in the first quadrant because of symmetry. I know the cosine values (x-coordinates) are negative in the second and third quadrants so the graph should be below the line for $\pi/2 \le x \le 3\pi/2$.

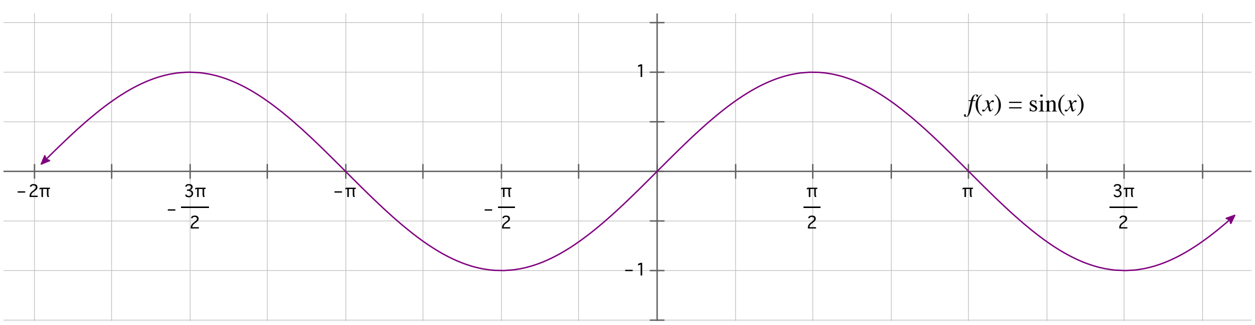
I also know the $\cos \pi = -1$, so I graphed that point, the $\cos 3\pi/2 = 0$, and the $\cos 2\pi = 1$

You might notice that both graphs look like waves. If I started at x = 0 to graph, I see the sine curve's wave goes through the origin. At x = 0, cosine curve's wave passes through (0, 1). If I was able to shift one of the graphs, they would coincide.

Remember when we looked at the unit circle and discussed reference angles. We indicated that an angle whose measure was 405˚ had a reference angle of 45˚. We got that by subtracting 360˚. Let me point out a couple of things, one, since we could make more than one rotation, the graph of the trig functions can go on infinitely, two, we said an angle whose measure is –210˚ has a a reference angle of 30˚ and that suggests the graph goes out infinitely in the negative direction, and three using the coordinates derived from the special right triangles, the waves keep the same size and shape.



So, the graph of the sine and cosine functions are continuous and go out to positive and negative infinity. And, just like on the unit circle, we see the repetition of the values.



Both graphs repeat at multiples of 2π. We say their period is 2π. As we saw on the unit circle and on these graphs, the maximum and minimum heights (y-values) are between one and negative one. We say their amplitude is 1.

Sec. 2 Graph y = *a*sin(*b*x + *c*) + *d*; *b = 1*

And, the really good news, these graphs can be moved around the coordinate system just as we have done with our other graphs; parabolas, circles, absolute value, etc., using the same types of rules and notation.

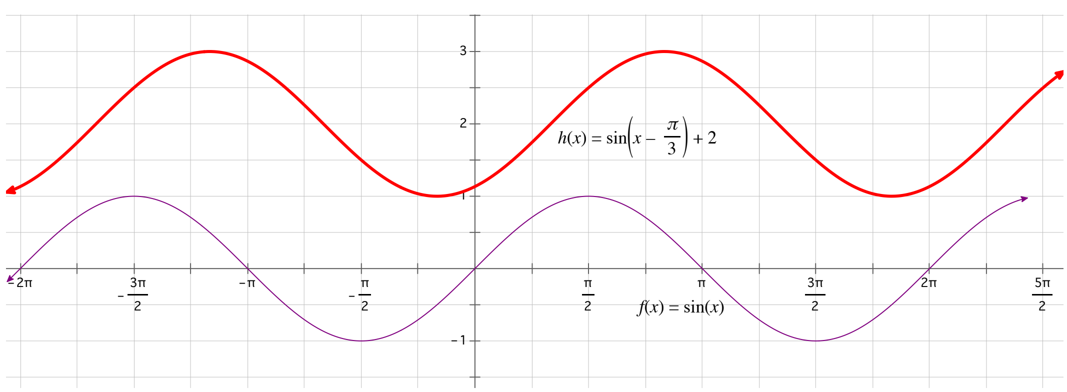
So, recall graphing a parabola, we know what $y = x^2$, looks like. If I changed the equation to $y = x^2 + 1$, the graph of the parabola is moved up one unit. The same is true for sine and cosine graphs, we know what the graph of y = sin x looks like. The graph of y = sin (x) + 1 moves the entire graph up one unit.

To move the graph horizontally, we looked at the value inside the parentheses. So, $y = (x - 2)^2$ moved the parabola over 2 units to the right. Similarly, the graph of y = cos (x – π/2) moves the graph of the cosine over π/2 units to the right.

And finally, if $y = 5x^2$ stretches the graph of the parabola. The same thing occurs with the sine and cosine graphs. That is, y = 5 sin x stretches the sine graph.

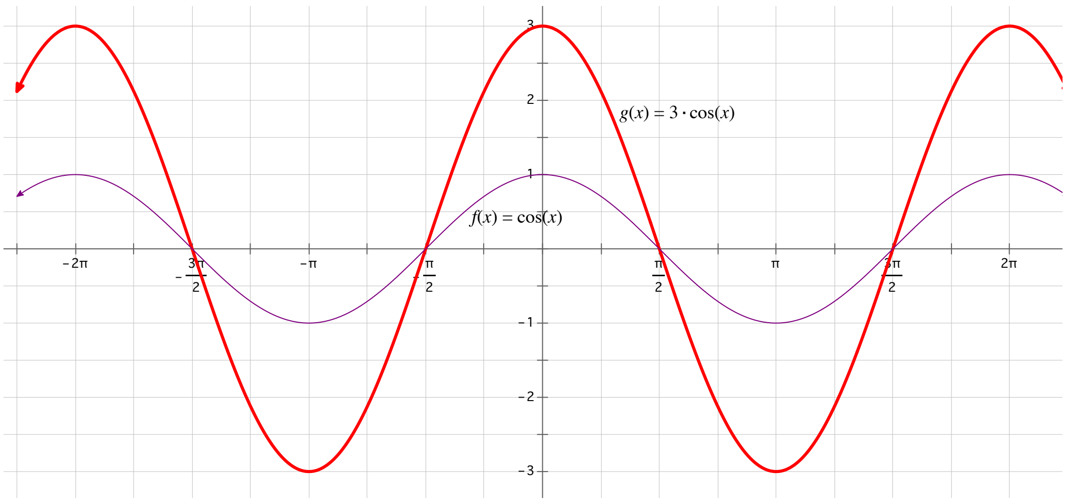
So, let's look at an example that translates the graph up 2 and over π/3 and see that graph looks exactly the same but is shifted.

Example 3 Graph y = sin(x – π/3) + 2



Notice the graph of the curve is the same, but all the points were moved up 2 and over π/3.

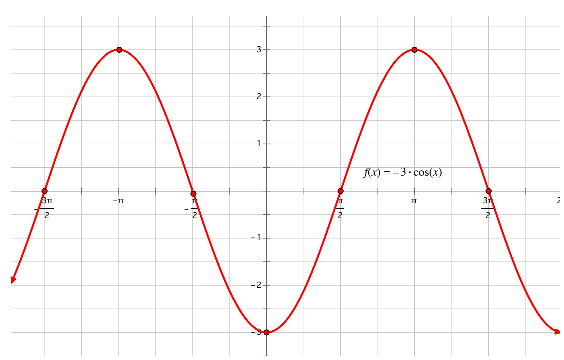
Example 4 Graph y = 3 cos x



In this example, we can still recognize the graph of the cosine, but we can also see it has been stretched by a factor of 3.

What do think might happen to the graph if we multiplied the cosine by negative 3? y = – 3cos x. If you are not sure, try a couple of convenient values of x and see what happens.

I graphed that by knowing it was going to be a wave that would typically start at (0, 1). But since the cosine was being multiplies by (–3), that my new graph should start at (0, –3). I then recalled my reference angles, at π/2 and 3π/2, x = 0, and at π, x = –1, but I was multiplying the cosine by (–3), so that would result in +3 on the graph.The rest was just symmetry

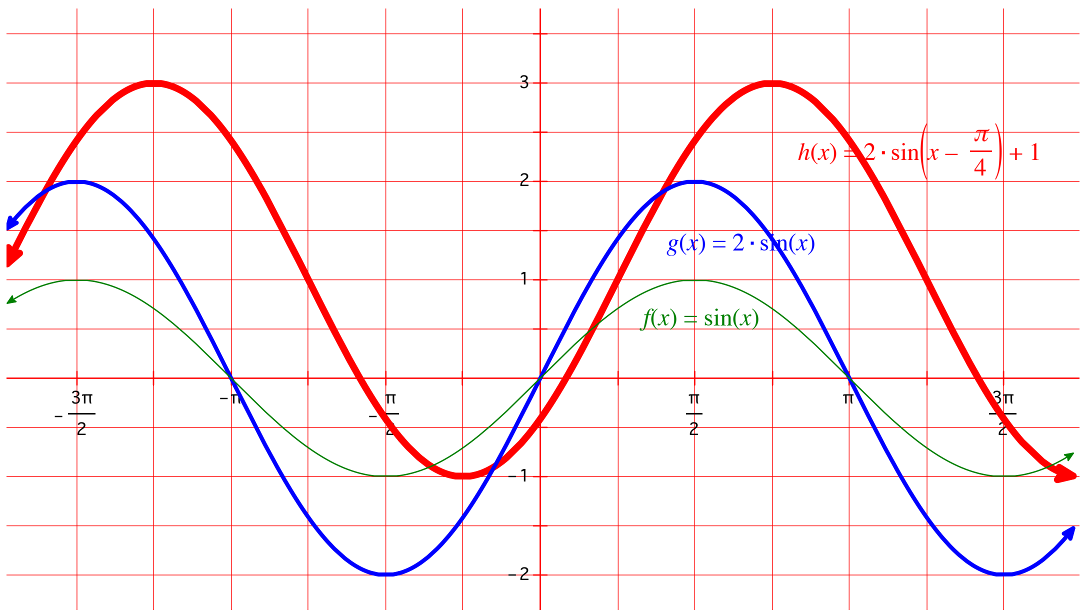


To graph these trig functions, you should be very aware of what the look like, then apply the stretch, and horizontal and vertical translations

Procedure for graphing:

1. Graph the parent function
2. Graph the vertical or horizontal stretch' y*
3. From y*, use the horizontal and vertical translations

Example 5 Graph y = 2 sin (x – π/4)+ 1

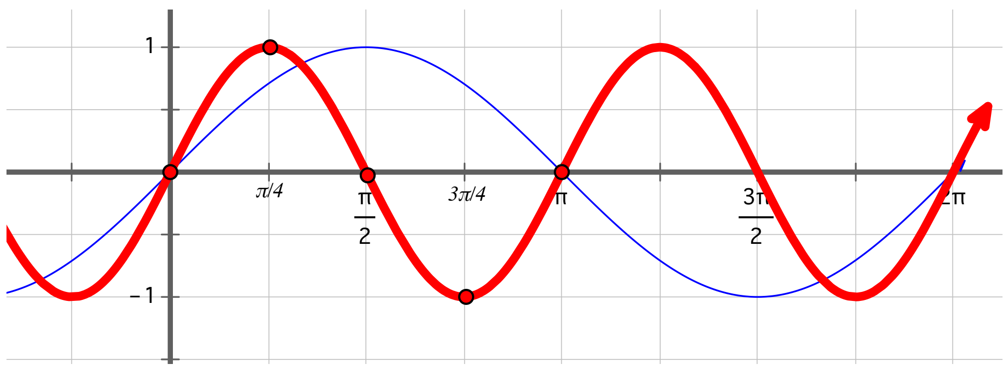


1. Graph the parent function in green, f(x) = sin(x)
2. Graph the vertical stretch, g(x) by multiplying all the y values by 2, the blue graph. Basically making the waves larger
3. Finally moving g(x), blue graph, over π/4 and up 1

Sec. 3 Graph y = $a\sin(bx + c) + d$

All the waves we studied up to this point repeat every 2π. Can we make them repeat more or less often?

Look at the following 2 equations; $f(x) = \sin(x)$ and $h(x) = \sin(2x)$. What is physically different in those two equations?



Now, we should be able to sketch f(x) pretty easily. Now, graph h(x) for values of 0, $\pi/4$, $\pi/2$ and π.

Look at the red graph, what is the period, how often does the wave repeat?

What we have already discovered is if $f(x) = \sin bx$, $b > 0$ ranges between 0 and 2π we get one complete sine wave with amplitude 1. When $b = 1$, the period (one complete sine wave) is 2π. As we can see above, when $b = 2$, sin 2x, we got one complete sine wave at $2\pi/2 = \pi$. This suggests that the period of a function $f(x) = \boldsymbol{a\sin(bx + c) + d}$ would be given by $2\pi/b$. This same argument could be used for graphing the cosine and later tangent functions.

And the phase shift (translation) would be determined by letting $bx + c = 0$ because the graph was defined between $0 \leq x \leq 2\pi$, so bx ranges between $-c$ and $2\pi - c$. That results in $x = -c/b$.

Example 5 Graph $f(x) = 3\sin(2x - \pi/2)$

Almost by inspection, up to this point should tell us we have a sine wave with amplitude 3 and phase shift $\pi/4$ by setting $2x - \pi/2 = 0$ to the right with period π by setting $2x = 2\pi$. That results in a complete sine wave between $\pi/4$ and $5\pi/4$.

Rather than just using that information, let's walk through our understanding.

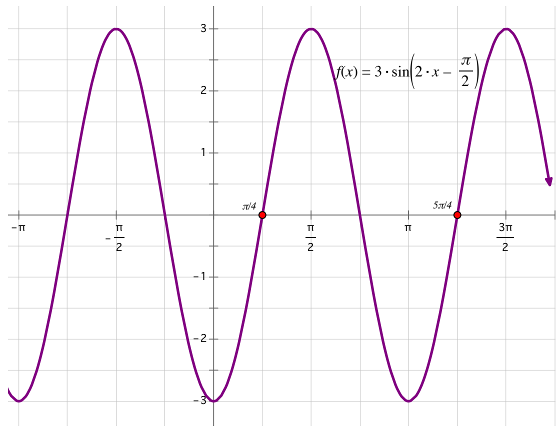
To obtain the interval containing the sine wave, we let $2x - \pi/2$ range from 0 to 2π. Solving, we have

$2x - \pi/2 = 0$ and $2x - \pi/2 = 2\pi$

$x = \pi/4$ $\quad 2x = 5\pi/2$

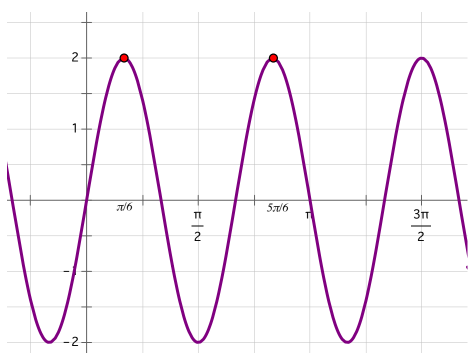
$x = 5\pi/4$

The amplitude is 3 and the period is $5\pi/4 - \pi/4 = \pi$



Example 6 Graph $y = 2 \cos (3x - \pi/2)$

Using inspection and mental math, the amplitude is 2, the period, $3x = 2\pi$ results in $2\pi/3$ and the phase shift is $3x - \pi/2 = \pi/6$.



So, from the graph you can see the period went from $\pi/6$ to $5\pi/6$, so the period is $4\pi/6 = 2\pi/3$ and the shift was $\pi/6$ to the right.

Graph the following without plotting points

1. $y = \sin x$
2. $y = 2 \sin x$
3. $y = \sin 2x$
4. $y = (1/3) \sin (x - \pi/4)$
5. $y = 3 \cos x$
6. $y = -2 \cos 2x$
7. $y = 2 \sin (2x - \pi/3)$
8. $y = 3 \cos (3x - \pi/2)$
9. $y = 2 \sin (2x - \pi/3) + 1$
10. $y = 3 \cos (3x - \pi/2) - 1$