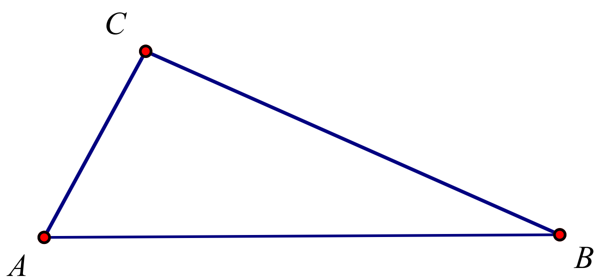
Chapter 6

Triangles & Polygons

A triangle (Δ) is the union of three segments determined by three noncollinear points.



Each of the three points, A, B and C is a **vertex** of the triangle.

$\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ are called the **sides** of the triangle. $\angle$A, $\angle$B, and $\angle$C are called the **angles** of the triangle.

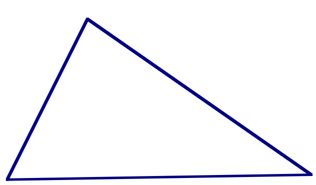
As there are parts to a triangle, we often look at different classifications of triangles for convenience in describing them. Sometimes they are classified by the number of congruent sides, sometimes by the number of congruent angles.

Triangle Classification by sides

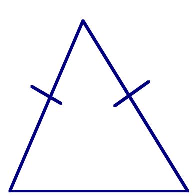
Scalene Δ - a Δ with no two sides $\cong$.

Isosceles Δ - a Δ with at least two sides $\cong$.

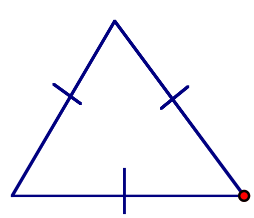
Equilateral Δ - a Δ with all sides $\cong$.



scalene Δ



isosceles Δ



equilateral Δ

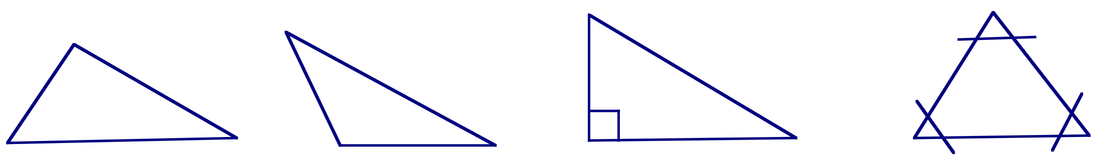
Triangle Classification by Angles

Acute Δ - a Δ with three acute ∠s.

Obtuse Δ - a Δ with an obtuse ∠s.

Right Δ - a Δ with a right ∠.

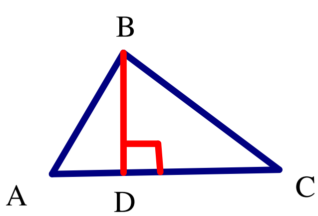
Equilangular Δ - a Δ with all ∠s ≅.



acute Δ *obtuse* Δ *right* Δ *equiangular* Δ

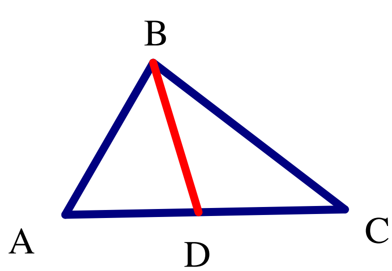
In later chapters, we will find more relationships when we examine segments and rays associated with triangles. For now, we will just identify a few of them; altitudes, medians and angle bisectors.

An altitude of a triangle is a segment that joins a vertex to a point on a line on the opposite side and is perpendicular to that line. An altitude can be drawn from each vertex.



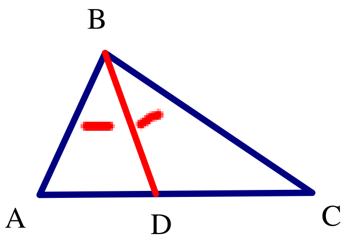
$\overline{BD}$ *is an altitude*

A median of a triangle is a segment whose endpoints are the vertex and the midpoint of the opposite side.



$\overline{BD}$ *is a median*

An angle bisector of a triangle is a segment contained in the ray bisecting an angle whose endpoints are the vertex and a point on the opposite side.

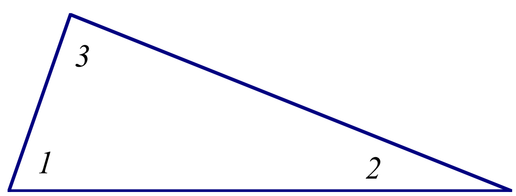


$\overline{BD}$ *is an angle bisector*

Angle Theorems: Triangles

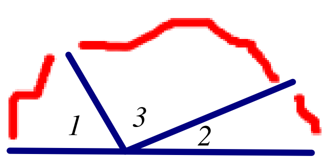
If I asked an entire class to draw a triangle on a piece of paper, then had each person cut out their triangle, we might see something interesting happen if they tore out the angles and placed them side by side.

Let's label the angles 1, 2, and 3 as shown.



First, we will cut the triangle out. Hopefully everyone's triangle will look different. Some might be scalene, right, or equilateral triangles. After cutting each triangle, we will number the angles and tear each angle from the triangle.

Placing those angles side by side, the three angles always seem to form a straight line. Neato!

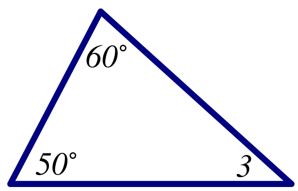


Since that would be occurring for everyone that drew the triangle, that might lead me to believe the sum of the three interior angles of a triangle is 180º.

While that's not a proof, it does provide me with some valuable insights. The fact is, it turns out to be true, so we write it as a theorem.

Theorem **The sum of the interior angles of a triangle is 180º.**

Example 1 Find the measure of $\angle 3$.



Since the sum of the two angles given is 110˚, $\angle 3$ must be 70˚

Drawing a triangle and cutting out the angles only suggests the sum of the interior angles of a Δ is 180˚. It is not a proof. Let's see what that proof might look like.

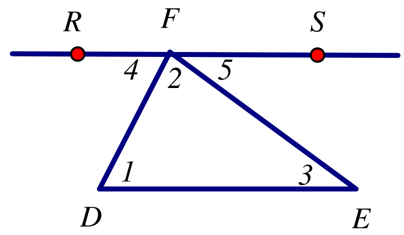
Now we will see the importance of knowing your theorems, postulates and definitions. If all I were to do was draw a triangle and label the three angles, I would not get far with the proof. What I need to do is use all the information I have learned before and apply it to this theorem.

Before this chapter, we were working with parallel lines. While triangles are certainly not made up of parallel lines, I can use my knowledge of geometry and draw a line parallel to the base of a triangle. I will use the reason as construction. But, we know, from a point not on a line, there is exactly one line parallel to that line through that point.

By introducing parallel lines, I will be able to use the relationships we learned previously about angles being congruent. You just gotta love math! Let's look at the proof of this theorem.

Triangle Sum Theorem

The sum of the measures of the interior angles of a triangle is 180º.



Given: ΔDEF
Prove: $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$

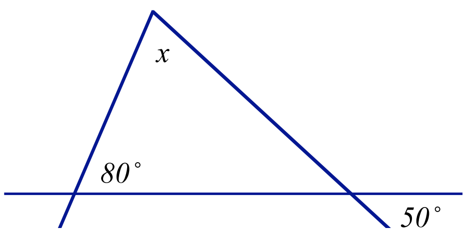
The most important part of this proof will be our ability to use the geometry we have already learned. If we just looked at the three angles of the triangle, we'd be looking for an awfully long time without much to show for it. What we will do is use what we just learned – we were just working with parallel lines, so what we will do is put parallel lines into our picture by constructing $\overleftrightarrow{RS}$ parallel to $\overleftrightarrow{DE}$ and labeling $\angle 4$ and $\angle 5$ that were formed. Now we have parallel lines being cut by transversals, we can use our knowledge of angle pairs being formed by parallel lines – alternate interior angles.

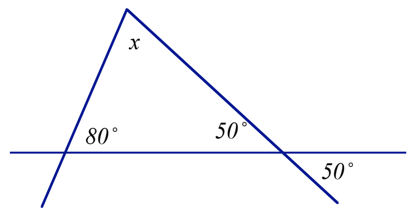
	Statements	**Reasons**
1.	Δ DEF	Given
2.	Draw RS || DE	Construction
3.	$\angle 4 \wedge \angle DFS$ are supp	Ext sides of 2 adj $\angle$'s - line
4.	$\angle 4 + \angle DFS = 180^\circ$	Def Supp $\angle$'s
5.	$\angle DFS = \angle 2 + \angle 5$	$\angle$Add Postulate
6.	$\angle 4 + (\angle 2 + \angle 5) = 180^\circ$	Substitution
7.	$\angle 1 = \angle 4$ $\angle 3 = \angle 5$	2 || lines cut by t, alt int $\angle$'s =
8.	$\angle 1 + \angle 2 + \angle 3 = 180^\circ$	Substitution in statement 6

Here's an important HINT; anytime a theorem, rule formula or postulate is given a name, it's because it's used often – so learn them. We just proved a theorem, it was given a name, Triangle Sum Theorem, it's important, know it!

Notice how important it is to integrate our knowledge of geometry into these problems. Step 3 would not have jumped out at you. It looks like angles 4, 2, and 5 form a straight angle, but we don't have a theorem to support that so we have to look at $\angle 4$ and $\angle DFS$ first.

Example 2 Find the value of x.





We just learned and proved the sum of the interior angles of a triangle measures 180˚. Let's use that information.

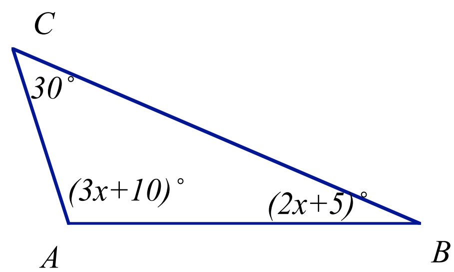
Notice in the diagram on the right, we were able to fill in one of the angles of the triangle because we know vertical angles are congruent.

Now, using the last theorem, the Triangle Sum Theorem, the sum of the interior angles of a triangle measures 180˚, we have

$$\begin{aligned} 80^\circ + 50^\circ + x &= 180^\circ \\ 130^\circ + x &= 180^\circ \\ x &= 50^\circ \end{aligned}$$

Let's look at the same type problem using algebra rather than numbers. Keep in mind, our mathematical relationships don't change. In other words, the sum of the interior angles of a triangle is still 180˚

Example 3 Find the $m\angle A$



From the Triangle Sum Theorem we know

$$m\angle A + m\angle B + m\angle C = 180^\circ.$$

Use that to find the value of x, then substitute that in the expression representing m∠A

$$\angle A \quad + \angle B \quad + \angle C = 180^\circ$$

$$\begin{aligned}(3x+10)^\circ + (2x+5)^\circ + 30^\circ &= 180^\circ \\ 5x + 45 &= 180^\circ \\ 5x &= 135 \\ x &= 27^\circ\end{aligned}$$

$$m\angle A = (3x+10)^\circ \rightarrow m\angle A = 3(27) + 10 = 91^\circ$$

If we did a few more problems with this theorem, *the sum of the interior angles of a triangle measures 180˚,* we would see some other relationships very quickly. When we see relationships that seem to follow directly from a theorem, we call those ***corollaries***.

Corollary 1 **If 2 ∠s of one Δ are ≅ to 2 ∠s of another Δ, the third ∠s are ≅.**

We will use this corollary quite frequently. This follows directly from the sum of the three interior angles must be 180˚.

Corollary 2 **The measure of each ∠ of an equilateral Δ must be 60˚.**

Again, if the sum is 180˚, then each equal ∠ must have measure 60˚.

Corollary 3 **In a Δ there can be at most one right ∠ or one obtuse ∠.**

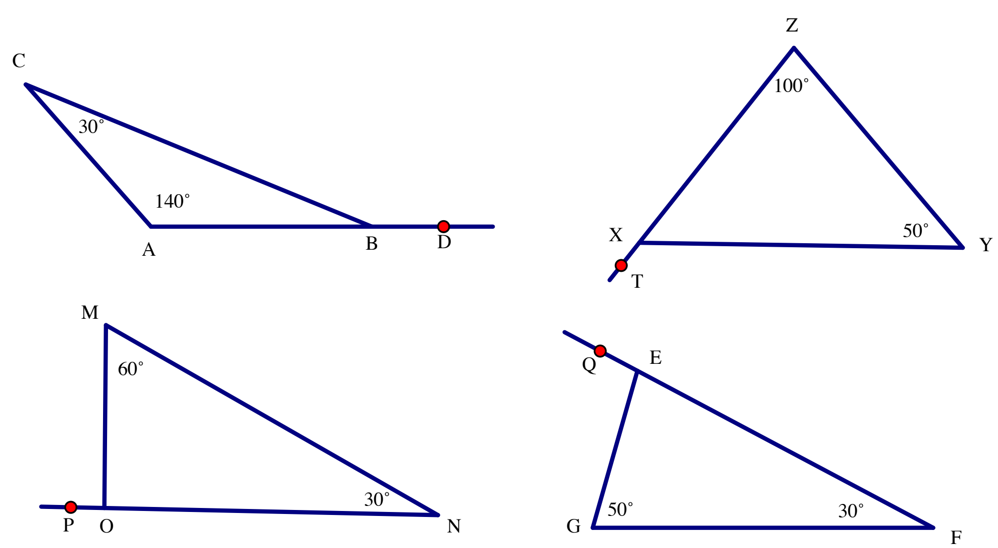
If there were more than one right or obtuse angle, the sum would be greater than 180˚.

Corollary 4. **The acute angles of a right Δ are complementary.**

Still again, this follows because the sum of the three ∠s must be 180˚, if one of the ∠s is 90˚, then the sum of the other two ∠s must be 90˚.

A theorem that seems to follows as a consequence from that theorem is one about the relationship between the exterior angle of a triangle and angles inside the triangle. If we drew three or four triangles and labeled their interior angles, we would see a relationship between the two remote interior angles and the exterior angle. Let's take a look at a few triangles and see if we can see the relationship. The two remote interior angles are the two non-adjacent angles in the triangle.

Example 4 Find the missing measure of each of the following interior triangle



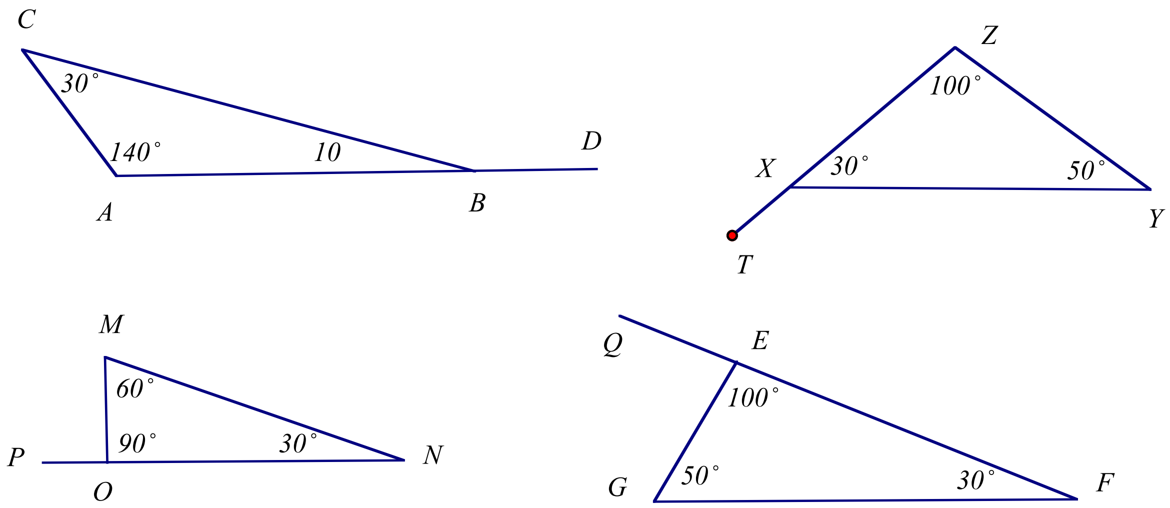
Using the Triangle Sum Theorem, we can determine the third angle of the triangles.

ΔABC, $m\angle B = 10^\circ$.
ΔXYZ, $m\angle X = 30^\circ$,
ΔMON, $m\angle O = 90^\circ$,
ΔEFG, $m\angle E = 100^\circ$

We can find those measures because we know the sum of the interior angles of a triangle measure 180°.

Using those measures, can we find the measure of the exterior angles drawn.

Example 5 Find the measure of the exterior angles shown: $\angle CBD$, $\angle TXY$, $\angle MOP$ and $\angle GEQ$



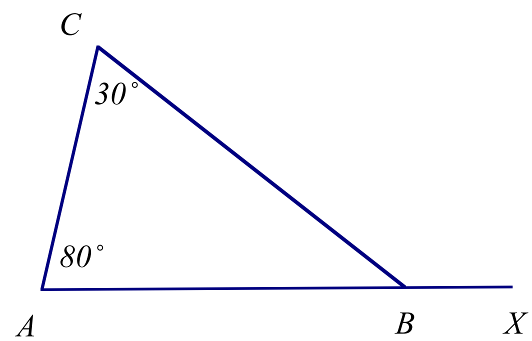
See any relationship between those angles? Filling in the measures of each of the angles we have found, we can find the measure of each exterior angle drawn by knowing a previous theorem, *if the exterior sides of two adjacent angles lie in a line, the angles are supplementary* – measure 180°.

That would mean $m\angle CBD = 170^\circ$, $m\angle TXY = 150^\circ$, $m\angle MOP = 90^\circ$, and $m\angle GEQ = 80^\circ$.

So, not only could we find the missing third angles, we could find the measure of the exterior angles given to us in the diagrams.

If we examined these problems long enough, we might also see a relationship between those exterior angles and the interior angles of a triangle. The relationship is the measure of the exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles. Look at those in the diagram.

Example 6 Find the measure of the exterior angle, $m\angle CBX$.

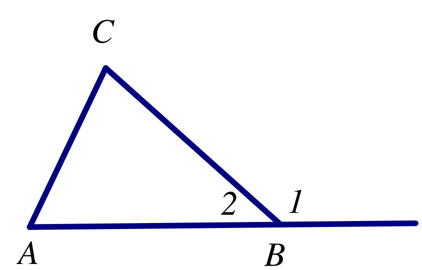


Using our observation, $m\angle CBX = 110˚$, the sum of the two non-adjacent angles, 80 and 30 = 110.

Or you could have done it the long way, find the $m\angle B$ using the Triangle Sum Theorem, $m\angle B = 70˚$. Then use supplementary angles to find $m\angle CBX$.

That suggests to prove this we will use those two relationships, the sum of the interior angles of a triangle is 180˚ and if the exterior sides of two adjacent angles lie in a lie, they are supplementary. Since both relationships equal 180˚, we will set them equal and solve the resulting equation. Now, let's look at the proof.

Theorem: **The exterior $\angle$ of a Δ is equal to the sum of the 2 remote interior $\angle$'s**



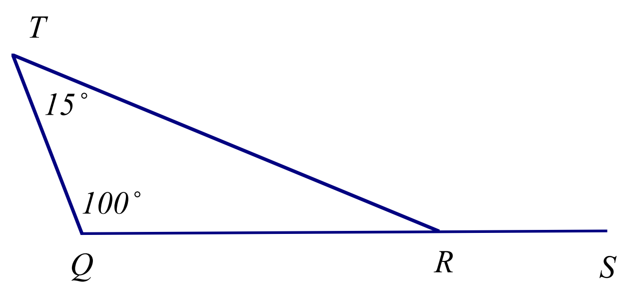
Given: ΔABC

Prove: $m\angle 1 = m\angle A + m\angle C$

Use the diagram and the geometry we know.

	Statements	**Reasons**
1.	$m\angle A + m\angle C + m\angle 2 = 180^\circ$	Triangle Sum Theorem
2.	$\angle 1$ and $\angle 2$ are supp $\angle$'s	Ext sides 2 adj $\angle$'s - line
3.	$m\angle 1 + m\angle 2 = 180^\circ$	Def Supp $\angle$'s
4.	$\angle A + \angle C + \angle 2 = \angle 1 + \angle 2$	Substitution or Transitive Prop
5.	$\angle A + m\angle C = m\angle 1$	Subrat Prop Equality

Example 7 Find the $m\angle TRS$



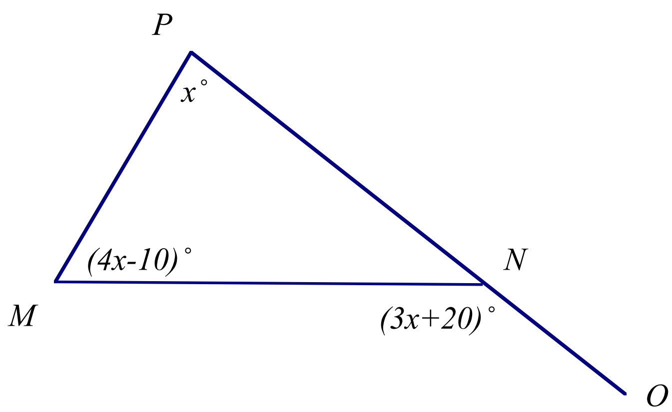
By the last theorem, we now know the exterior angle of a triangle is equal to the sum of the 2 remote interior angles

$$m\angle TRS = m\angle T + m\angle Q$$
$$m\angle TRS = 15^\circ + 100^\circ$$
$$m\angle TRS = 115^\circ$$

Knowing that theorem, it's clear you could do this problem mentally.

Now, let's do the same type problem using variables. The good news in math, nothing changes. If we know the theorem, we just set the problem the sme way and solve the resulting equation.

Example 8 Find the $m\angle M$



To find $m\angle M$, we have to find the value of x, then substitute that back into the expression representing $m\angle M$.

Using the theorem, the ext $\angle$ of a Δ is = to the sum of the 2 remote int. $\angle$s, we have

$$m\angle MNO = m\angle M + m\angle P$$
$$3x + 20 = (4x - 10) + x$$
$$3x + 20 = 5x - 10$$
$$30 = 2x$$
$$15 = x$$

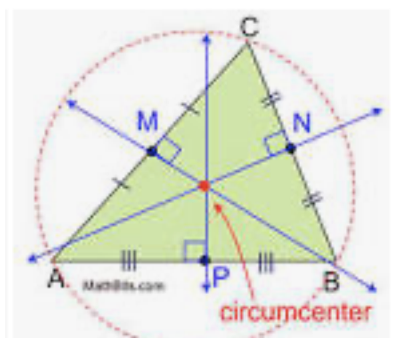
If x = 15 and $m\angle M = (4x-10)^\circ$, then $m\angle M = 4(15) - 10 = 50^\circ$

4 Measures of Center
Triangles

1. Circumcenter
2. Incenter
3. Centroid
4. Orthocenter

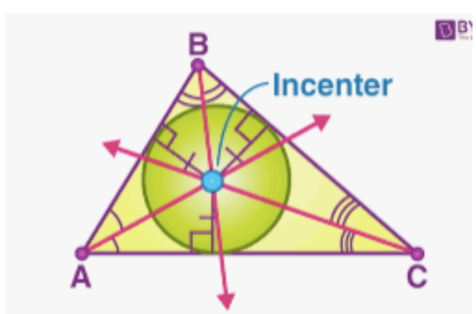
Circumcenter

The circumcenter is the point of intersection found by constructing the **perpendicular bisectors** of each side of a triangle. That point represents the center of a circle whose radius is the distance between that point any vertex of the triangle. The circle is "circumscribed" about the triangle.



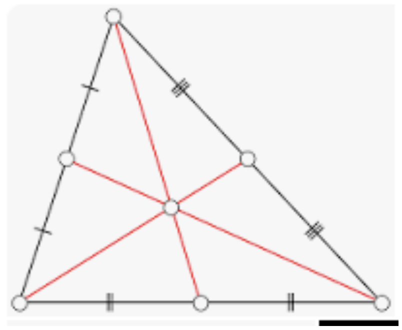
Incenter

The incenter is the point of intersection found by constructing the **angle bisectors** of each angle of a triangle. That point represents the center of a circle whose radius is the distance between that point and the perpendicular distance to any side of the triangle. The circle is inscribed in the triangle.



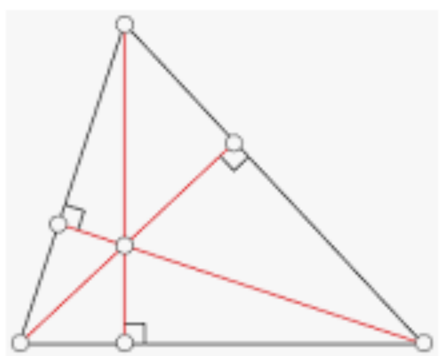
Centroid

The centroid is the point of intersection found by constructing the **medians** of each side of a triangle. That point represents the center of a circle that balances the triangle. That point separates the median into two parts with a 2:1 ratio.



Orthocenter

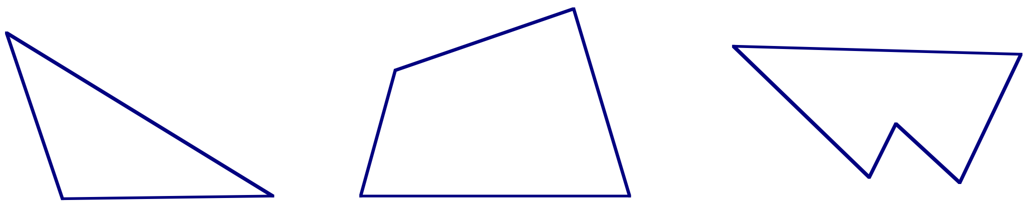
The orthocenter is the point of intersection found by constructing the **altitudes** of each side of a triangle. The products of the lengths of the segments into which the orthocenter divides each altitude is a constant.



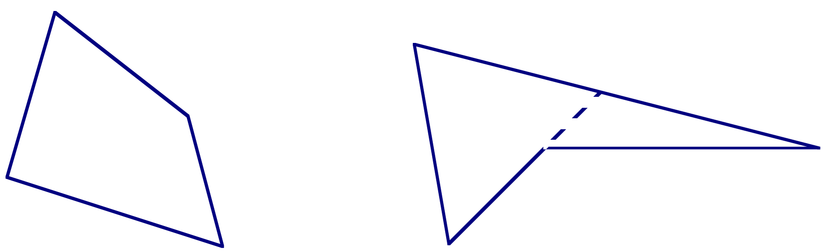
Polygons

A **polygon** can be described by two conditions:

1. No two segments with a common endpoint are collinear.
2. Each segment intersects exactly two other segments, but only on the endpoints.



Convex polygon - **a polygon such that no line containing a side of the polygon will contain a point in the interior of the polygon.**

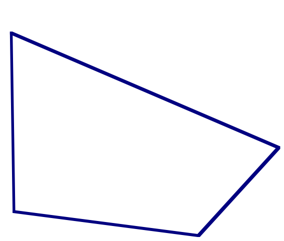


convex polygon *concave polygon*

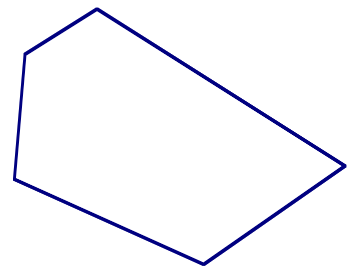
Notice, if I extend a side (dashed line) on the right polygon, the line contains interior points. Therefore, it is not a convex polygon, it is called a **concave** polygon.

What's the sum of the interior angles of a quadrilateral? a pentagon? an octagon?

The answer is, I just don't know. But …, if I draw some pictures, that might help me discover the answer.

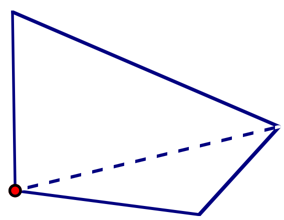


quadrilateral

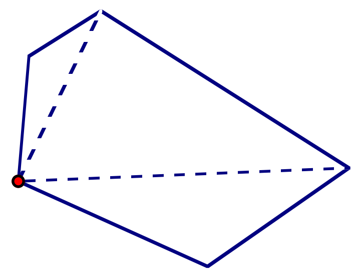


pentagon

By drawing as many diagonals as I can from a single vertex, I can form triangles. A diagonal is a line segment that connects two non-consecutive vertices.



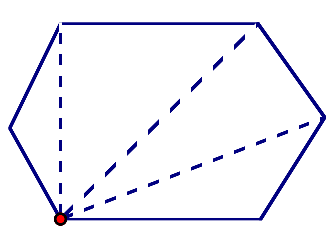
4 sides - 2 triangles



5 sides - 3 triangles

Those observations might lead me to believe the sum of the interior angles of a quadrilateral is 360º, Triangle Sum Theorem, because there are two triangles formed. In a five-sided figure, a **pentagon**, three triangles are being formed. Since the sum of the interior angles of each triangle is 180º, then the sum of the interior angles of a pentagon must be 3(180˚) which is 540º.

The number of triangles being formed seems to be two less than the number of sides in the polygon. Try drawing a hexagon and see if the number of triangles formed is two less than the number of sides. Look at the diagrams.

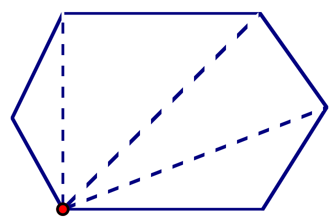


6 sides - 4 triangles
formed from one vertex

These examples suggest a polygon with ***n*** sides would have **(*n* – 2)** triangles formed. So, by multiplying the number of triangles formed by 180º, that should give me the sum of the interior angles. Sounds like a theorem to me.

Theorem **The sum of the interior angles of a convex polygon is given by (n – 2) 180º**

Example 1 Find the sum of the interior angles of a hexagon.



From the last theorem, we know the sum of the interior $\angle$s of a polygon is $(n–2)\ 180^\circ$

A hexagon has 6 sides, so $n = 6$ and the number of triangles formed is two less than the number of sides.

$$\begin{aligned}\text{Sum int } \angle\text{s} &= (6–2)180^\circ \\ &= (4)180^\circ \\ &= 720^\circ\end{aligned}$$

That's just too easy!

Regular polygon - a convex polygon with all angles and segments congruent.

Example 2 Find the measure of each angle of a regular octagon.

Knowing that an octagon has eight sides, I don't need to draw the picture. All I need to do is find the sum of the interior angles and divide that result by the number of angles – 8.

Sum int ∠s octagon = $(n–2)180^\circ$

$= (8–2)180^\circ$

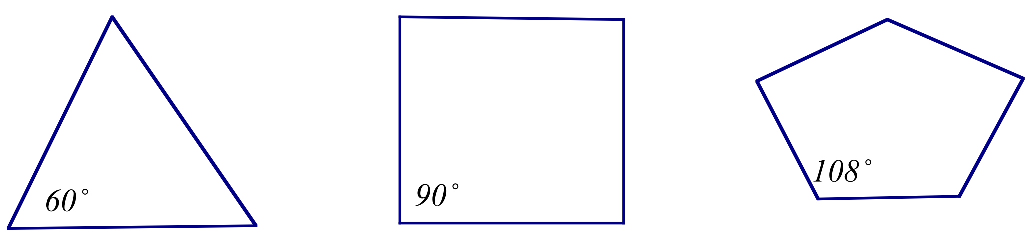
$= (6)180^\circ$
$= 1080^\circ$

Now that I know the sum of the interior angles, I divide that by the number of sides/angles.

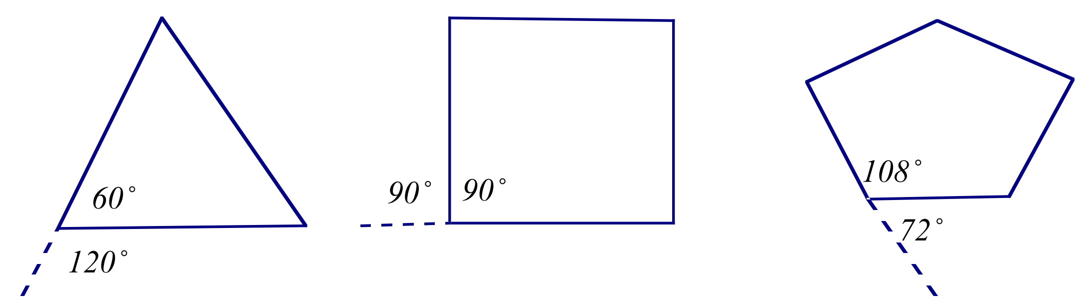
Each ∠ of a regular octagon = $1080/8 = 135^\circ$

Exterior Angles – Polygons

If we played with these pictures of regular polygons longer, we'd find more good news that would lead to another theorem.



Let's say we drew a number of regular polygons, polygons whose sides and angles are congruent as we just did. We could find the measure of each exterior angle of the triangle, one angle at each vertex, because we have two adjacent angles whose exteriors sides lie in a line – they are supplementary.



If each interior angle of a triangle measures 60˚, then each exterior angle would have measure 120˚. And the sum of the exterior angles would be 360˚.

Looking at the square, each interior angle would measure 90˚, each exterior angle would also measure 90˚. And the sum of the exterior angles would be 360˚.

And finally, in the pentagon, each interior angle measures 108˚, then each exterior would measure 72˚. And the sum of the exterior angles would be 360˚.

Is there a pattern developing there?

Let's summarize, in the triangle there are 3 exterior angles each measuring 120˚, their sum is 360˚

In the square, there are four exterior angles each measuring 90˚, their sum is 360˚.

In the pentagon, there are five exterior angles each measuring 72˚, their sum is 360˚.

It would appear the sum of the exterior angles of the examples we used seem to be 360˚. That might lead us to the following theorem.

Theorem **The sum of the exterior angles of a convex polygon, one angle at each vertex, is 360º.**

The sum of the interior angles of a polygon changes with the number of sides. But the sum of the exterior angles always measure 360˚. That's important to know.

Assume I have a regular polygon whose exterior angle measured 40˚ and I wanted to know how many sides the polygon had, could I make that determination?

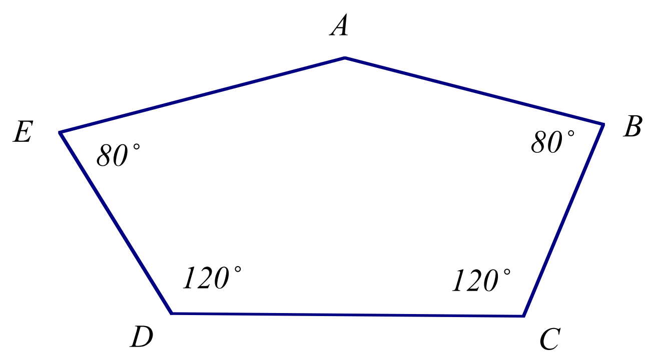
Since the polygon is regular, we know all angles must be congruent. If each exterior angle measured 40˚ and the sum of the exterior angles must be 360˚, then $360^\circ \div 40^\circ = 9$. The polygon must have 9 sides!

Let's make a minor change, let's say the interior angle of a regular polygon measured 150˚, how many sides would it have? To make this determination, I would have to know the measure of the exterior angle. Since the interior angle measures 150˚, the exterior angle must measure 30˚since they would form a straight angle.

If each exterior angle measured 30˚, the 360˚ ÷ 30˚ = 12.

The polygon would have 12 sides.

Example 1 Given the diagram, find m∠A



Since this is a pentagon, the sum of the interior angles will be given by the formula (n–2)180˚, where n = 5. Substituting, we have (3)180˚ = 540˚

The sum of the given angles is:

$$m\angle B + m\angle C + m\angle D + m\angle E = 400^\circ$$

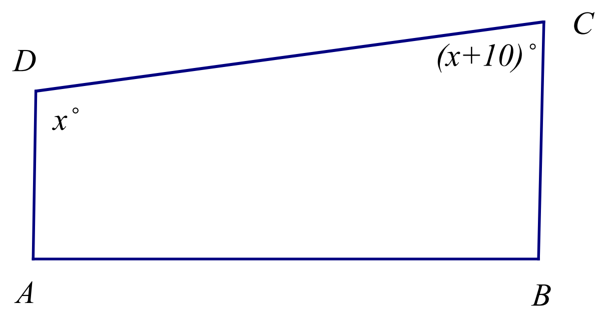
Since the sum of the 5 ∠s must be 540˚, and the 4 ∠s = 400,

$$m\angle A + [m\angle B + m\angle C + m\angle D + m\angle E] = 540^\circ$$

$$m\angle A + 400^\circ = 540^\circ$$

$$m\angle A = 140^\circ$$

Example 2 Given $\overline{AD} \parallel \overline{BC}$, find the value of x.



In this example, two lines were given as parallel. That should make you take note. What do I know about angles when two parallel lines are cut by a transversal?

One of the theorems we learned was *if two parallel lines are cut by a transversal, the same side interior angles are supplementary.*

$\angle$D and $\angle$C are same side interior angles.

That tells me that $m\angle D + m\angle C = 180^\circ$

$$x + (x+10) = 180$$

$$2x + 10 = 180$$

$$2x = 170$$

$$x = 85^\circ$$

Example 3 If the interior angle of a regular polygon measures 120˚, how many sides does it have?

An interior and exterior angle of a polygon form a straight line, thy are supplementary. If the interior angles measures120˚, then the exterior measures 60˚.

Since the sum of the exterior angles of a polygon measure 360˚ and one exterior angles measures 60˚, then finding the quotient; 360/60 = 6. The polygon has 6 sides.

<u>Angles of a Polygon</u>

The sum of the interior angles of a triangle is 180˚

The exterior ∠ of a triangle is equal to the sum of the 2 remote int ∠'s

The sum of the interior angles of a convex polygon is given by (n – 2)180˚

The sum of the exterior angles of a convex polygon is 360˚

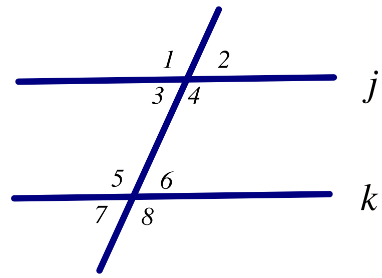
Polygon classification

Polygons are often classified by the number of sides.

Triangles –	3 sides
Quadrilaterals –	4 sides
Pentagons–	5 sides
Hexagons–	6 sides
Heptagons–	7 sides
Octagons–	8 sides
Nonagons–	9 sides
Decagons–	10 sides

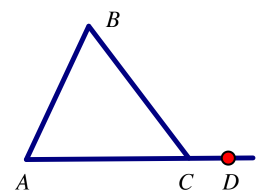
Angles; Parallel lines & Polygons

1. If j || k, then ∠ 5 is congruent to what other angles? Give a reason for each.



2. If $j \parallel k$ and $m\angle 1 = 100^\circ$, then $\angle 6 =$

3. The number of obtuse angles in an obtuse triangle is



4. If $m\angle A = 50^\circ$ and $m\angle B = 80^\circ$, then $\angle BCD$ equals

5. The sum of the measures of the interior angles of a convex octagon is

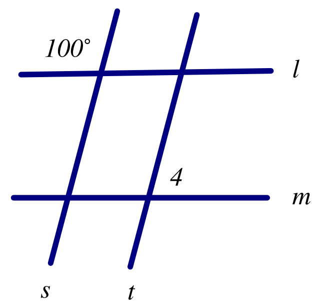
6. The sum of the measures of the exterior angles of a convex pentagon is

7. The total number of diagonals that can be drawn from one vertex of a hexagon is

8. In Δ ABC, $\overrightarrow{BX}$ bisects ∠ ABC, $m\angle A = 110^\circ$ and $m\angle C = 40^\circ$, then $m\angle ABX$ is

9. If the measure of the interior angle of a regular polygon is 140˚, how many sides does the polygon have?

10. Given the figure; $l \parallel m$ and $s \parallel t$ find the $m\angle 4$.



11. Explain why it is not possible for the measure of an exterior angle of regular convex polygon to be 22˚