CHAPTER 15 Equations with Radicals

Sec. 1 Simplifying Radicals - Square Roots

From grade school, you can probably remember how to take square roots of numbers like $\sqrt{25}$, $\sqrt{64}$, and the $\sqrt{100}$. The number on the inside of the radical sign is called the **radicand**.

Finding those answers are easy when the radicand is a perfect square. But what happens if the radicand is not a perfect square? Knowing you were thinking just that very thought, you must be very pleasantly surprised to know I am going to tell you how to simplify those expressions.

Let me just tell you how, then I'll use an example to make it more clear.

To simplify a square root (second root)

1. rewrite the radicand as a product of a perfect square and some other number,
2. take the square root of the number you know.
3. The number you don't know the square root of stays inside the radical.

Perfect Squares; 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …

Example 1. Simplify $\sqrt{75}$

We don't know the $\sqrt{75}$ so we'll rewrite the radicand as a product of a perfect square and some other number. Since 25 is a perfect square that is a factor of 75, we'll rewrite.

$$\sqrt{75} = \sqrt{25x3}$$

$$= \sqrt{25} \text{ x } \sqrt{3}$$

$$= 5\sqrt{3}$$

That's pretty straight forward. If you are not familiar with perfect squares, you should write them down by multiplying the counting numbers by themselves as was done on the previous page in red.

Perfect Squares

$1^2 = 1$	$6^2 = 36$
$2^2 = 4$	$7^2 = 49$
$3^2 = 9$	$8^2 = 64$
$4^2 = 16$	$9^2 = 81$
$5^2 = 25$	$10^2 = 100$

Example 2. **Simplify** $\sqrt{72}$

$$\sqrt{72} = \sqrt{36x2} = 6\sqrt{2}$$

What would have happened if you rewrote 72 as 9 x 8?

$$\sqrt{72} = \sqrt{9 \cdot 8}$$
$$= 3\sqrt{8}$$

Oh, oh, that's not the same answer we got before! How can that be? Well, one reason is we're not finished, we can simplify $\sqrt{8}$.

$$3\sqrt{8} = 3\sqrt{4 \cdot 2}$$

$$= 3(2)\sqrt{2}$$

$$= 6\sqrt{2}$$

Now, that's the same answer we got before.

Whew! You would not have able to go to bed tonight thinking about that.

When simplifying radicals, it would be a good idea to become very comfortable with perfect squares. And just as importantly, when you rewrite the radicand as the product of a perfect square and some other number, use the **LARGEST** perfect square you can. Otherwise, you'll end up doing a lot more steps.

Remember simplifying fractions, you'd use the largest factor that divided into the numerator and denominator because that meant you ended up with less work. The same applies here, the larger the perfect square you use, the shorter the problem.

In algebra, nothing changes except we add variables. To take the nth root of a variable – all we do is divide the exponent on the variable by the index.

Example 3 Find the $\sqrt{x^{10}}$

If index is not written, so it is understood to be 2.

Divide the exponent by the index , $\frac{10}{2}$, the $\sqrt{x^{10}} = x^5$

As always, I cannot make these any more difficult – only longer. To simplify radicals, we rewrite the radicand as a product of a perfect nth root and some number.

Example 4 Find the $\sqrt{x^{10}y^6z^5}$

Since the index is not written, it's understood to be 2.

Divide each exponent by 2, we have $x^5y^3\sqrt{z^5}$, the exponent 5 is not evenly divisible by 2. So, let's simplify it by rewriting it as the product of a perfect square and a number - $\sqrt{z^5} = \sqrt{z^4z} = z^2\sqrt{z}$

Ok, reworking example to make sure the index divides into the exponents evenly, we have

$$\sqrt{x^{10}y^6z^5} = \sqrt{x^{10}y^6z^4z}$$
$$= x^5\ y^6\ z^2\ \sqrt{z}$$

Sec. 2 Simplifying Other Roots

When we simplified square roots, written as $\sqrt[2]{n}$, we often did not write the index 2, signifying the second root. We accept $\sqrt{n}$, as the second root when the index is not written.

We know $\sqrt{16} = 4$, but do you know $\sqrt[4]{16}$? The fourth root is asking what number times itself FOUR times is 16. The answer is 2. Our comfort with square or second roots comes from our experience with the multiplication facts. But with other roots, we might have to use a factor tree to get those roots.

Now, what happens if we are not using square roots and we include variables as part of the radicand?

The good news is the strategy stays the same, but if we are dealing with cube roots, we have to rewrite the radicand as a product of perfect cubes and some number. If we have a fourth root, the radicand has to be written as a product on numbers to the 4^{th} power and some number.

Procedure for Simplifying n^{th} roots:

1. **Rewrite the radicand as a product of numbers and variables raised to the power or multiple of the index, n, and some other numbers.**
2. **Take the n^{th} root of the factors raised to the nth power by dividing the exponent by the index.**
3. **Leave the other numbers and variables in the radical.**

Note, simplifying variables is very easy. You rewrite the variable in the radicand using a factor of the index, and simplify by dividing by the index.

Example 1 $\sqrt[2]{x^{11}} = \sqrt[2]{x \cdot x^{10}}$ 10 because 2 is a factor of 10

Example 2 $\sqrt[3]{x^{17}} = \sqrt[3]{x^{15}x^2}$ 15 because 3 is a factor of 15.

Example 3 **Simplify** $\sqrt[3]{\mathbf{16x^2y^7}}$

In the radicand, 16 and y^7 are NOT perfect cubes. That means I have to rewrite them as a product of a perfect cube and some other number.

8 is a perfect cube, so we can rewrite 16 as 8 (2)
y^6 is a perfect cube, so we can rewrite y^7 as y^6 y

Let's substitute those factors into the radicand.

$$\sqrt[3]{8(2)x^2\,y^6y}$$

The cube roots of 8 is 2 and the cube root of y^6 is y^2, so taking those out of the radicand, we are left with

$$= 2y^2\sqrt[3]{2x^2y}$$

Example 4 Simplify $\sqrt[5]{32x^{10}y^{17}z^{39}}$

Rewriting the radicand using exponents that are multiples of 5.

$$= \sqrt[5]{2^5x^{10}y^{15}y^2z^{35}z^4}$$

Now, divide each exponent that is a multiple of 5 by the index 5. Everything else stays in the radical

$$= 2x^2y^3z^7\sqrt[5]{y^2z^4}$$

Sec. 3 Radical Operations

Add - Subtract

You add, subtract, multiply and divide radicals in much the same way we did with algebraic expressions. That is, for addition and subtraction, we combined like terms. With radicals, combining like terms means we combine (add or subtract) radicals with the SAME radicand and index.

If they don't have the same radicand and index, we can either not combine them or we can simplify them, then combine them.

These next two examples have the same radicand and index, they are "like" terms, so we combine them.

Example 1 $3\sqrt{5} + 4\sqrt{5} = 7\sqrt{5}$

Example 2 $4\sqrt{11} - 9\sqrt{11} = -5\sqrt{11}$

In this next example, we don't have the same radicands. So, we will try and rewrite them.

Example 3 $\quad \mathbf{3\sqrt{12} + 5\sqrt{20} - 8\sqrt{12} + 3\sqrt{20}}$

$$\mathbf{= 3\sqrt{4\cdot 3} + 5\sqrt{4\cdot 5} - 8\sqrt{4\cdot 3} + 3\sqrt{4\cdot 5}}$$

$$\mathbf{= 6\sqrt{3} + 10\sqrt{5} - 16\sqrt{3} + 6\sqrt{5}}$$

$$\mathbf{= -10\sqrt{3} + 16\sqrt{5}}$$

Multiply Radicals

You multiply radicals the same way you multiply algebraic expressions. That is, use the Distributive Property. But remember to simplify the radicals after you have completed the multiplication.

You need to know, $\quad \sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$

Let's look at an algebra and a radical multiplication together so we see the similarity in finding the product. Notice, the coefficients are the same.

Example 4

$(3x + 2)(4x + 1)$	$(3\sqrt{x} + 2)(4\sqrt{x} + 1)$
$12x^2 + 3x + 8x + 2$	$12\sqrt{x^2} + 3\sqrt{x} + 8\sqrt{x} + 2$
$12x^2 + 11x + 2$	$12\sqrt{x^2} + 11\sqrt{x} + 2$

Please note everything is done the same in these multiplications.

I do have to simplify the radical $\sqrt{x^2} = x$

$$12x + 11\sqrt{x} + 2$$

Compare the answers using just algebra to radicals, the coefficients are the same!

Dividing with Radicals – Rationalizing the Denominator

Using the same ideas we have used in regular division, we will clear the denominator (divisor) of radicals by multiplying by 1.

When dividing by decimals, you might remember we moved the decimal point as far to the right as possible in the divisor, then moved it the same number of places to the right in the dividend. Mathematically, we were multiplying the dividend/divisor; numerator/denominator by 1 in the form of some power of 10. That gave us an equivalent expression with a denominator without a decimal point. In other words, we got rid of the decimal in the divisor (denominator). That made division make sense so we could actually subtract out exact numbers, then round answers later.

We are going to use the same concept with radicals. Having a radical in the denominator results in having a repeating decimal in the denominator. To get rid of that, I will multiply by 1 in such a way as to get rid of the radical in the denominator.

Rationalizing the Denominator

When you have a single radical in the denominator, you multiply the expression by 1 in the form of that radical. That works because we know that $\sqrt[n]{x^n} = x$. That gets rid of the radical.

Example 5 Rationalize the denominator $\frac{2}{\sqrt{3}}$

To get rid of the radical, I will multiply that expression by 1 in the form of $\frac{\sqrt{3}}{\sqrt{3}}$.

$$\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{9}} = \frac{2\sqrt{3}}{3}$$

Example 6 Rationalize the denominator $\frac{5}{\sqrt{7}}$

Multiply by 1 in the form of $\frac{\sqrt{7}}{\sqrt{7}}$

$$\frac{5}{\sqrt{7}} \frac{\sqrt{7}}{\sqrt{7}} = \frac{5\sqrt{7}}{7}$$

Let's look at a slight variation recalling that the index and exponent must match to get rid of the radical; $\sqrt[n]{x^n} = x$.

Example 7 Rationalize the denominator $\frac{3}{\sqrt[7]{x^5}}$

In order to get rid of the radical, I have to have $\sqrt[7]{x^7}$. So, what can I multiply $\sqrt[7]{x^5}$ by? Well, x^5 times what results in x^7? If you said x^2, we are in business. Now, let's do the problem,

$$\frac{3}{\sqrt[7]{x^5}} \frac{\sqrt[7]{x^2}}{\sqrt[7]{x^2}} = \frac{3\sqrt[7]{x^2}}{\sqrt[7]{x^7}} = \frac{3\sqrt[7]{x^2}}{x}$$

Example 8 Rationalize the denominator $\frac{5}{\sqrt[4]{x}}$

Again, what do I multiply by to make the exponent equal the index. x ( ) = $\sqrt[4]{x^4}$? Hopefully you said x^3.

$$\frac{5}{\sqrt[4]{x}} \frac{\sqrt[4]{x^3}}{\sqrt[4]{x^3}} = \frac{5\sqrt[4]{x^3}}{\sqrt[4]{x^4}} = \frac{5\sqrt[4]{x^3}}{x}$$

Rationalizing the denominator means we are getting rid of the radical in the denominator. To get rid of a simple division by a single radical, we just saw we multiplied by 1 so the denominator becomes $\sqrt[n]{x^n}$ **which equals x.**

What happens if you don't have a single factor in the denominator which is a radical?

Well, to continue to do these successfully, you do need to know your special products – specifically the **Difference of 2 Squares**.

$$(a - b)(a + b) = a^2 - b^2$$

Knowing that special products means we square the first and last terms and the middle terms that would have radicals will subtract out.

$$(\sqrt{3} + 5)(\sqrt{3} - 5) = 3 - 25 = -22$$

Applying that to rationalizing the denominator, we will multiply by 1 by using the Difference of 2 Squares pattern.

Example 9 Simplify $\frac{7}{4+\sqrt{2}}$

In this problem, we have a binomial in the denominator. If I use my knowledge of the Difference of 2 Squares, I change the sign in the middle and use that to multiply by 1. That changed number is called the conjugate.

$$\frac{7}{4+\sqrt{2}} \cdot \frac{4-\sqrt{2}}{4-\sqrt{2}} = \frac{7(4-\sqrt{2})}{16-2} = \frac{7(4-\sqrt{2})}{14} = \frac{4-\sqrt{2}}{2}$$

Example 10 Rationalize the denominator $\frac{3}{5-\sqrt{x}}$

The conjugate of $5 - \sqrt{x}$ is $5 + \sqrt{x}$. Multiplying by 1 using the conjugate, we have

$$\frac{3}{5-\sqrt{x}} \frac{5+\sqrt{x}}{5+\sqrt{x}} = \frac{3(5+\sqrt{x})}{25-x}$$

Sec 3. Solving Equations containing Radicals

A radical equation is an equation in which there is a variable inside the radical sign (radicand). Our strategy, just as we have continually used is to get rid of what we don't like and transform the equation into one we already know how to solve.

Example 1 $\sqrt{x} - 2 = 4$

The strategy we'll use to solve equations containing radicals will be based on raising each side to the power of the index.

$$\boldsymbol{\sqrt[n]{a} = b, \text{ then } a = b^n}$$

In other words, to get rid of a square root, we square both sides of the equation. To get rid of a third root, we cube each side of the equation. To get rid of any root, we raise both sides of the equation to the same power as the index. Remember, when an index is not written. it is understood to be 2.

Algorithm for Solving Equations Containing Radicals

1. Isolate the radical
2. Raise both sides to a power equal to the index
3. Solve the resulting equation
4. Always, always check your answer!

Example 2 Solve for x; $\sqrt{x} - 4 = 3$

Isolate the radical $\sqrt{x} = 7$

Square both sides $x = 49$ *Check the answer.*

Example 3 Solve for x; $\sqrt{x} - 2 = 3$

$\sqrt{x} = 5$

$x = 25$ *Check the answer!*

Example 4 Solve for x; $\sqrt{x+4} + x = 8$

Isolate the radical $\sqrt{x+4} = 8 - x$

Square both sides $x + 4 = 64 - 16x + x^2$

Squaring both sides results a quadratic equation, put everything on one side, zero on the other side. Factor, set each factor equal to zero.

$0 = x^2 - 17x + 60$
$0 = (x - 5)(x - 12)$
$x = 5$ or $x = 12$ *Check the answer*.

Notice that 12 does NOT work and is called an **extraneous root**. The only answer is 5.

When we squared both sides of an equation, the degree is raised. There is a corollary to the **Fundamental Theorem of Algebra** that states an equation of degree *n* will have *n* roots. So, when we solved the quadratic equation, degree 2, that corollary suggested we would have two roots. However, the original equation was of degree 1. In other words, by squaring both sides of an equation, we may introduce solutions to the higher degree equation that do not work for the original equation. Therefore, it is always necessary to check your solutions in the original equation to ensure we did not introduce extraneous solutions.

You'll notice the "solving" part can sometimes result LINEAR or QUADRATIC equations. You need to be able to identify different types of equations and the strategies for solving them to be able to solve radical equations.

To solve radical equations, all you do is isolate the radical and raise to the power of the index. The rest is determined by the type of equation you have.

Example 5 Solve $\sqrt[3]{3x-9} - 2 = -4$

$\sqrt[3]{3x-9} = -2$ Isolate radical

$3x - 9 = -8$ Cube both sides

$3x = 1$

$x = 1/3$

Example 6 Solve $\sqrt{x^2 - 8} = 2 - x$

$x^2 - 8 = 4 - 4x + x^2$ Square both sides

$-12 = -4x$

$3 = x$

Check your answer. 3 does not work

What that means is there is no real value of x that will satisfy the original equation. We say the answer is the **null set** or **empty set**. We write it this way $\emptyset$

Up to this point, we have only examined equations that had only one radical. If there are two radicals in an equation, isolate one of them and get rid of that radical, then isolate the other radical and get rid of that

That does not make the problem more difficult to solve, it only makes it longer.

Example 7 **Solve**: $\sqrt{x+7} - 1 = \sqrt{x}$

1. Isolate one of the radicals $\sqrt{x+7} = \sqrt{x} + 1$
2. Square both sides $x + 7 = x + 2\sqrt{x} + 1$
3. Isolate the other radicals $6 = 2\sqrt{x}$
4. Simplify $3 = \sqrt{x}$
5. Square both sides $9 = x$
6. Check your answer

Notice, if there are two radicals, you must isolate one at a time. The problem is longer, not more difficult and the same strategy is used.

The greatest difficulty experienced by many students is squaring a binomial containing a radical. That brings us back to the special products learned earlier.

$$(a + b)^2 = \underline{a}^2 + 2ab + \underline{b}^2$$

Remember, square both the first and last terms, then find the product of the two terms and double.

Examples

$$(\sqrt{5} + 3)^2 = 5 + 2\sqrt{5} \cdot 3 + 9 = 14 + 6\sqrt{5}$$

$$(\sqrt{3} - 7)^2 = 3 - 2\sqrt{3} \cdot 7 + 49 = 52 - 14\sqrt{3}$$

$$\left(\sqrt{x} + 4\right)^2 = x + 2\sqrt{x} \cdot 4 + 16 = x + 8\sqrt{x} + 16$$

You should do enough of these so you can do the multiplication in your head. Otherwise the multiplication becomes the problem rather than solving the radical equation. I know, you want to do another problem.

Example 8 Solve for x; $\sqrt{x-5} - \sqrt{x} = -1$

Isolate one of the radicals first. My preference is to isolate the radical with the longest expression in the radicand first.

$$\sqrt{x-5} = \sqrt{x} - 1$$
$$x - 5 = x - 2\sqrt{x} + 1$$
$$-6 = -2\sqrt{x}$$
$$3 = \sqrt{x}$$
$$9 = \mathrm{x}$$

Check you answer!

To solve radical equations:

1. **Transform the equation so that the radical is alone on one side.**
2. **Raise both sides to a power equal to the index of the root.**
3. **Solve the resulting equation by any of the standard procedures.**
4. **Check carefully each apparent root in the original equation, rejecting any which extraneous.**

Solve the following, check your answers!

1. $\sqrt{x} = 2$
2. $\sqrt{x} = 3$
3. $\sqrt{x-1} = 5$
4. $\sqrt{x+2} = 10$
5. $2\sqrt{x+3} = 8$
6. $\sqrt{x} - 4 = 3$

7. $\sqrt{x} + 3 = 5$

8. $\sqrt{x - 4} - 2 = 8$

9. $3\sqrt{x} - 1 = \sqrt{x} + 1$

10. $\sqrt{x + 8} + \sqrt{x} = 8$

11. $\sqrt{x + 7} = 2 - \sqrt{x - 5}$

12. $\sqrt{4x + 3} = 2\sqrt{x - 1} + 1$