Ch. X4 Number Theory

Divisibility

The phrase *number theory* sounds impressive, but the chapter is just an acknowledgement of a great deal of observations, patterns, and logic that will help us in our work in mathematics.

Primarily concerned with the relationships among integers. Many topics from number theory are incorporated in elementary mathematics. They include multiples, factors, rules for divisibility, prime numbers, prime factorization, greatest common factors, least common multiples, as well with operations with rational numbers in the form of a/b, a, b $\in$ J.

Let's begin with some common understanding; b divides a, written as $b|a$, translates to b is a factor of a or b is a divisor of a. The definition, if a and b are integers, then b divides a, $b|a$, if and only if there is an integer c such that $a = cb$.

Knowing that, let's take a look at a relatively simple example. Consider two bags of marbles and the number of marbles in each bag can be shared (divided) equally between three people. Mathematically, we'd say the number of marbles in each bag is a multiple of three. Now, if all the marbles were placed in one bag, is it still possible to share the marbles equally between the three people? Hopefully, you answered yes.

That suggests that if the number of apples in the first bag is a and the number of apples in the second bag is b, then we can still share (divide) the apples equally if they are all placed in the same bag.

Mathematically, we have: $3|a$ and $3|b$, then $3|(a + b)$. That leads to a couple of theorems:

Theorem: For any integers, a, b and d

1) If $d|a$ and $d|b$, then $d|(a + b)$
2) If $d|a$ and $d\nmid b$, then $d\nmid(a + b)$

Subtraction is defined in terms of addition, therefore

Theorem: For any integers, a, b and d

1) If $d|a$ and $d|b$, then $d|(a - b)$
2) If $d|a$ and $d\nmid b$, then $d\nmid(a - b)$

Theorem: For any integer a and d, if $d|a$ and k is an integer, then $d|ka$

As we go through this chapter, we will see how these theorems play a role with a number of applications as we continue our study of mathematics.

Rules of Divisibility

Let's begin by looking at our rules of divisibility for twos, fives, and tens. When you learned your multiplication facts in third grade, you probably made some observations that helped you in your memorization. When you multiplied a number by 10, the answer always ended in zero. When you multiplied by 5, the answer either ended in a zero or a five. And when you multiplied by 2, the number was always even.

Using those observations, if we looked at a product (answer) in a multiplication computation and saw the answer ended in zero, we might surmise that 10 was a *factor* of that number. A factor is a number in a multiplication problem. Another way of saying that is that 10 would go into that number without a remainder (evenly). In other words, that number was divisible by ten.

By looking at those observations, we can come up with three rules of divisibility that are probably very familiar to you and will help us when working with larger numbers.

"Ends-in" Rules of Divisibility

Rule 1. Divisibility by 10. A number is divisible by 10 if it ends in a zero.

Rule 2. Divisibility by 5. A number is divisible by 5 if it ends in a zero or five.

Rule 3. Divisibility by 2. A number is divisible by 2 if it ends in an even number.

Ex. 1. Is 720 divisible by 10?

Since 720 ends in a zero, 720 is divisible by 10.

If a number ended in a five or a zero, using the same logic, we could surmise that 5 is a factor. Or that the number is divisible by 5.

Ex. 2. Is 720 divisible by 5?

Since 720 ends in a zero, 720 is divisible by 5.

Ex. 3. Is 435 divisible by 5?

Since 435 ends in a five, 435 is divisible by 5.

Continuing, we could see from our multiplication facts that any number multiplied by two always results in an *even* number. That is, the last digit of a number ends in 0, 2, 4, 6, or 8.

Ex. 4 **Is 826 divisible by 2?**

Since 82<u>6</u> ends in an even number, 826 is divisible by 2.

"Sums" Rules of Divisibility

The first three rules for divisibility were easy enough to see based on our observation of multiplication facts. Let's look at some other numbers, like 3 and 9.

If we picked numbers at random and multiplied them by 3, we might see some other patterns that would allow us to look at numbers and determine if they were divisible by those numbers.

We will start with 3. We will pick numbers at random like 11, 15, 20, 23, and 145 and multiply them by 3.

3 x 11 = 33 3 x 15 = 45 3 x 20 = 60 3 x 23 = 69 3 x 145 = 435

Upon first observation, nothing seems to jump out at me that would suggest those products are divisible by 3. But, if I were to look longer, think, and *try* to find a hint, I might begin to notice that the <u>sum of the digits in the products are divisible by 3</u>. I would know this from my memorization of the multiplication facts.

33 → 3 + 3 = 6

45 → 4 + 5 = 9

60 → 6 + 0 = 6

69 → 6 + 9 = 15

435 → 4 + 3 + 5 = 12

Using that observation, I might begin to think that if the sum of the digits of a number are divisible by 3, then the number is divisible by 3.

These observations and previous examples lead me to another rule of divisibility.

Rule 4. Divisibility by 3. A number is divisible by 3 if the sum of the digits is divisible by 3.

Using that same type of observation and logic, we will look at numbers whose products were formed by multiplying by 9.

We will again pick numbers at random; 20, 34, 443, and 657, multiply those by 9, then look at their products.

9 x 20 = 180 9 x 34 = 306 9 x 443 = 3987 9 x 657 = 5913

180 → 1 + 8 + 0 = 9

306 → 3 + 0 + 6 = 9

3987 → 3 + 9 + 8 + 7 = 27

Looking at these numbers, we see a pattern that suggests the if the sum of the digits of a number is divisible by 9, then the number is also divisible by 9.

Rule 5. Divisibility by 9. A number is divisible by 9 if the sum of the digits is divisible by 9.

The next rule is based more on logic than on observation. If a number is divisible by 2 and also divisible by 3, that means it has factors of 2 (even) and 3, thus the number would be divisible by 2 x 3 or 6.

"Combo" Rules of Divisibility

Rule 6. Divisibility by 6. A number is divisible by 6 if it is divisible by 2 and 3.

That means we can look at a number very quickly and determine if it is divisible by 6. First it would have to be even (divisible by 2) and second the sum of the digits would have to be divisible by 3.

Ex. 7. Is 354 divisible by 6?

354 is even so it is divisible by 2 and 3 + 5 + 4 = 12, 12 is divisible by 3 so 354 is divisible by 3. Since it is divisible by 2 and 3, it is divisible by 6.

You might ask, why would I want to know these rules for divisibility? Well, an immediate use would be for simplifying fractions. If you were asked to simplify 111/123, that might be considered more difficult than simplifying 3/12 by some. But knowing the rules for divisibility, I could determine very quickly that a common factor of the numerator and denominator of 111/123 is 3. Thus making the problem simpler.

We could extend the reasoning we used for divisibility by 6 for other numbers. For instance, if a number is divisible by 5 and by 3, do you think it might be divisible by 15? What do we know about how numbers end when they are multiplied by 25, could you come up with a rule for divisibility of 25?

"Last Digits" Rules of Divisibility

If we continued to look at patterns by multiplying by 4, then look at products, we would come up with other rules of divisibility.

Rule 7. Divisibility by 4. A number is divisible by 4 if the last two digits of the number is divisible by 4.

Rule 8. Divisibility by 8 A number is divisible by 8 if the last three digits of the number is divisible by 8

Other Rules

Rule 9. Divisibility by 7 if the integer represented without the units digit, minus twice the units digit is divisible by 7

Rule 10. Divisibility by 11 if the sum of the digits in the places that are even powers of 10 minus the sum of the digits in the places that are odd powers of 10 is divisible by 11

Using the rules of divisibility, we can quickly look at numbers and determine if they are divisible by 2, 3, 4, 5, 6, 9, and 10.

Let's put those rules together:

"Ends-in" Rules of Divisibility

Rule 1. Divisibility by 10. A number is divisible by 10 if it ends in a zero.

Rule 2. Divisibility by 5. A number is divisible by 5 if it ends in a zero or five.

Rule 3. Divisibility by 2. A number is divisible by 2 if it ends in an even number.

Sums Rules

Rule 4. Divisibility by 3. A number is divisible by 3 if the sum of the digits is divisible by 3.

Rule 5. Divisibility by 9. A number is divisible by 9 if the sum of the digits is divisible by 9.

"Combo" Rules of Divisibility

Rule 6. Divisibility by 6. A number is divisible by 6 if it is divisible by 2 and 3.

"Last Digits" Rules of Divisibility

Rule 7. Divisibility by 4. A number is divisible by 4 if the last two digits of the number is divisible by 4.

Other RULES

Rule 8. Divisibility by 7 if the integer represented without the units digit, minus twice the units digit is divisible by 7

Rule 9. Divisibility by 11 if the sum of the digits in the places that are even powers of 10 minus the sum of the digits in the places that are odd powers of 10 is divisible by 11

With a little extra thought, we can construct numbers that are divisible by some or all of those numbers as well.

Ex. 10. Write a 5-digit number that is divisible by 2, 3, 4, 5, 6, 9, and 10.

Using logic, we know the number must end in zero if it is to be divisible by 10.

_ _ _ _ 0

If a number ends in 0, then it is divisible by 10, 5, and 2.

For a number to be divisible by 4, the last two digits have to be divisible by 4.

_ _ _ 4 0

What are other numbers could I have placed in that position so the last two numbers are divisible by 4?

Up to this point we have constructed a number that is divisible by 2, 5, 10, 6 and 4. If we continue with this construction so that the number is divisible by 9, then it is automatically divisible by 3.

2 0 3 4 0

If the number is divisible by 3 and 2, then it is divisible by 6. Therefore we have constructed a number that is divisible by 2, 3, 4, 5, 6, 9, and 10.

If we continued to play with numbers and rules of divisibility, we might notice other patterns that will help us later with larger numbers. We know, for example, 3 is a factor of 12 and 3 is a factor of 21. The sum of 12 and 21 is 33, and 3 is a factor of 33. That is to say that 33 is divisible by 3. Another example might be 30 is divisible by 6, 24 is divisible by 6, and if we found their sum, 30 + 24 = 54, we see that is also divisible by 6. While that makes sense, it follows from the theorems we learned earlier, 3|12 and 3|21, then 3|(12 + 21) or 3|33.

Do you think that could be extended to subtraction? In other words, if 12 is a factor of 72 and 96, would 12 be a factor 96 –72 = 24? Would 24 be divisible by 12? Oh yes, we had a theorem for just that.

Now, let's make sure we can use these rules to find factors.

1. Identify factors of the following numbers using the rules of divisibility

a.	102	b.	105	c.	210
d.	1024	e.	111	f.	1080
g.	2103	h.	3012	i.	81,324
j.	1,025	k.	4,311	l.	1,134
m.	16,425	n.	42,516	o.	213

1. Write a 5 digit number that is divisible by 2, 3, 4, 5, 6, 9 and 10.

2. Write a 5 digit number that is divisible by 2, 3, 4, 5, 6, 8 and 10, but not 9

While we looked at these rules of divisibility by seeing patterns develop, let's take a little deeper look. Let's look at the rule for divisibility by three by looking at an example. Using that rule, we know 5721 is divisible by 3, 3|5721, can we show that for any number? For our purposes, any four-digit number.

Let $\boldsymbol{n} = \mathbf{a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d}$**;** a four-digit number, Let's rewrite that so we can see how that rule can be derived by first picking a number that is divisible by 3 close to 1000, like 999. We will let $10^3 = 1{,}000$ which equals (999 + 1), 100 = 10^2 which is (99 + 1) and 10 is (9 + 1) and substitute that back into $\boldsymbol{n}$

$\boldsymbol{n} = \mathbf{a\,(999 + 1) + b\,(99 + 1) + c(9 + 1) + d.}$ Multiplying that out, we have
$\mathbf{= 999a + a + 99b + b + 9c + c + d}$
$\mathbf{= 999a + 99b + 9c + a + b + c + d}$

This number is not necessarily divisible by 3, but all those coefficients of a, b, and c are divisible by 3 the theorems we introduced in the beginning of the chapter, namely **$d|a$ and $d|b$, then $d|(a + b)$.**

Since 999a + 99b + 9c is divisible by 3, then in order to have $a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d$ divisible by 3, then a + b + c + d must be divisible by 3. Which leads to the sum of the digits be a multiple of 3.

We could use a similar explanation for divisibility by 4. Again, for convenience, we will use a four digit number.

$\boldsymbol{n} = \mathbf{a \cdot 10^3 + b \cdot 10^2 + c \cdot 10 + d}$**;** we know 4|100 and 4|1000, so 4|a(100) and 4|b(1000), which by theorem $4|(a10^3 + b10^2)$. That means the first the first two digits are divisible by 4. The last two numbers are 10c + d. So, for $\boldsymbol{n}$ to be divisible by 4, 10c + d, the last two digits must also be divisible by 4.

What we can see is those rules came from some thought that followed the theorems we studied. While those derivations are nice to know, what you need to know are the rules for divisibility.

Let's move on prime and composite numbers.

Prime and Composite Numbers

The Natural (Counting) Numbers {1, 2, 3, 4, …} can be broken into three categories; prime, composite and neither.

By definition,

Prime numbers have exactly two factors, one and itself. Examples include 2, 3, 5, 7, 11, 13, 17, and so on.

Composite numbers are numbers that have more than two factors. Examples include 4, 6, 8, 9 10, 12, and so on.

Notice the number one (1) has only one factor. One is neither prime nor composite.

How can you determine if a number is prime? One way is to use the **Sieve of Erostosthenes** by writing all the numbers from 1 to however high you want to go, then underline all multiples of primes beginning with 2.. Underline the first prime number - 2, then color multiples of 2 in green. Next, go to the next prime number after 2 that has not been underlined , underline it and write those multiples of 3 in red. Continue this process with the next number not in color – 5 underline it and color multiples in blue. Continue to the next number not colored for 7, underline it and color multiples in orange. All the numbers underlined are prime. The numbers in color are composite.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Fundamental Theorem of Arithmetic

All composite numbers can be written as a product of prime numbers in one and only one way (order does not matter).

Ex. 1. **Write 12 as a product of primes.**

From our knowledge of the multiplication facts, we know that 12 can be written as 4 x 3. That's a product, but 4 is not prime. But 4 can be written as 2 x 2, so 12 = 2 x 2 x 3 - a product of primes.

What if 12 was written as 6 x 2 instead of 4 x 3? Well 6 x 2 is not a product of primes because 6 is not prime. But 6 can be written as 3 x 2, so 12 = 3 x 2 x 2. Notice those two answers are the same – just written in a different order!

$$2 \times 2 \times 3 = 3 \times 2 \times 2$$

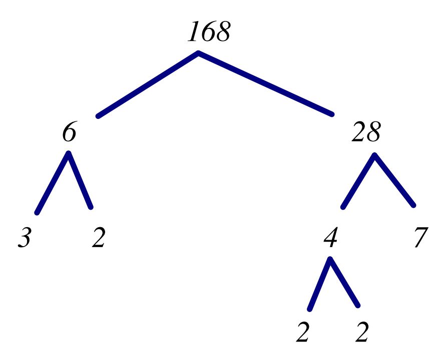
By convention, not rule, we usually write the product by writing the smallest numbers first. That is, to help people read them, we would normally see 12 written as 2 x 2 x 3.

Prime Factorization

Prime factorization is the process of re-writing numbers as product of primes. There are different ways of finding prime factors. One method is using a factor tree, use your knowledge of multiplication facts or rules of divisibility to find factors and continue to rewrite the numbers until you **<u>only</u>** have primes.

Ex. 2. **Write 168 as a product of primes.**

Write 168 at the top middle of the page and rewrite as a product of two factors.



The factors are 3 x 2 x 2 x 2 x 7 or $3 \times 2^3 \times 7$. By convention, we would write that as

$$2^3 \times 3 \times 7$$

Another method is to divide the original number by the smallest prime numbers until the last quotient is prime – called Division by Primes.

Ex. 3. Write 168 as a product of primes.

Divide 168 by 2, then continue dividing by prime numbers.

$$\begin{array}{r} 2\,\underline{)1\,6\,8} \\ 2\,\underline{)8\,4} \\ 2\,\underline{)4\,2} \\ 3\,\underline{)2\,1} \\ 7 \end{array}$$

The prime factors are 2^3 x 3 x 7, just like before.

Number of Factors in a Number

Besides writing numbers as products of primes, another question might be how many factors does a number have? For examples how many factors does 12 have? You might be able to answer that because of your knowledge of the multiplication facts. 1x12, 2x6, and 3x4. So the factors of 12 are 1, 2, 3, 4, 6 and 12. Twelve has 6 factors. That was easy because of our familiarity with the number 12.

How about 84? How many factors are there in 84? Again, I could find factors of 84 using the multiplication facts or rules of divisibility.

1x84, 2x42, 3x28, 4x21, 6x14, 7x12. So the factors of 84 are 1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, and 84. There are 12 factors.

Rather than looking at these factors by listing as we have done, let's use prime factorization and use exponents.

12 = 2x2x3 or 2^2x3^1. 12 has 3x2 = 6 factors.

40 = 2x2x2x5 or 2^3x5^1. 40 has 4x2 = 8 factors.

84 = 2x2x3x7 or 2^2x3^1x7^1 84 has 3x2x2 =12 factors

We don't normally write the exponent when it is 1, but to make our observation clearer I did.

Now, stay with me on this because this pattern does not just jump out at you. One way to determine the number of factors in a number is to write the product of primes in exponential notation as we just did. That is, $12 = 2^2 x 3^1$. We said 12 had 6 factors. If I played long enough, I might see a pattern set up that would tell me the number of factors a number had – not the factors themselves. If I add one to each exponent, then find the product of the exponents gives the number of factors. In 12, the exponents are 2 and 1. Adding one to each exponent, I have 3 and 2 and the product is 6. That's the number of factors.

In the number 84, rewritten as $2^2 x 3^1 x 7^{1,}$ the exponents are 2, 1, and 1 respectively. If I add one to each exponent and then find their product, I have 3, 2, and 2. The product of those numbers is 12. That is how many different factors are in 84.

Theorem **The number of factors in the number *p* that can be written as a product of primes, $a^m \cdot b^n$, is given by the formula (m+1)(n+1)**

Ex. 4. **How many factors are in 200?**

$200 = 5x2x5x2x2 = 2^3 x 5^2$.

Using the theorem, add one to each exponent, then multiply.

The exponents are 3 and 2. Add one to each exponent, (3 +1)and(2 +1)
Find the product. 4x3 = 12.
There are 12 factors.

We just don't what the factors are, but ….

Finding Factors

To find all the factors of a number we will use Dividing by Primes in conjunction with the Multiplicative Inverse. That is, we will start with a number, then write that number as a product of the number and one. Then, using the Multiplicative Inverse and Identity for multiplication, we will rewrite that product by dividing and multiplying by 2 (a prime) until those divisions are exhausted, then begin again by dividing by the next prime number – 3.

Ex. 1. Find all the factors of 48.

Rewriting 48 as a product

48	
48	1
24	2
12	4
6	8
3	16

The factors of 48 are {1, 2, 3, 4, 6, 8, 12, 24, 48}

Ex. 2. Find all the factors of 60.

Rewriting 60 as a product

60		
60	1	
30	2	
15	4	
20	3	div 3
10	6	
5	12	

The factors of 60 are {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60}

Knowing these theorems and rules are used in in math. That using the rules of divisibility will help us simplifying fractions, to determine if a fraction would convert to a terminating or repeating decimal, how many factors were in a number, factoring polynomials, and finding common denominators – *the more math you know, the easier math gets!*

Determine if a Number is Prime

How can we determine if a number is prime? We know from the Fundamental Theorem of Arithmetic, that every composite number can be written as a product of primes.

Suppose p is the *least prime* factor of a number n, then we know that p times some other prime number q is equal to n. Since p is the least prime, that would suggest the $p \cdot p \leq n$ or $p^2 \leq n$.

Looking at that from a slightly different angle, we have

Theorem **If n is an integer greater than one and not divisible by any prime number p, where $p^2 \leq n$, then n is prime.**

Ex. 1. **Is 397 prime or composite?**

Look for the possible prime numbers p so that $p^2 \leq 397$.

Taking the square root of a perfect square larger than 397, 400 is close.

I only have to check the primes numbers less than 20. The primes are 2, 3, 5, 7, 11, 13, 17 and 19 are the only primes where $p^2 \leq 397$, none of them are factors of 397, so 397 is prime.

Ex. 2 **Is 43 a prime number?**

Checking all the possible prime numbers so when I square them they are less than or equal to 43, I have 2, 3, and 5. None of those numbers divide into 43, so 43 is prime.

Why didn't I continue checking the other primes, 7, 11, 13, etc.? Because I only had to check the primes that are less than or equal to the square root of 43.

Ex. 3. **Is 637 prime or composite?**

The square root of 637 is less than 25

The prime numbers we want to check are 2, 3, 5, 7,11, 13, 17, and 23.

$23^2 \leq 637$. 637 is not even, so it is not divisible by 2, the sum of the digits is 16, so it is not divisible by 3, it does not end in 5 or 0, so it is not divisible by 5. The next number prime we check is 7, 7 is a factor. Therefore, 637 is composite.

Let's move on to Greatest Common Factors (GCF), sometimes referred to as Greatest Common Devisors (GCD) and Least Common Multiples (LCM) and referred to as Least Common Denominators (LCD) when working with fractions.

As always, to understand math, we need to know definitions. So let's start by defining factors.

Factor: is a number in a multiplication problem.

Greatest Common Factors

Common factor: A number that is a factor of two or more nonzero numbers.

Ex. 1. Find common factors of 18 and 24.

Factors of 18: **1**, **2**, **3**, **6**, 9, 18

Factors of 24**:** **1**, **2**, **3**, 4, **6**, 8, 12, 24

The common factors are written in red; 1, 2, 3 and 6.

Greatest Common Factor (GCF): Factors shared by two or more numbers are called *common factors*. The largest of the common factors is called the ***greatest common factor***.

In the last example, the greatest common factor, GCF, of 18 and 24 is 6. That is the largest factor in both of those numbers.

There are a number of ways of finding the GCF.

Method 1 – Intersection Method - Writing out the Factors

To find the GCF, list all the factors of each number. Look at the common factors and the largest one is the GCF.

Ex. 2. Find the GCF of 24 and 36.

Factors of 24 1, 2, 3, 4, 6, 8, 12, 24

Factors of 36 1, 2, 3, 4, 6, 9, 12, 18, 36

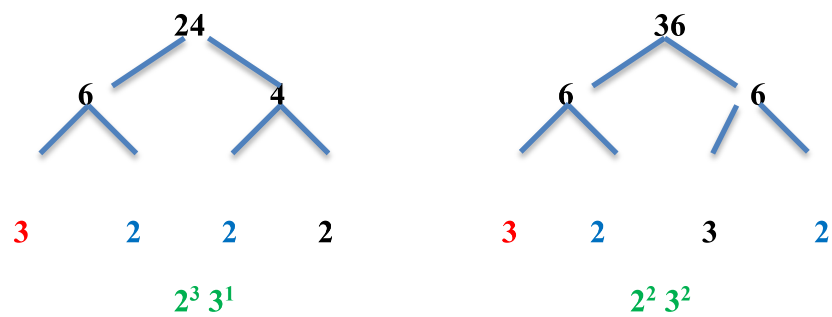
The GCF is the greatest factor that is in both lists; 12.

Method 2 – Using a Factor Tree

To find the GCF, write the prime factorization of each number and identify which factors are common in each number.

Ex. 3. Find the GCF of 24 and 36. – same problem as ex. 2

Looking at the last problem using factor trees, we 24 and 36



Now looking at the tree diagram and the prime factorization of each number, I pair numbers from the first tree with numbers in the second tree. In the example, I did that by color, the 3's match (pair) in red, blue matches 2 (pairs) with a blue 2, the green at the bottom is the prime factorization

So the numbers that are paired (matching) are 3, 2, and 2. When multiplied equals 12.

Another way to find the GCF using the factor tree is to use the factors in exponential form.

$36 = 2^2\,3^2$

$24 = 2^3\,3^1$

Each number has two 2's and one 3, therefore the GCF is $2^2 x 3^1$ or 12.

Ex. 4. Find the GCF of $3^4 x 5^4 x 7^2$ and $3^2 x 5^6$.

The common factors are 3 and 5. Use the smallest exponent on each of these numbers to find the GCF. Since the number with the smallest exponent is in both numbers, those are the common factors. The smallest exponent on the 3's is 2 and the smallest exponent on the 5's is 4. **Therefore, the GCF is $3^2 x 5^4$.**

Generally, the GCF problems are not given to you written as a product of primes as in example 4. You have to find the factors either using the Intersection Method or by a Factor Tree.

Then, identify factors each numbers has in common, then use the smallest exponent of each to find the GCF.

Least Common Multiples

Least common multiple is a lowest multiple (number) of two or more numbers. That means all the factors of each number have to be contained in a common multiple.

Methods of finding the LCM

There are three methods for finding the Least Common Multiple (LCM) between two numbers:

Method I: Intersection Method - Make a list.
Write the multiples of each numbers until there is a common number.

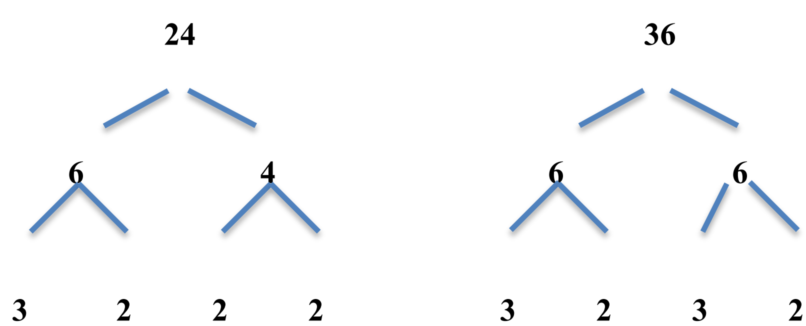
Ex. 1. **Find the LCM of 12 and 16.**

Multiples of 12: 12, 24, 36, 48, 60, 72, …
Multiples of 16: 16, 32, 48,…
48 is the smallest multiple of both numbers, therefore 48 is the LCM.

Method II: Prime factorization.

Ex. 2 **Find the LCM of 24 and 36**

Using the same factor tree I used for the GCF problem for 24 and 36, we have



$2^3\ 3^1$ $2^2\ 3^2$

To find the LCM, we have to have all the prime factors of BOTH numbers using the greater exponent.

So, using all the factors from each tree, write those factors. We know that those factors work for the first number 24 because 24 goes into 24.

Does it work for 36? In other words, are all the prime factors of 36 included in the prime factors of 24?

The prime factors of 24 are 3(2)(2)(2), we have two 3's and three 2's.

For 36 we need two 3's, we only have one, so we have to add another factor of 3. For 36, we need two factors of 2, I have that, so I am done.

So, the LCM is 3(3)(2)(2)(2) = 72

Write the prime factorization of both numbers. **The LCM has to contain ALL the factors of BOTH numbers using the greatest exponents**. Write the prime factors, use the highest exponent because we want to find the greatest factor in EACH number.

Ex. 3. Find the LCM of 72 and 60.

Prime factors of 72: 2x2x2x3x3=72
Prime factors of 60: 2x2x3x5=60

The Prime factors of 60 and 72 are made up of 2's, 3's, and a 5. How many of each prime factors are used to make up the LCM?

The prime factorization of 72 has two prime factors – 3^2 (2^3).
The prime factorization of 60 has three prime factors – 2^2 (3)(5)

Write all the factors that appear in each number - (2) (3) (5)

Now, determine the exponents, the highest exponent of each factor. The highest exponent on 3 is 2. The highest exponent on 2 is 3, and the 5 has exponent 1.

So, the answer is 2^3 $(3^2)(5^1)$

The LCM = $2^3 \text{x} 3^2 \text{x} 5^1$=360.

Method III: Reducing Method

Another way of finding the LCM is to write the factors as a fraction, then reduce and cross multiply. The product is the LCM.

Ex. 4. **Find the LCM of 60 and 72.**

The factors of 60 are **2x2x3x5**
The factors of 72 are **2x2x2x3x3**

Simplify the fraction

$$\frac{2x2x3x5}{2x2x2x3x3} = \frac{5}{2x3}$$

Now, cross multiplying we have (2x2x3x5)(2x3) = 360

The good news is we really don't have to write the prime factors to simplify a fraction, we can simplify the fraction, then just cross multiply.

Ex.5. **Find the LCM of 18 and 24.**

Make a fraction using 18 and 24. Order does not matter.
Reduce that fraction which is now called simplifying the fraction

$$\frac{18}{24} = \frac{3}{4}$$

Cross multiply. Either 18x4 or 3x24. Both products equal 72.
The **LCM** is 72.

Note: When adding or subtracting fractions, the **LCM** is referred to as the Least Common Denominator or **LCD**. Method III works great!

Converting Fractions - Repeating or Terminating Decimal?

From our Rules of Divisibility, we know numbers that end in 0 are divisible by 2, 5, and 10. Powers of 10; 10, 100, 1000, etc. are divisible by 2, 5, and 10. But, since we want to convert fractions to decimals, we can see a that only numbers that have factors of 2, 5 or 10 will ever go into a power of 10.

In other words, 3 will not go into a power of 10. Notice, the sum of the digits in a power of ten will never add up to a multiple of 3.

That means, when I factor a number into primes, if the number has only prime factors of 2 or 5, then the number will divide into a power of 10 and will be a terminating decimal.

Example 1. Is 3/20 a terminating or repeating decimal?

3/20, will be a terminating decimal because the prime factorization of 20, the denominator, is 2^2x5. Only 2's and 5's.

Example 2. Is 1/8 a terminating or repeating decimal?

1/8 will be a terminating decimal because the prime factorization of 8 is 2^3, only 2's.

Example 3. Is 5/12 a terminating or repeating decimal?

5/12 will be a repeating decimal, because the prime factorization of 12 is 2^2x3. To be terminating, we can only have factors of 2 or 5. A number with a prime factor containing 3 will never divide into a power of 10.

Example 4. Is 2/15 a terminating or repeating decimal?

2/15 will be a repeating decimal, because the prime factorization of the denominator is 5x3. It has a prime factor other than 2 or 5.

Example 5. Is 3/35 a terminating or repeating decimal?

3/35 will be a repeating decimal, it has a prime factor of 7.

When converting a fraction to a decimal, it will either terminate or repeat within the number of places determined by the denominator. In other words, the fraction 1/7 will either terminate or repeat within 7 places.

Example 6. Convert 1/7 to a repeating decimal.

1/7 = 142857142857.. or $.\overline{\mathbf{142857}}$

Notice it started repeating within 7 places.

Hopefully, you can see the usefulness of the study of Number Theory and how it impacts your study of mathematics and makes math easier to do.