Areas

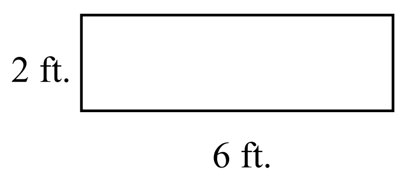
Rectangles

One way to describe the size of a room is by naming its dimensions. So a room that measures 12 ft. by 10 ft. could be described by saying its a 12 by 10 foot room. In fact, that is how it is often written in blue prints.

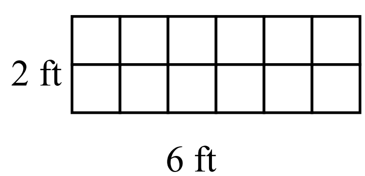
In geometry, rather than talking about a room, we might talk about the size of a polygonal region.

For instance, let's say I have a closet with dimensions 2 feet by 6 feet. That's the size of the closet.

Someone else might choose to describe the closet by determining how many square tiles it would take to cover the floor.



To demonstrate, let me divide that closet into one-foot squares.



By simply counting the number of squares that fit inside that region, we find there are 12 equally sized squares.

Now, if I repeated this process for a smaller closet, let's say a 4 foot by 3 foot closet. We could draw the shape, divide the closet into one foot squares, then count the number of equally sized squares formed. Doing this, we'd see there 12 one foot squares.

If I were to continue this process for larger rooms, we might want to know the area of a 12 foot by 10 foot room. Again, I could draw the picture, divide the picture into equally sized squares, then count the number of squares. That would result in 120 squares.

Placing the figures side by side with their dimensions and number of square tiles needed to cover the room, we might see a pattern develop. The number of equally sized squares that cover the diagram is equal to the product of the dimensions.

So this is almost too good to be true, rather than drawing the picture, dividing the diagram into equally sized squares, then counting them, it looks like I can determine the number of equally sized squares just by multiplying their dimensions.

Before we go to far, let's identify the shape of the room as a **RECTANGLE**.

Rectangle is a quadrilateral (4-sided polygon) with 4 right angles and the opposite sides are parallel and have equal lengths.

Rectangle

Rather than saying the number of equally sized squares that cover the rectangle, let's agree that will be called the **AREA** from now on. That is to say, if I ask for the area of a rectangle, I am asking how many equally sized squares will fit inside.

To find the number of equally sized squares - Area, we multiply the dimensions. The dimensions are made of two numbers. We will call the larger number the **LENGTH** and the smaller number the **WIDTH**.

Okay, we introduced some vocabulary you need to be familiar with; rectangle, dimensions, area, length and width.

Seeing the pattern develop with our examples and using the vocabulary leads us to a shortcut described by the following postulate.

Postulate The area of a rectangle is equal to the product of the length and the width, given by the formula $\mathbf{A} = \boldsymbol{l} \times \boldsymbol{w}$

Putting this into perspective, we see the number of squares that fit inside the rectangle is referred to as the **area**. A shortcut to determine that number of squares is to multiply the length by the width.

Most books refer to the longer side of a rectangle as the length (l), the shorter side as the width (w). Other books might show that same relationship by saying base x height, or

A = b x h We will use

$$\mathbf{A = \mathit{lw}}$$

The answer in an area problem is always given in square measure because we are determining how many equally sized squares fit inside the region. Square measure can be written as sq. units or units^2.

CAUTION!

When finding the area, the dimensions (length & width) must be given in the same unit.

Example 1 Find the area of a rectangle whose length is 5 yards and width is 12 feet.

To do this problem, you first must convert the 12 feet to yards or the 5 yards to feet so you have the same units.

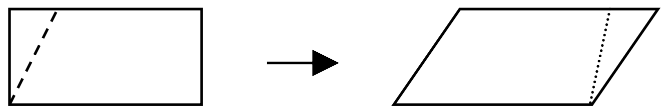
Using the formula, we have $A = lw$

Method 1, using feet	**Method 2**, using yards
5 yards = 15 feet	12 feet = 4 yards
A = 15 x 12	A = 5 x 4
A = 180 square feet	A = 20 square yards

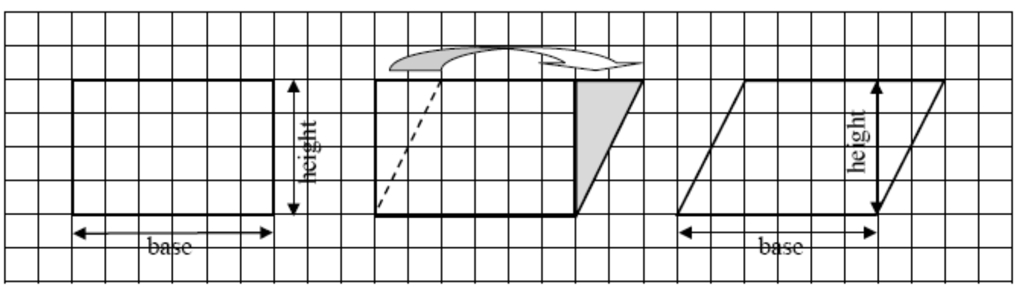
Both these answers are correct. Which answer is preferable might be determined by your teacher or your occupation. While many of us measure length using feet, when you buy carpet, they sell it by the square yard.

Parallelograms

If I were to take a rectangle and cut a corner off one side and attach it the other side, do you think the area would change? The answer is no, all we did was rearrange the area.

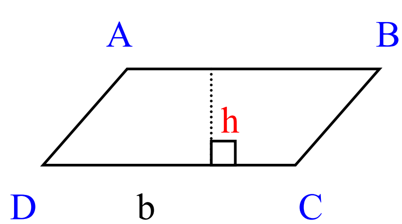


That's good news because we can use the previous formula to find the area of this polygon. Since the shape has changed, so we need to name that differently.



In the first picture we have a rectangle, so the area is described by length multiplied by the width width. What we will do is change that notation to base x height as shown. The new figure (polygon) will be called a **PARALLELOGRAM**.

PARALLELOGRAM A quadrilateral in which both pairs of opposite sides are parallel.



Since all we did was rearrange the area, the area of the parallelogram should be given by the same formula, base x height. The height of a parallelogram was the width of the rectangle that formed it and the base is the length.

Theorem The area of a parallelogram is equal to the product of the length of the base and the length of the corresponding height.

$$\mathbf{A} = \boldsymbol{bh}$$

CAUTION!
The height is always the **shortest** distance from top to bottom.

Example 2 Find the area of a parallelogram if the base is 3 meters and the height is twice the base.

$$\mathbf{A} = \boldsymbol{bh}$$

b = 3 and h is twice b. So, h = 6.

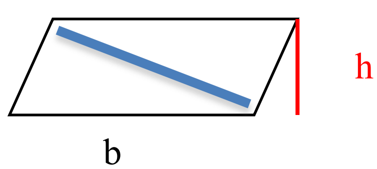
A = 3 x 6
A = 18 square meters

Area – Triangles & Trapezoids

Triangles

In the last section, we found the formulas to find areas of rectangles and parallelograms. More importantly, we saw the formula for a parallelogram came directly from the area formula of a rectangle.

If I were to play with parallelograms, I might find that if I cut a parallelogram along one of its diagonals, it would cut the parallelogram into two congruent polygons called TRIANGLES.



That means these two triangles would fit right on top of each other, they coincide, they would have the same area. If the area of the parallelogram is given by the formula **bh**, then each triangle would be half that. So, we find the formula for the area of a triangle is given by the formula:

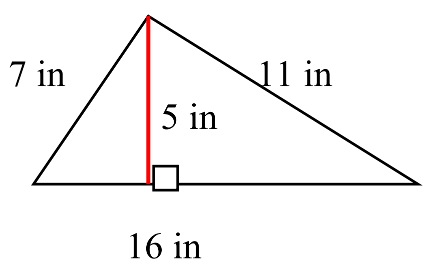
$$\mathbf{A = \frac{1}{2} bh}$$

Triangle is a polygon made up of three sides.

Theorem The area of a triangle is equal to one-half the product of the base and height.

Just like for parallelograms, the height is always the shortest distance from top to bottom.

Example 3 Find the area of the following triangle.



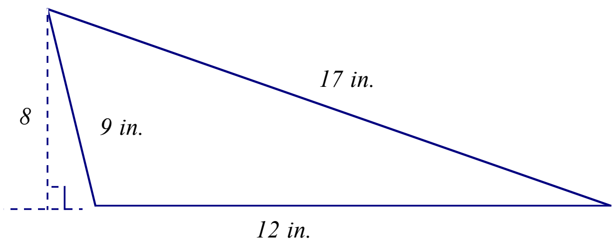
First thing you have to know is the polygon is a triangle. The area formula for a triangle is

$$\mathbf{A = \frac{1}{2} bh}$$

The base is 16 inches, the height is 5 inches. Filling those values into the formula, the area is 40 square inches. Notice the answer is given in square measure. That's because we are measuring how many squares will fit inside the triangle.

Also notice I did not use the 7 or 11 that were given in the problem. The 5 is the shortest distance from top to bottom, therefore 5 is the height.

Example 4 Find the area of the triangle.



$$A = \frac{1}{2} bh$$

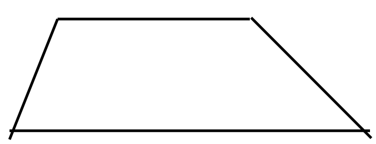
The base is 12 and the height, shortest distance from top to bottom, is 8.

A = ½ (12)(8)
A = 48 sq. in.

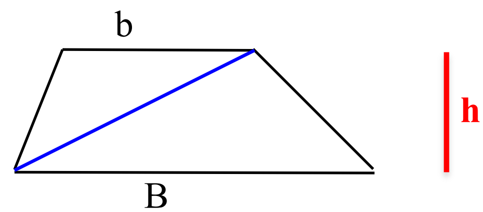
Notice the 9 and 17 inches had nothing to do with finding the area of the triangle.

Trapezoids

Trapezoid is a quadrilateral with one set of parallel sides. The parallel sides are called the **bases,** the other two sides are called the **legs**.



Get out your scissors. If I were to cut a trapezoid along one of its diagonals, two triangles would be formed. In this case, you can see they are not congruent.



However, I do notice I have two triangles formed. That means if I could find the area of each triangle, then add them together, I would have found the area of the trapezoid. Don't you just love math?

What's interesting is the two triangles formed have the same height - **h**.

The area of the triangle on the left is $\frac{1}{2}$ **bh.**

The area of the triangle on the right is $\frac{1}{2}$ **Bh**.

That means the area of the trapezoid is given by

$$A_{\text{trap}} = \frac{1}{2}\mathbf{Bh} + \frac{1}{2}\mathbf{bh}$$

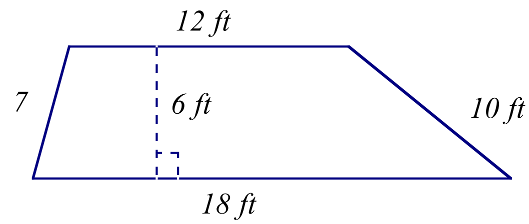
Factoring out $\frac{1}{2}$h, I have $\frac{1}{2}$h(B + b). Putting the variables in alphabetical order, we now have the formula for a trapezoid

$$\mathbf{A} = \frac{1}{2}\,\mathbf{(B + b)\ h}$$

Theorem The area of a trapezoid is the product of half the sum of the bases and the height.

Notice that this formula is related to the area formula for a triangle, the triangle was derived from the parallelogram, and the parallelogram from the shortcut we learned about rectangles.

Example 5 Find the area of the trapezoid.



$$\mathbf{A} = \frac{1}{2}\,(\mathbf{B} + \mathbf{b})\,\mathbf{h}$$

In this example, the bases are 18 and 12 ft, the height is 6 ft. Substituting those into our formula, we have:

$A = ½ (12 + 18)6$
$A = ½ (30)6$
$A = 90$ sq. ft

Area – irregular & shaded regions

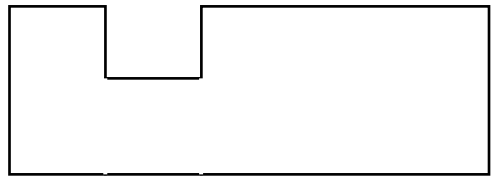
Irregular Shapes

If you can find areas of polygons like rectangles, triangles, parallelograms, and trapezoids, then finding areas of irregularly shaped figures should not be a big deal.

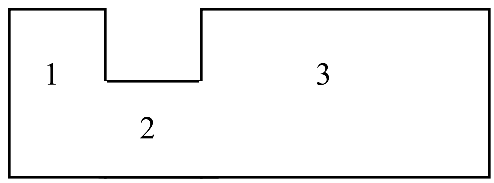
In fact, if you think about it, you have already found areas of irregular polygons. We will use the same method to find irregular shapes that we used in finding the area of a trapezoid.

Remember, we said a trapezoid was made up of two triangles, we found the area of each, and then added them together. Piece of cake, right?

We will do the same thing for irregularly shaped polygons.

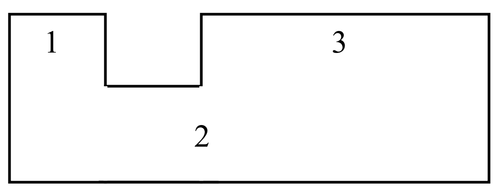


Looking at the picture, we realize that we do not know a formula to find the area. But, we can divide that polygon into shapes we do recognize and have area formulas.



That gives us three rectangles, all of which have an area given by LW. If we add them together, we have the area of the irregular polygon.

Someone else may have divided the region like this.

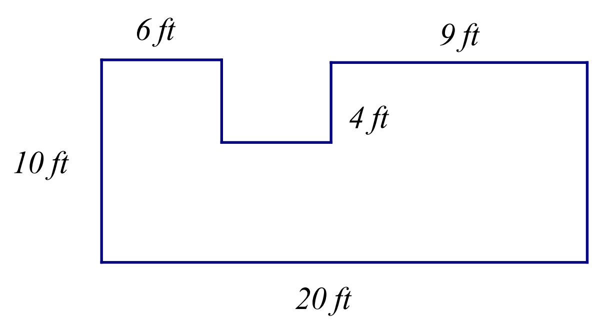


Again, we see that the figure that we did not recognize is made up of polygons for which we know area formulas.

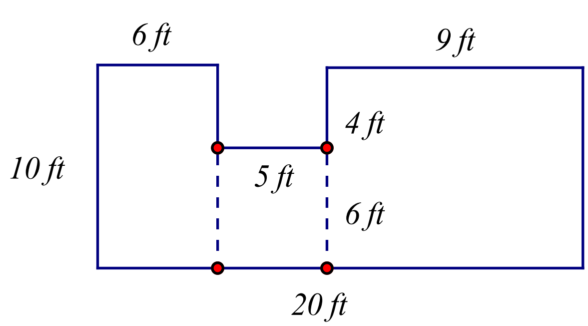
So, we do what we always do in math. That is, we take something we don't recognize or immediately know how to solve and change it into a problem that we do recognize and are able to do. So, we will break irregular shapes into shapes we recognize, find each area, then add them together to find the area of the irregular shape. That's the method we used for finding the area of a trapezoid.

Areas of Irregular shapes divide the irregular shaped polygon into shapes you recognize, find the area of each, then add the areas together.

Example 6 Find the area of the following polygon.



The irregular shaped polygon can be broken into smaller rectangles and we should be able to find any missing measurements. Notice, the length of the base is 20 ft and the two measurements across the top are 6 ft and 9 ft for a total of 15 ft. What is the length of the missing segment? Well. 20 – 15 = 5. So the missing segment is 5 ft.



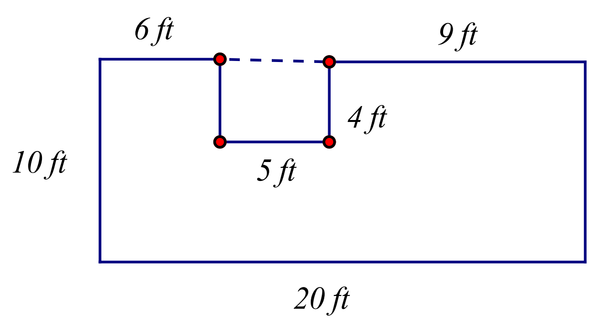
Now we have 3 rectangles and we filled in the missing measurements. The area of the rectangle on the right is A_r, the area of the middle rectangle is A_m, and the area of the rectangle on the left is A_l.

$A_r = 10 \text{ x } 6 = 60$ $A_m = 5 \text{ x } 6 = 30$ $A_l = 9 \text{ x } 10 = 90$

$$A_{\text{irregular polygon}} = A_r + A_m + A_l$$
$$A_{\text{irregular polygon}} = 60 + 30 + 90$$
$$A_{\text{irregular polygon}} = 180 \text{ square ft.}$$

Somebody else could have divided the irregular polygon horizontally, the arithmetic would have been different, but the answer would be the same.

Still, another person may have addressed the problem from a completely different viewpoint by filling in the missing piece as shown.



Doing the problem this way, we find the area of the completed rectangle which has dimensions 20 ft by 10 ft, that area is 200 square ft.

Subtract from that area, the area of the piece we added. That's a rectangle with dimensions 5 ft by 4 ft, the area is 20 square ft.

$$A_{\text{irregular polygon}} = 200 - 20 = 180 \text{ square ft.}$$

Notice, that is the same area we calculated before by breaking the irregular polygon into rectangles.

Area – circles

To find the area formula of a circle, we will use the same techniques we used to find areas of other polygonal regions. That is, we make it look like something we already have studied.

Since a circle does not have sides, it is not a polygon. So, the distance around it is not called perimeter, it's called a **circumference**.

Interestingly, it turns out that the circumference of any circle divided by the diameter always equals 3.1415926545….

$$\frac{\mathbf{c}}{\mathbf{d}} = 3.1415926545\ldots$$

Since 3.1415926545…is a non-repeating non terminating decimal which we can only approximate using decimals, we'll give that number a special name, π (pi).

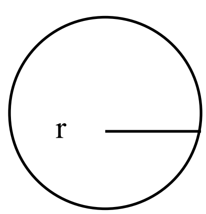
That means we can rewrite that formula as

$$\frac{\mathbf{c}}{\mathbf{d}} = \pi$$

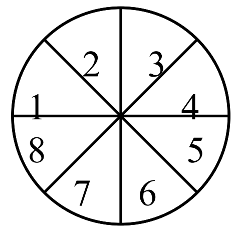
An equivalent expression is $\mathbf{C} = \pi\mathbf{d}$. Since a diameter has the same measure as two radii, we can also write the circumference formula in terms of the radius.

$$\mathbf{c = 2}\ \pi\mathbf{r}$$

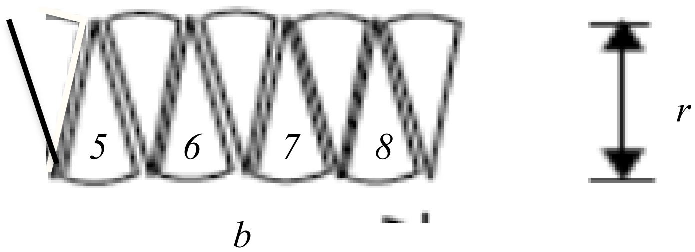
With that information, we can now try to find a way to determine the area of a circle with radius **r**



To do that, I will break the circle up in eighths.



Now, I'm going to cut out those 8 sections and rearrange them as shown.



I need you to notice a couple of things before we move forward. Note from the circle, the bottom, sections compose half the distance around the circle – sections 5, 6, 7, & 8. That means that is equal to half the circumference. Also note that the rearrangement begins to look like a parallelogram. It would look more and more like a **parallelogram** if I continued to break up the circle into more sections.

Remember, the area of a parallelogram is given by the formula **bh**. The base in our rearrangement is composed of the bottom sections, which is half the circumference of 2 πr or πr The height of the parallelogram is the radius.

Using the formula for parallelogram, we have

$$\mathbf{A = bh}$$

Substituting π**r** for **b** and **r** for **h**, we have

$$\mathbf{A = (\pi r)r} \quad \text{OR} \quad \mathbf{A = \pi r^2}$$

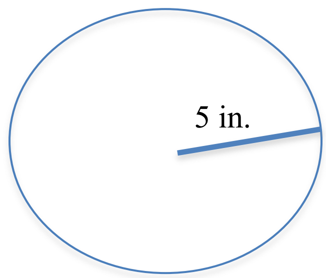
That means the area of the sections is $\pi \mathbf{r^2}$. Putting those sections back together, we have the area of a circle.

Theorem The area of a circle is equal to the product of π and the square of the radius of the circle.

$$\mathbf{A = \pi r^2}$$

Because we are measuring area, the answer is always written in square measure.

Example 7 Find the area of the circle.

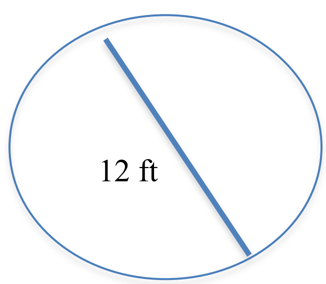


The radius is 5 in. Using the formula $A = \pi r^2$, r = 5 in.

$$\pi\, 5^2 = 25\,\pi \text{ sq. in.}$$

If you substitute 22/7 or 3.14 for π, but if you should not use an "=" sign.

Example 8 Find the area of the circle.

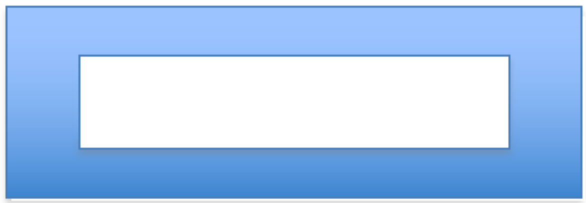


In this circle, the diameter is 12 ft. That means the radius is 6 ft.

Therefore we have $A = \pi\, 6^2$
$A = 36\,\pi$ square ft.

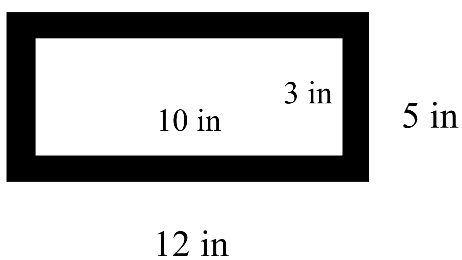
Areas of Shaded Regions

Finding the area of a shaded region is pretty straight-forward.



Areas of Shaded Regions The best way to attack this problem is by the finding the area of the larger region, then taking (subtracting) out the area of the smaller region.

Example 1 Find the area of the shaded region.

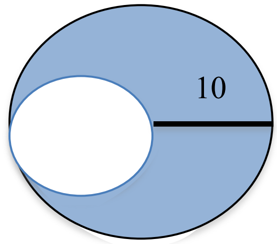


Doing that, the area of the larger rectangle is 60 sq. inches. Taking out the area of the smaller rectangle, I have

Area = 60 sq. in. – 30 sq. in.

= 30 sq. in.

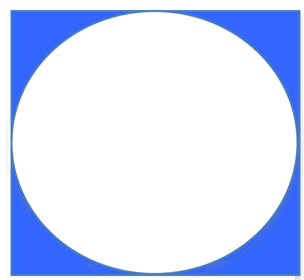
Example 2 Find the area of the shaded region



The area of the large circle, πr^2, is 100 π sq.'. The radius of the smaller circle is 5. A = 25π

Subtracting 100 π sq.' – 25 π sq.' = 75 π sq. units

Example 3 Find the area of the shaded region



The area of the square with side s; $A_s = s^2$

$A_s = 100$ sq in.

The area of the circle with r = 5 in.; $A_c = \pi r^2$

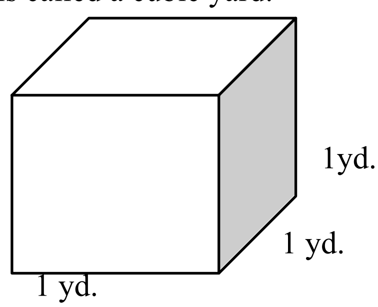
$A_c = \pi\ 25$

Area $_{\text{shaded region}} = 100 - 25\pi$ sq. in.

Volume

Let's talk volume. The volume of a three dimensional figure measures how many cubes will fit inside it. If you were to rent a truck, you'd want to know how much it could hold, you'd want to know the volume.

If you were to buy dirt for your yard, it's typically sold in cubic feet, that's describing volume. If you were laying a foundation for a house or putting in a driveway, you'd want to buy cement, cement is sold by the cubic yard. In other words, you'd want to know how many boxes, cubes, that measured one yard, by one yard, by one yard, you'd need to purchase. A box that has those dimensions is called a cubic yard.



1 cubic yard

Let's say that I have five rows of boxes with three boxes in each row, how many boxes would I have? Hopefully, you'd say 15 boxes, 5x3. How many boxes would I have if I put a second layer on the boxes? Well, 15 Boxes on the bottom plus 15 more on top, that's 30 boxes. How many boxes would I have if I stacked them four high? Since each layer has 15 boxes, I would end up with 15 x 4, or 60 boxes. This almost makes sense, doesn't it?

Well, that leads us to another discovery. If I know how many boxes are on the first layer, then to find the total number of boxes in a stack, I'd multiply the number on the first layer by the height of the stack.

The discovery is a way of finding volume. But, rather than call them boxes, we'll call them cubes. And, as usual, we'll also have to introduce some terminology.

We will be finding volumes of prisms and pyramids. So, let's go on and find out how prisms and pyramids are defined.

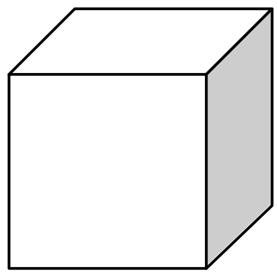
Prism — Is a three dimensional figure with two Parallel Congruent bases which are polygons.

Congruent — Figures are congruent if they can be made to look exactly alike.

Pyramid — Is a three dimensional figure with any two dimensional figure for the base that meets in a point.

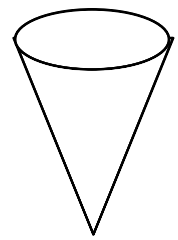
Prism

V = Bh



Pyramid

V = 1/3 (Bh)



B - area of the base.

H - height of the object.

The "B" in volume problems represents the formulas for area that we have already studied.

Let's say, for example, I have to find the volume of a circular cylinder. A prism because the top and bottom are congruent, and circular, because the bases are circles.

The formula then is $V = B\,h$ or $V = \pi r^2 h$, we can see in this case that $B = \pi r^2$

A wedge, piece of cheese, might be referred to as a triangular prism. The triangular describes the congruent bases. The formula then would be V = B h, where B is the area of the base. Since the base is a triangle, the formula we would use is 1/2 (bh). Therefore, in order to find the Volume of a triangular prism, we would use

$$V = B\,h$$
$$1/2\,(b h_t) h$$

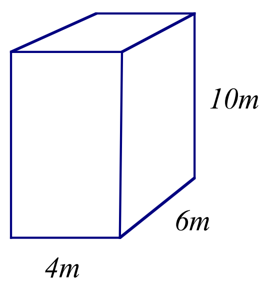
I know, you think this is neat. By the way, the h_t means the height of the triangle. The h without the subscript is the height of the prism.

Because using a small "h" for the height in a triangle or trapezoid, I will use a capital "H" for the height of a prism or pyramid.

In the following chart, please notice that if you know the area formulas, then the volume formulas are found by multiplying by the height "H" of the prism or pyramid.

	Area	Volume Prism	Volume Pyramid
Rectangle	lw	lwH	lwH/3
Parallelogram	bh	bhH	bhH/3
Triangle	½ bh	½ bhH	½ bhH/3
Trapezoid	½(B+b)h	½ (B+b)hH	½ (B+b)hH/3
Polygon	½ ap	½ apH	½ apH/3
Circle	πr^2	$\pi r^2 H$	$\pi r^2 H/3$

Example 1 Find the volume rectangular prism.



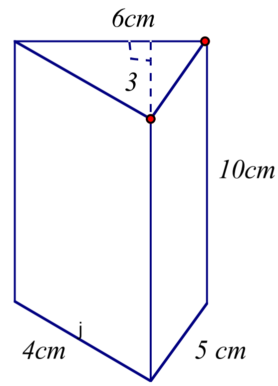
Because the base is a rectangle, this is called a rectangular prism. To find the volume, I find the area of the base (B), and multiply that by the height of the prism (h),

$$\begin{aligned} V &= BH \\ &= (lw)H \\ &= (24)(10) \end{aligned}$$

V = 240 cubic m.

Cubic because we are determining how many 1 meter cubes will fit inside the prism.

Example 2 Find the volume triangular prism.



Because the base of the prism is a triangle, this is called a triangular prism. Some might refer to it as a wedge. The volume of a prism is found by multiplying the area of the base

by the height of the prism. Same as with the rectangular prism.

$$V = BH$$
$$= (1/2\ bh_t)H$$

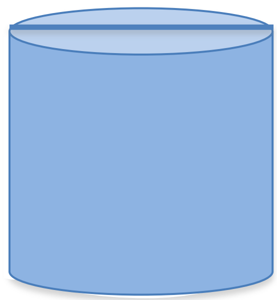
Notice, I have an H and h_t. The H is the height of the prism, the h_t is the height of the triangle.

The area of the base is ½ bh, b= 6, $h_t = 3$, and the height of the prism, H = 10.

Substituting those values into our equation, we have:

$$V = [(1/2)\ 6)\ (3)]10$$
$$V = 90 \text{ cubic cm}$$

Example 3 Find the volume of the cylinder.



This is not a prism because the base is not a polygon, we treat it like one to find the volume. It's called a cylinder.

Again, V = BH, where B is the area of the base and H is the height of the cylinder.

$$V = (\pi r^2)\ H$$

Since the diameter is given as 10, the radius is 5'.
Substituting, we have:

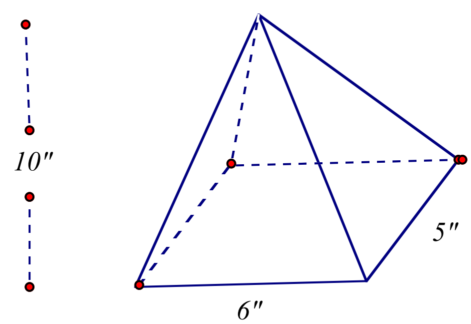
$$\begin{aligned} V &= (\pi\ 5^2)10 \\ &= (25\pi)10 \\ &= 250\pi \text{ cubic feet.} \end{aligned}$$

Volume – Pyramids

A pyramid meets in a point. We have rectangular and triangular pyramids just like we had prisms. The formula is almost the same as the volume of a prism, except we divide BH by 3 or take 1/3 of BH.

$$V = (1/3)\ BH$$

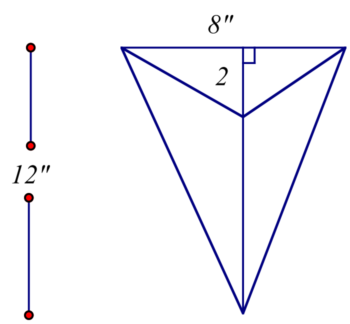
Example 4 Find the volume of the rectangular pyramid.



The base is a rectangle with sides 6 and 5, the height is 10. Substituting those values into the formula, we have:

$$\begin{aligned} V &= 1/3\ ((6)(5)\ \ (10) \\ &= 1/3\ (30)\ \ (10) \\ &= 100 \text{ cubic inches} \end{aligned}$$

Example 5 Find the volume



In this problem, the base is a triangle. That means we have h_t and H as we did before.

$$V = (1/3)\ BH$$

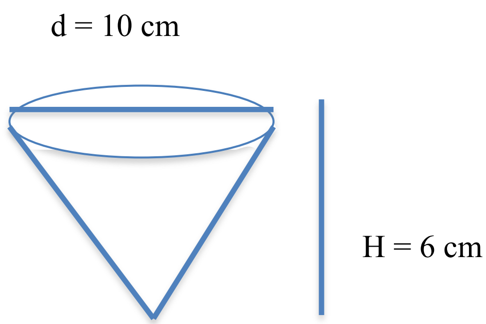
The area of the base is ½ bh_t

Substituting our values, we have:

$$V = (1/3)\ \underline{[1/2\ (8)(2)]}\ (12)$$

$$= (1/3)\ \underline{[8]}\ (12)$$
$$= 32 \text{ cubic “}$$

Example 6 Find the volume.



We treat this cone just like it was a pyramid, find the one third the area of the base times the height.

$$V = (1/3)\ BH$$

Substituting the formula for the area of the base, if the diameter is 10, then the radius is 5 cm

$$= (1/3)\ \pi r^2\ H$$
$$= (1/3)\ \pi 5^2\ 6$$
$$= (1/3)\ 25\pi\ 6$$
$$= 50\pi \text{ cubic cm}$$

Please note all these formulas are related. We noticed a pattern to find the area of a rectangle, saw how that was used to find the area of a parallelogram, cut a parallelogram along its diagonal to get the area of a triangle and used the triangle to find the area of a trapezoid.

We then noticed to find the volumes, we used those area formulas to find the area of the base and multiplied those by the height (H) of the prisms and pyramids. Bottomline, if you memorize the area formulas, then everything else is just too easy.