3qaChapter 4 Relations and Functions

Sec. 1 Relations and Functions as Ordered Pairs

Students who have read a menu have experienced working with ordered pairs. Menus are typically written with a food item on the left side of the menu, the cost of the item on the other side as shown:

Hamburger …………$3.50
Pizza...………………..2.00
Sandwich……………..4.00

Menus could have just as well been written horizontally;

Hamburger, $3.50, Pizza, 2.00, Sandwich, 4.00.

But that format (notation) is not as easy to read and could cause confusion. Someone might look at that and think you could buy a $2.00 sandwich. To clarify that so no one gets confused, I might group the food item and its cost by putting parentheses around them:

(Hamburger, $3.50), (Pizza, 2.00), (Sandwich, 4.00)

Those groupings would be called ordered pairs, pairs because there are two items. Ordered because food is listed first, cost is second.

By definition, **Relation - any set of ordered pairs**.

Another example of a set of ordered pairs could be described when buying cold drinks. If one cold drink cost $0.50, two drinks would be $1.00, three drinks would be $1.50. I could write those as ordered pairs:

(1, .50), (2, 1.00), (3, 1.50), and so on

From this you would expect the cost to increase by $0.50 for each additional drink. What do you think might happen if one student went to the store and bought 4 drinks for $2.00 and his friend who was right behind him at the counter bought 4 drinks and only paid $1.75?

My guess is the first guy would feel cheated, that it was not right, that this was not working, or this was **not *functioning***. The first guy would expect anyone buying four drinks would pay $2.00 - just like he did.

Let's look at the ordered pairs that caused this problem.

(1, .50), (2, 1.00), (3, 1.50), (4, 2.00), (4, 1.75)

The last two ordered pairs highlight the malfunction, one person buying 4 drinks for $2.00, the next person buying 4 drinks for a $1.75.

For this to be fair or functioning correctly, we would expect that anyone buying four drinks would be charged $2.00. Or more generally, we would expect every person who bought the same number of drinks to be charged the same price. When that occurs, we'd think this is functioning correctly. So, let's define a function.

Function is a special relation in which no two different ordered pairs have the same first element.

Since the last set of ordered pairs have the same first elements, those ordered pairs would not be classified as a function.

Sec. 2 Functions and Relations as Rules

If I asked students to determine the cost of 10 cold drinks at $0.50 per drink, many might realize the cost would be $5.00. If I asked them how they got that answer, eventually, with prodding, someone might tell me they multiplied the number of cold drinks by $0.50. That shortcut can be described by a rule

Cost = $0.50 x number of cold drinks

$$c = .50n$$

or the way you see written it in your math book

$$y = .50x \text{ or } y = \frac{1}{2}x$$

That rule generates more ordered pairs. So, if I wanted to know the price of 20 cold drinks, I would substitute 20 for x. The result would be \$10.00. Written as an ordered pair, I would have (20,10)

Let's look at another rule.

Example 1 $y = 3x + 2$

If we substitute 4 for x, we get 14 out, represented by the ordered pair (4, 14). If we substitute 0 for x, we get 2, represented by (0, 2). There are an infinite number of numbers I can substitute for x and find corresponding values of y, and those can be graphed on the coordinate plane. So, we can see, a rule can generate ordered pairs and a graph.

$$(0, 2); (1, 5); (2, 8); (3, 11); (4, 14), \ldots$$

RELATION is any set of ordered pairs. The set of all first members of the ordered pairs is called the **DOMAIN** of the relation. The second numbers are called the **RANGE** of the relation.

Sometimes we put restrictions on the numbers we can substitute into a rule, the domain. Those restrictions may be placed on the relation so it fits real world situations.

For example, using our cold drink problem. If each drink costs \$0.50, it would not make sense to find the cost of –2 drinks. You can't buy negative two drinks, so we would put a restriction on the domain. The only numbers we could substitute are positive whole numbers; 0, 1, 2, 3…

The restriction on the domain also effects the range. If you can only use positive whole numbers for the domain, what values are possible for the range?

We defined a **Function as a special relation in which no two ordered pairs have the same first member.**

What that means is that for every member of the domain there is one and only one member of the range. That means if I give you a rule, like $y = 2x - 3$, when I substitute $x = 4$, I get 5 out. Represented by the ordered pair (4, 5). Now if I substitute 4 in again for x, I have to get 5 out. That makes sense, it's expected so just like the rule for buying cold drinks, this is working, this is functioning as expected, it's a function.

Another example of a rule might be going to an amusement park with an entrance fee of \$15 and then you pay \$2 for each ride. Mathematically, we would write that like this:

$$C = \$15 + \$2r \quad \text{or} \quad C = \$2r + \$15$$

Or using x to represent the number of rides and y to represent the cost, we'd have

$$y = 2x + 15$$

Now, you are thinking, no big deal, isn't that what we would expect?

I can determine if ordered pairs are functions by just looking at the values of x in the ordered pair. If no x values are repeated, then the set of ordered pairs represents a function.

For example, in this list of ordered pairs, (2, 3), (3, 5), (4, 7), (11, 13), I can identify the domain, the range and determine if the ordered pairs represent a function.

The domain is represented by the values of x; {2, 3, 4, 11}
The range is represented by the values of y; {3, 5, 7, 13}

We represent ordered pairs by using parentheses, ( )'s
We represent domains and ranges using brackets, { }'s

It's customary to list domains in ranges from smallest to largest.

Now, does that list of ordered pairs represent a function? To make that determination, we notice that no x values are repeated. That means that list of ordered pairs represents a function.

Let's look at another example of a set of ordered pairs and again identify the domain, range and determine if they represent a function.

Given: (2, 5), (3, 1), (5, 11), (6, 8) and (3, 7)

The domain (x-values) is { 2, 3, 5, 6}
The range (y-values is {1, 5, 7, 8, 11}

Notice the 3 is written twice and has different y-values, therefore that set of ordered pairs does not represent a function – it's a relation.

Who Cares if a Rule is a Relation or Function?

A good reason to know if a rule is a function is that some rules we can apply operations, like adding, subtracting, and other rules we cannot. A rule that is a function will allow us to combine functions. Other rules that are just relations will not provide us that opportunity.

Looking at another rule might give us a clue, $x^2 + y^2 = 25$.

Solving that for that y, we get $y^2 = 25 - x^2$

$$y = \pm\sqrt{25 - x^2}$$

Now if we substitute a number like 3 in for x, we get two answers, (3, 4) and (3, –4). You can see there is **not one and only one** member in the range for each member in the domain. Therefore this rule describes a relation that is not a function.

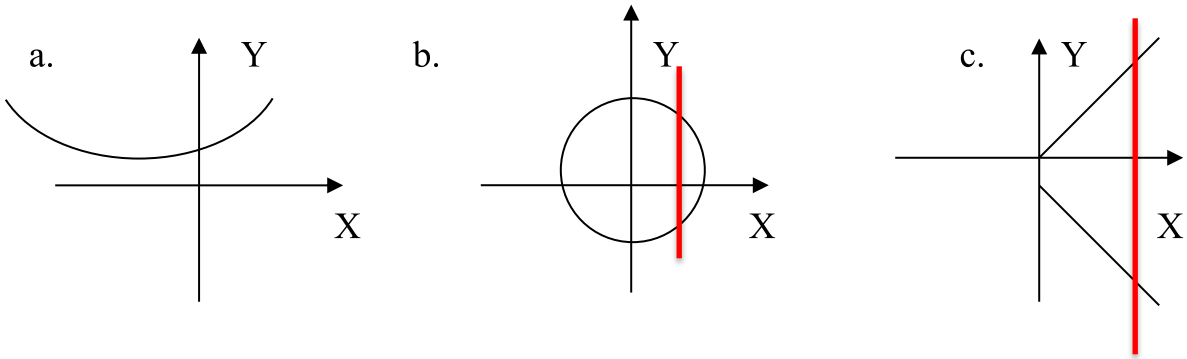
Vertical Line Test - Function

To determine if a graph describes a function, you use what we call the **Vertical Line Test**. That is, you try to draw a vertical line through the graph so it intersects the graph in more than one point. If you can do that, then those two ordered pairs have the same first element, but a different second element. Therefore the graph would not describe a function.

If there does not exist any vertical line which crosses the graph of the relation in more than one place, then the relation is s function. That is called the VERTICAL LINE TEST.

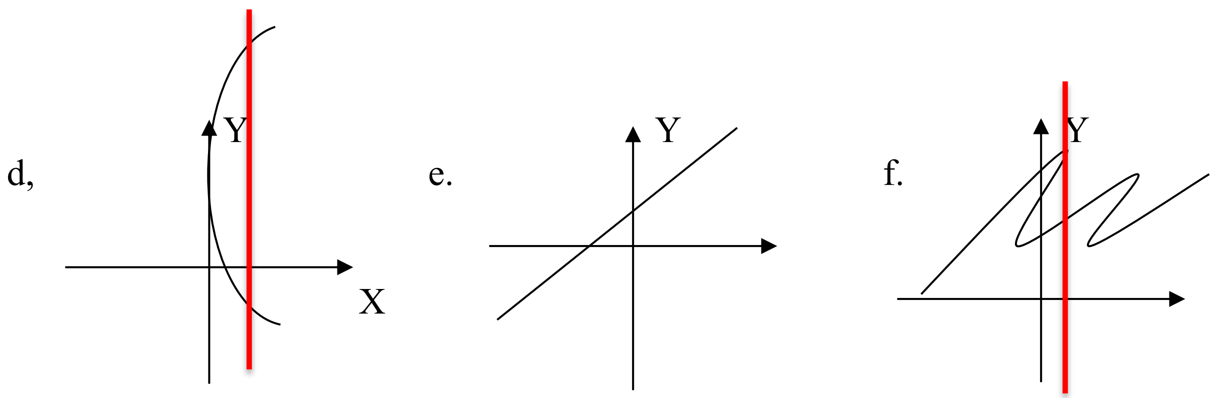
What we try to do is draw a vertical line so it intersects the graph in more than one place. If we can't then we have a graph of a function.

Try these-



Why does this work? Well if we think about it for a few minutes we can see if there is more than one intersection, then that particular X has two different Y's associated with it.

Label the following as relations or functions.



Only a. and e. are functions as can be seen by the Vertical Line Test.

We use functions almost every day in our lives, because we are just living, sometimes we don't think of it as mathematics.

Example 2. Let's say you have a cell phone, the phone company charges you \$10.00 per month plus \$0.05 per text. Find the domain and range.

Without a lot of math, you know if you don't use the cell phone that month, you will be charged a flat rate of \$10.00. If you send one text, the charge will be \$10.05, 2 texts will be \$10.10, 3 texts \$10.15. If you sent 60 texts, you would be charged \$13.00.

Thinking of this mathematically, the number of texts I could send is zero or greater; $x \geq 0$. The costs would range from \$10 and up; $y \geq 10$.

Domain = {x is an integer and $x \geq 0$}
Range = {$y \geq 10$}

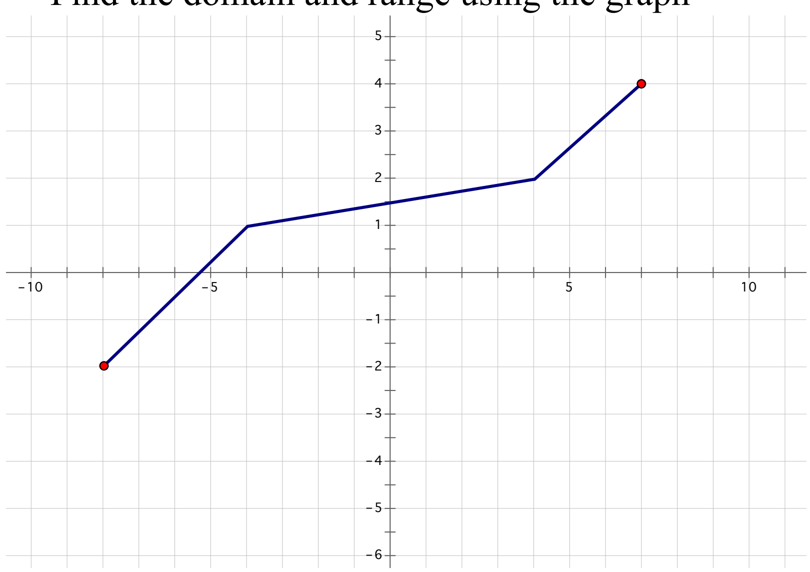
Being able to identify the domain and range of functions plays greater importance later on in math. So, let's make sure we can identify them by examining ordered pairs, looking at graphs, and later examining equations.

If a function is being described by a set of ordered pairs, we know the domain is the first element, the range is the second in the ordered pair.

To identify the domain and range looking at a graph is just as simple. We look to see all the possible values of x on the horizontal axis for the domain. Then for the range, we look at all possible values of y on the vertical axis.

Since these graphs are made up of an infinite number of points, they can't be listed, so we will have to use inequalities to describe them.

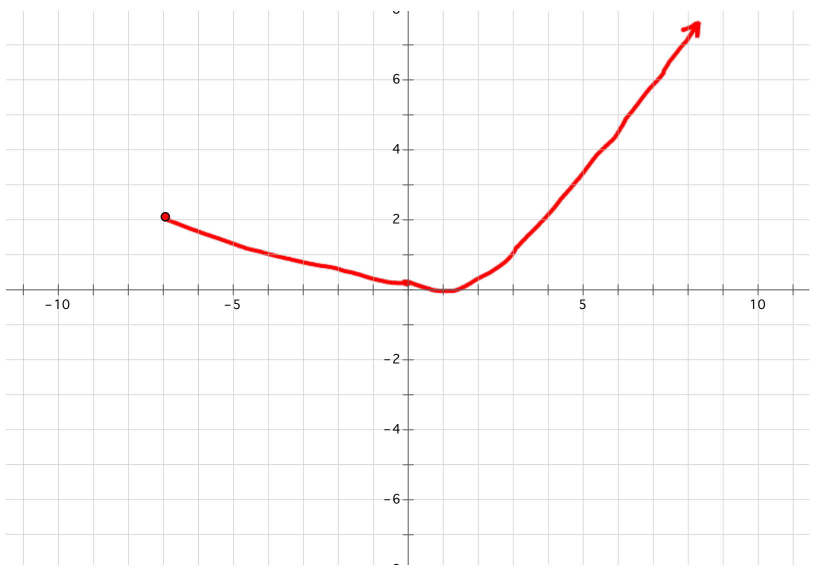
Example 4 Find the domain and range using the graph



Looking at the graph from left to right, **horizontally**, the x values begin at–8 and continue to +7. Since there are closed circles at those numbers, they are included in the domain. The domain, the x's are all the values between those 2 numbers. Mathematically, we have the domain is $\{-8 \le x \le 7\}$

Now, looking at the graph from bottom to top, **vertically**, the y values begin at –2 and continue up as high as +4. Again, since they are closed circles at the endpoints, those numbers are included in the range. Mathematically, we have the range is $\{-2 \le y \le 4\}$

Example 5 Find the domain and range using the graph



To find the domain, we look at the x-axis, the horizontal axis to see where the values begin and end from left to right. We see the smallest value of x is at –7 and the greatest value of x is not shown on the graph because it continues to the right (designated by the arrow). So, we say the domain is $\{ x \ge -7\}$

To find the range, we look at the y-axis, the vertical axis from bottom to top. We see the smallest value of y is 0 and it keeps going up also. So, we say the range is $\{y \ge 0\}$

Sec. 4 Using Functional Notation

Let's go back to the text problem, example 2. I could describe that mathematically

$$\text{cost} = \$10.00 + \$0.05\text{texts}$$

$$c = 10 + .05t \quad \text{or} \quad c = .05t + 10$$

or the way you see it written in math class with x's and y's

$$y = .05x + 10$$

Where y represents the cost and x represents the number of texts sent on the phone.

Using fractions, that equation might also look like $y = \frac{1}{20}x + 10$

Since those rules generate ordered pairs, all I need to do is substitute in numbers to find the cost of using my phone for any number of texts.

Using the rule $y = .05x + 10$, I might ask how much it would cost to use the phone if I sent ten texts.

Someone else might ask how much it might be if they sent 20 texts. Since in our math classes, we often use x's and y's when dealing with equations and their graphs, this could lead to confusion later. So, developing a notation that allows us greater ease in recognizing different events (rules) will make our life easier.

Still more people could ask the cost of operating the phone for any number of texts.

As you can see, to ask these questions a full sentence has to be written, then we substitute the desired number of texts into our rule; $y = .05x + 10$
Rather than wasting all that time and paper writing sentences, it might be nice to develop a shorthand method (notation) to describe the same situation.

Let's go back to $c = .05x + 10$ to describe the cost of using the phone. Remember c represent the cost, x represents the number of texts.

$$c = .05x + 10$$

Rather than writing sentences to find the cost of sending 10 texts, 20 texts or any number of texts , I could write these using mathematical shorthand.

$c(x) = .05x + 10$ read the value of c at x is $.05x + 10$

$c(10) = .05(10)x + 10$ read the value of c at 10 is $.05(10) + 10$
$c(20) = .05(20) + 10$ read the value of c at 20 is $.05(20) + 10$
$c(30) = .05(30) + 10$ read the value of c at $30 = .05(30) + 10$

and for x texts it would be $c(x) = .05x + 10$, *read the value of c at x = .05x + 10.*

Some like to say c(x) as *c **of** x* or *c **at** x = .05x + 10*

What this notation allows us to do is find values of a function using notation just by substituting numbers into an expression (rule).

Example 6 If $f(x) = 4x – 3$, find f(10)
$f(10) = 4(10) – 3 = 37$………………(10, 37)

Example 7 If $h(x) = x^2 – 3$, find h(6)……………(6, 33)
$h(6) = 6^2 – 3 = 33$

Example 8 If $p(x) = 2x^2 – 3x + 5$, find p(0)………(0, 5)
$p(0) = 2(0)^2 – 3(0) + 5 = 5$

Later, when graphing, we will see using functional notation, instead of just y, will help us interpret the graphs a lot easier and quicker.

Sec. 5 Operations with Functions

Not all rules can be combined, rules that are functions can be.

If f and g are two functions with a common domain, then the sum of f and g, is defined to be: $(f + g)(x) = f(x) + g(x)$

The difference of f and g is defined by: $(f - g)(x) = f(x) - g(x)$ and the quotient of f and g is defined by $(f/g)(x) = \frac{f(x)}{g(x)}$ where g(x) cannot be zero.

Let's see what all that means.

$$\text{If } f(x) = 3x \quad \text{and} \quad g(x) = x - 4,$$

Now we will substitute a number for f and g, if x = 2, then

$$f(2) = 6, \qquad g(2) = -2$$

Adding those together,

$$\begin{aligned} f(x) + g(x) &= 3x + (x - 4) \\ f(2) + g(2) &= 6 + (-2) \\ &= 4 \end{aligned}$$

Let's trying adding the f and g rules together first, then find (f + g) (x)
Now, if I substitute 2 into that function, I get 4, just like I did before.

$$\begin{aligned} f(x) + g(x) &= 3x + (x - 4) \\ (f + g)(x) &= 4x - 4 \\ (f + g)(2) &= 4(2) - 4 \\ (f + g)(2) &= 4 \end{aligned}$$

What we can see from this example is I can find the value of f(2) and the value of g(2) and add those two results together to find an answer or I could have combined f and g first into one rule, then found (f + g) (2). The answer is the same.

You can ONLY do that if the rule is a function! Why? Because each value of x has only one value of y.

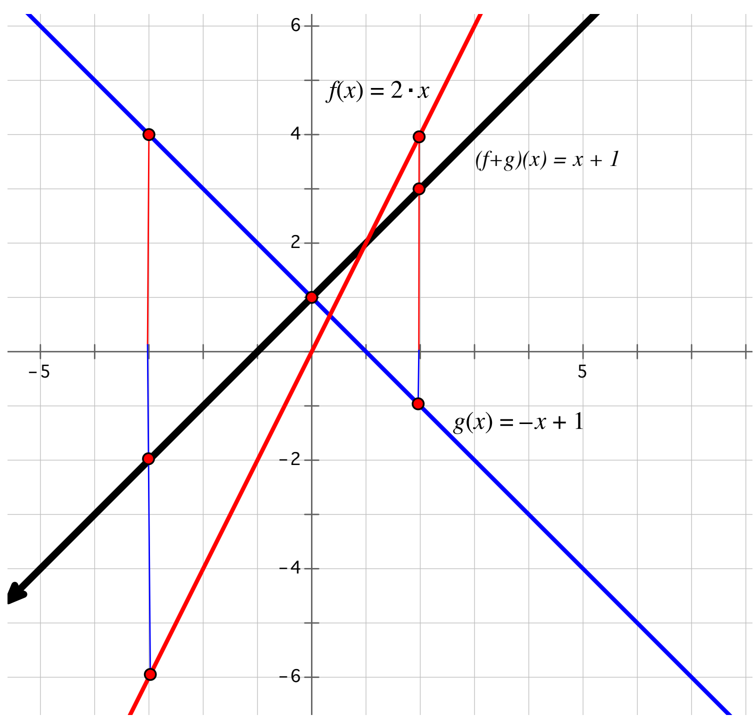
$(f+g)(x)$ is the same as $f(x) + g(x)$

While you may be wondering why someone might want to even bother with this, later it will help us in our graphing and save us some time computationally. By combining rules as we did with f and g, we can cut our work down considerably – particularly if we had a lot of computations to do.

Graphically, what that means is that I can graph $(f+g)(x)$ by combining the graphs of $f(x)$ and $g(x)$ by adding the y-coordinates. That process is called "addition of ordinates".

Notice in this next graph, rather than labeling the equations in terms of x and y, I'm labeling them in terms of x and f(x) and x and g(x). Using that just developed notation makes it easier for us to see what is happening without confusion.

Example 1 Graph $f(x) = 2x$ and $g(x) = -x + 1$ on the same coordinate system.



Looking at the blue and red lines when x = 2, we see the y-coordinates (ordinates) are – 1 and 4. When I add those ordinates together, we get –3, that results in a new point at (2, 3). Look at the graph blue and red graphs when x = – 3, we see the y-coordinates are 4 and – 6, the result of adding those together is the point (– 3, – 2).

When I connect those two new points, (2, 3) and (–3, –2) we get the graph of the black line. And notice that is the graph of the result of adding f and g; (2x) + (–x+ 1) = **x + 1**

In a nutshell, when we have functions we want to combine, we choose different values of x, look at the graphs for those values and add the y-coordinates of each function to find the new points – the new graph.

Let's take a look at the x-y charts for f(x) = 2x + 1, g(x) = x – 2 and combining those, (f + g) = 3x – 1 with the same domain { 0, 1, 2).

x	f(x)
0	1
1	3
2	5

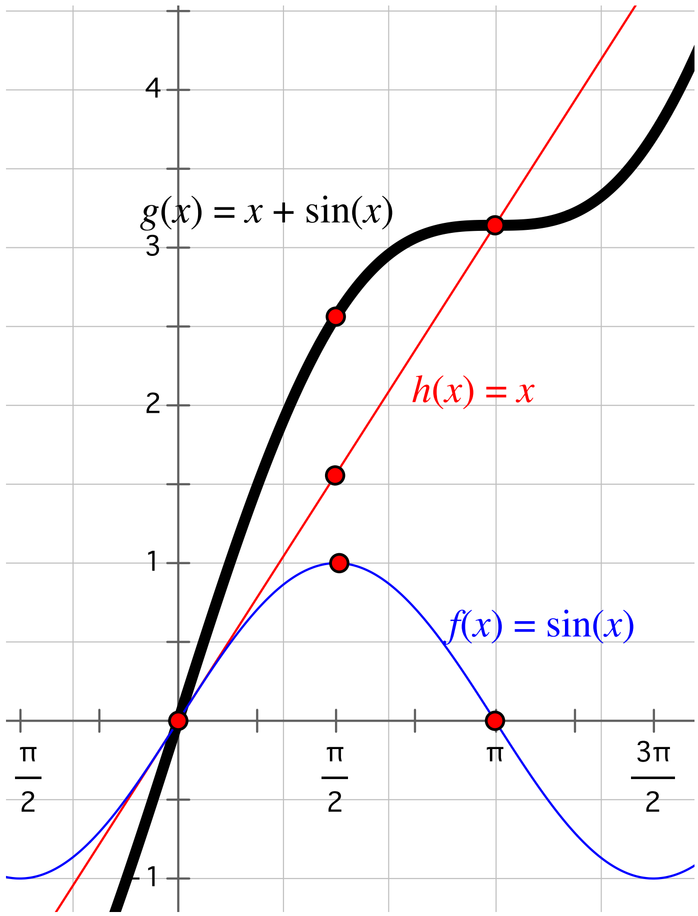
x	g(x)
0	–2
1	–1
2	0

x	(f +g)(x)
0	–1
1	2
2	5

Notice, when I add the values for f and g (y-values) for each member of the domain, that results in the sum of the y-values in the (f + g).

That can make graphing of functions easier by breaking them apart as in the next example.

Example 2 Graph $f(x) = \sin(x)$ and $h(x) = x$ by addition of ordinates.



In this graph, we will choose the most convenient values of x, then add the y coordinates. If I choose x = 0, both f and h pass through the origin. Both y values are 0, so when I add them f + h = 0. So (0, 0) is a point on the combined graph.

The next value of x I will choose is π/2. The sin π/2 = 1 as shown on the blue graph. h(x) at π/2 is π/2. Let π = 3.14, then π/2 = 1.57 as shown on the red graph.

Adding those ordinates together at x = π/2, we have 1 + 1.57 = 2.57, the point (π/2, 2.57) as shown on the black graph. We continue picking convenient values of x and adding their ordinates to find the graph of their sum – the black graph.

Rather than adding the ordinates for each function, we could have combined the rules for f and h, and graphed g(x) = x + sin x and graphed as one equation.

Sec. 5 Composition of Functions

There are times when we may not want only to add or subtract functions as we just did, but compose a new function because the data we substitute in the first will be used to find the result of a second function.

For instance, let's say you are billed for your cell phone at a rate described by the following function (rule).

$$c(x) = 0.05x + 10$$

In other words the cost of your cell phone is \$10.00 per month plus five cents for each minute you speak.

Let's suppose you spoke for twenty minutes, you would be billed \$11.00 for the month.

Now, let's say you are taxed at 8% on that amount and that is added to your bill. Well, that's easy enough, I find the cost of the cell phone, take 8% of that number and add that sum to the bill. In our case, 8% of \$11.00 is \$0.88. So our bill is \$11.88.

Now, if I had one thousand customers and I wanted to find their monthly bill. To accomplish that, I would have to find the monthly charge, then take 8% and add that to the monthly charge. While that's not hard work, there's two steps of computation that have to be completed.

Wouldn't it be nice if I could find a way of combining those functions into one rule – eliminating one of the computations?

Let's rewrite these two rules using mathematical notation. We'll let f describe the cost of the cell phone as previously described:

$$f(x) = 0.05x + 10$$

And g describe the amount of tax to be paid based upon that bill.

$$g(x) = .08x$$

As we have just done, to find the cost of the cell phone plus tax, I would have to plug into f the number of minutes I spoke, take that result and plug that into g to find the tax, and finally, add those two numbers together.

As you can see, for each customer I have to perform three computations, find f, find g, then find the sum of **f** and **g**.

Composition of functions allows me to combine functions when the second function depends upon the value of the first function. As we saw, **g**, the tax was dependent upon the monthly phone charge – **f**.

Let's do a quick review before going any further. We introduced functional notation as a shorthand notation to determine values for various data points.

So $h(x) = 4x + 5$ was read as "h at x is equal to 4x + 5"

If I wanted to determine the value of h at 6, I would write

$$h(6) = 4(6) + 5 \text{ or } h(6) = 29.$$

The point I want to drive home is – everywhere I saw an x, I substituted a 6.

$$h(x) = 4x + 5$$

$$h(6) = 4(6) + 5$$

To compose a new function that will allow me to combine these two rules into one, I need to use this very same strategy of substitution.

So, let's go back to our phone problem, we have

$f(x) = 0.05x + 10$ representing the phone cost

$g(x) = .08x$ representing the tax on the usage

If all I want to do is find the tax described by g, then I'm going to substitute f(x) everywhere I see an x in g.

$g(x) = .08x$ Given

$g\{f(x)\} = .08(f(x))$ Substituting f(x) for x

$g\{f(x)\} = .08\ (.05x + 10)$ Substituting .05x + 10 for f(x)

$g\{f(x)\} = .004x + .80$ Simplifying (D-Property)

This new function, this composition, allows me to find the tax without first finding the monthly phone charge. So, to find the tax on using the phone 20 minutes, I merely substitute 20 into my new formula.

$$\begin{aligned} g\{f(20)\} &= .004(20) + .80 \\ &= .08 + .80 \\ &= \$0.88 \end{aligned}$$

Now, if I wanted to use one rule to find the monthly cost plus tax, I would add the tax rule (.004x + .80) that we just found to the **f**, the monthly cost of the phone. Let's call that rule T.

$$T(x) = .004x + .80 + f(x)$$
$$T(x) = .004x + .80 + (.05x + 10)$$
$$T(x) = .054x + 10.80$$

What this allows us to do is use one rule to find the cost of using a cell phone plus tax. Otherwise, you'd have to find the monthly cost of the phone, find the tax based on that cost, then add those two amounts together. While that would not be much work if you only had to calculate a few bills, it would be a real pain in the neck if you had to calculate 1000 phone bills.

Composing the new rule might take a couple of minutes up front, but it will save a lot of time in the long run. Let's see how the new rule works for a monthly bill when you spoke for 20 minutes.

$$T(x) = .054x + 10.80$$
$$T(20) = .054(20) + 10.80$$
$$T(20) = 1.08 + 10.80$$
$$T(20) = 11.88$$

Composition of Functions Algorithm f(g(x))

1. **Write the f and g functions**
2. **Write the f(x) rule**
3. **In f(x), substitute g(x) for every x**
4. **Substitute the g(x) rule**
5. **Simplify**

So, in the last example, we were able to combine two rules to get a tax, a composition, then able to add two rules together to get a total cost. Let's just do a simple composition.

Example 1 If $f(x) = 2x + 5$ and $g(x) = 3x + 1$, find $f\{g(x)\}$

$$\begin{aligned} f(x) &= 2x + 5 \\ f(g(x)) &= 2\, g(x) + 5 \\ &= 2\,(3x + 1) + 5 \\ &= 6x + 2 + 5 \\ f(g(x)) &= 6x + 7 \end{aligned}$$

Composing functions just allows us to make one rule instead of doing a number of separate rules and combining them. And the good news, in essence all we are doing is using the Substitution principle.

Example 2 If $t(x) = 4x$, $h(x) = 2x + 1$, find $h(t(x))$

$$\begin{aligned} h(x) &= 2x + 1 \\ h(t(x)) &= 2(t(x)) + 1 \\ &= 2(4x) + 1 \\ &= 8x + 1 \end{aligned}$$

Example 3 In example 2, is $h(t(x)) = t(h(x))$?

We just found $h(t(x))$, let's find $t(h(x))$ and see.

$$\begin{aligned} t(x) &= 4x \\ t(h(x)) &= 4\, h(x) \\ &= 4(2x + 1) \\ &= 8x + 4 \end{aligned}$$

We can see $h(t(x)) \neq t(h(x))$

Sec 5. Inverse Functions

We have used inverse operations to "undo" an expression when solving equations. If, for example, we wanted to solve for x:

$$5x + 3 = 48$$

Our first step would be to isolate the variable by adding the Additive Inverse of 3 to both sides of the equation using the Addition Property of Equality.

$$5x + 3 + (-3) = 48 + (-3)$$

To solve equations, once we had the equation in ***ax + b = c*** format, we used the **Order of Operations** in reverse to "undo" the expression to isolate the variable.

With respect to inverse functions, we will again use the Order of Operations to "undo" a rule.

When we have a function defined by paying \$15 to enter an amusement park and then paying \$2 for each ride, such as $f(x) = 2x + 15$, that function (rule) allows us to substitute values for x (Domain) the number of rides and find corresponding values for y (Range) the cost.

By undoing the function, we can determine the number of rides we rode based on the money we spent. In essence, going from the range to the domain.

Looking at $f(x) = 2x + 15$, we can determine using that rule that if we rode 4 rides, the cost would be determined by

$$f(4) = 2(4) + 15$$
$$f(4) = 23$$

which could be described the ordered pair (4, 23)

Easy enough. The question becomes, is there another rule, a function, that would allow us to "undo" that rule? In other words, is there a function, rule, that would allow us to determine if we spent \$23, we would have gone on 4 rides?

Well, if we think of this, we might realize if we used the Order of Operations to get from 4 rides to \$23, to undo that, we would use the Order of Operations in reverse using the inverse operation – just like we did when solving equations in the ax + b = c format.

Order of Operations
1. Grouping
2. Exponentials
3. Multiply/Divide
4. Add/Subtract

So let's walk through this: If f(x) = 2x + 15, we will substitute y for f(x)

$$y = 2x + 15$$

Notice, when x = 4, then y = 23 —≥ (4, 23)

Using y = 2x + 15, solve for x using the Order of Operations in reverse using the inverse operations – just like we did with literal equations.

$$\begin{aligned} y &= 2x + 15 \\ y - 15 &= 2x \\ \frac{y-15}{2} &= x \end{aligned}$$

Our original rule, f(x) = 2x + 15 described 4 rides resulting in a cost of \$23, (4, 23). Does this new rule take us back from spending \$23 to knowing we had 4 rides? In other words, when we substitute 23 for y do we get 4 for x?

Our new rule $\frac{y-15}{2} = x$ is written in functional notation as $\frac{y-15}{2} = f(y)$

And since it takes us backwards, take the values of the range and find values of the domain, f(y) is the inverse function.

By convention, we normally write our function rules in terms of x, so I'd like to rewrite

$$\frac{y-15}{2} = f(y) \quad \text{as} \quad \frac{x-15}{2} = f(x)$$

But that f(x) would be confused with the original f(x) = 2x + 15, so to differentiate the function and its inverse, we will use the following notation, $f^{-1}(x)$ to represent the inverse function of f(x).

$$f^{-1}(x) = \frac{x-15}{2}$$

That is read the inverse of f is $\frac{x-15}{2}$.

Now, let's put these ideas into perspective. We started with a function, a rule, for determining the cost of going to the amusement park and going on rides. To determine costs, we substituted the number of rides we would ride, the domain, and found the corresponding costs, the range.

The inverse function, a rule found by using the Order of Operations in reverse, allows us to go backwards. That is, if we know the cost (range), we could determine the number of rides (domain).

Summing this up. To find an inverse function, I used the Order of Operations in reverse using inverse operations – very much like we did when solving equations in the ***ax + b = c*** format.

We then created ***notation*** to help clarify between a function and its inverse – rather than write

$$f(y) = \frac{y-15}{2} \quad \text{we would write it as} \quad f^{-1}(x) = \frac{x-15}{2}$$

and call that the ***inverse of f***.

Also notice, if I made an x-y chart for the original function f(x) = 2x + 15

x	y
0	**15**
1	**17**
2	**19**
3	**21**
4	**23**
5	**25**

That results in the following ordered pairs:

(0, 15); (1, 17); (2, 19); (3, 21); (4,23); (5, 25)

$f^{-1}(x) = \dfrac{x-15}{2}$ resulted in the following ordered pairs

(15, 0); (17, 1); (19, 2); (21, 3); (23, 4); (25, 5)

Simply stated, the numbers we substitute into a function are called the DOMAIN, the outcomes are called the RANGE.

An inverse function interchanges those x's and y's, the domain and range.

Using that notion of interchanging the x's and y's, we can do that generally using the rule instead of just playing with the ordered pairs.

Graphically, if we interchange the domain and range of a function **f**, we see the graph of $\mathbf{f^{-1}}$ is a reflection of **f** in the line y = x.

Algorithm To Find an Inverse Function

1.	**GIVEN**	**f(x) = 3x – 1**
2.	Substitute y for f(x)	y = 3x – 1
3.	Interchange the x and y	x = 3y – 1
4.	Solve for y	$\frac{x+1}{3} = y$
5.	Rewrite y using $f^{-1}(x)$	$\frac{x+1}{3} = f^{-1}(x)$

Example 1 Find the inverse of h(x) = 4x + 3

y = 4x + 3	Given
x = 4y + 3	Interchange x and y
x – 3 = 4y	Solve
$\frac{x-3}{4} = y = h^{-1}(x)$	Use fct notation

Example 2 Find the inverse of $t(x) = x^2$

$$y = x^2$$
$$x = y^2$$
$$\sqrt{x} = y = t^{-1}(x)$$

If I have a function f which takes the number of rides (x) and determines a cost (y) and I have it's inverse which takes the cost (y) and determines the number of rides (x), then I should also be able to show that mathematically by substituting $f^{-1}(x)$ into f(x) and getting the x – the original values back.

$$f(x) = 2x + 15 \quad \text{and} \quad f^{-1}(x) = \frac{x-15}{2}$$

$$f(x) = 2x + 15$$

$$f(f^{-1}(x)) = 2\left(\frac{x-15}{2}\right) + 15$$

$$f(f^{-1}(x)) = x - 15 + 15$$

$$f(f^{-1}(x)) = x$$

So, the inverse function "undoes" the original function. That is we go from x to y, back to x. By doing this, I actually did a composition of functions.

We showed when we substituted $f^{-1}(x)$ into f(x) and got back x, we did a composition of functions.

If f and g are functions with domains D_f and D_g, respectively, and for each all $x \in D_f$

$$g(f(x)) = x$$

and for each $x \in D_g$ $\quad f(g(x)) = x$

then f and g are inverse functions

Big takeaway – not all rules can be combined, to operate with rules, they must be functions. Otherwise you will get more than one value for each x. Find the inverse of each of the following functions.

Example 3 If f(x) = 6x – 2 and g(x) = $\frac{x-2}{6}$, are f(x) and g(x) inverses?

Determine if f(g(x)) = g(f(x)) = x, then they are inverses.

$$\begin{aligned} f(g(x)) &= 6(g(x)) - 2 \\ &= 6\left(\frac{x-6}{6}\right) - 2 \\ &= x - 6 - 2 \\ &= x - 8 \end{aligned}$$

f and g are NOT inverses because f(g(x)) ≠ x. I can stop here.

Example 4 p(x) = $x^2 - 3$ and q(x) = $\sqrt[2]{x+3}$. Are p and q inverses?

We need to know of p(q(x)) = q(p(x)) and equal to x.

$$
\begin{aligned}
p(x) &= x^2 - 3 \\
p(q(x)) &= [q(x)]^2 - 3 \\
&= [\sqrt[2]{x+3}]^2 - 3 \\
&= x + 3 - 3 \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
q(x) &= \sqrt[2]{x+3} \\
q(p(x)) &= \sqrt[2]{p(x)+3} \\
&= \sqrt{(x^2-3)+3} \\
&= \sqrt[2]{x^2} \\
&= x
\end{aligned}
$$

Since p(q(x)) = q(p(x)) = x, these are inverse functions.

Later, when we work with logarithms, we will find the inverse of an exponential is still found by interchanging the domain and range, we will write that inverse as a logarithm.

As always, I cannot make finding inverses more difficult – only longer.

As an example, find the inverse of:

$$y = \sqrt[4]{\frac{3^x + 5}{7}}$$

Interchanging the x's and y's, we have:

$$x = \sqrt[4]{\frac{3^y + 5}{7}}$$

Now, we have to solve for y, by isolating the 3^y

$x^4 = \frac{3^y+5}{7}$ Raising each side to the power of 4

$7x^4 = 3^y + 5$ Multiplying both sides by 7

$7x^4 - 5 = 3^y$ Subtracting 3 from both sides

And now, by isolating the 3^y, we will learn in a later chapter how to rewrite this exponential as a logarithm and thereby writing its inverse.

$y = log_3(7x^4 - 5)$ Converting to a log