

## Ch. 25 Graphing Techniques

### Sec. 1 Parent Functions

In this chapter, we will take our knowledge of graphs of basic functions and expand our ability to graph polynomial and rational functions using common sense, zeros, y-intercepts, stretching & shrinking, translations, reflections, symmetry, and asymptotic lines.

The most basic way of graphing is by plotting points. Graphing becomes a lot easier if you have an idea of what the graph of a function might look like. We will still plot some **convenient** points, but they will be strategic so we can see what happens in certain areas of the graph. So, let's look at some basic graphs that should be immediately recognizable based on our previous experiences.

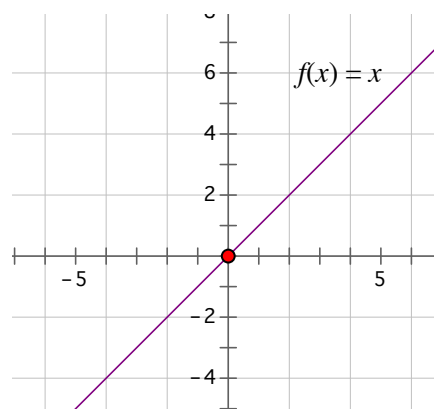
#### Identity Function

$$f(x) = x$$

Domain  $(-\infty, \infty)$

Range  $(-\infty, \infty)$

To graph use  $x = -1, 0, 1, 2$



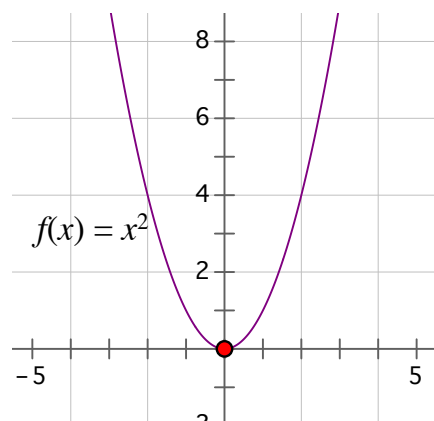
#### Squaring Function

$$f(x) = x^2$$

Domain  $(-\infty, \infty)$

Range  $[-\infty, \infty)$

To graph use  $x = -2, -1, 0, 1, 2$



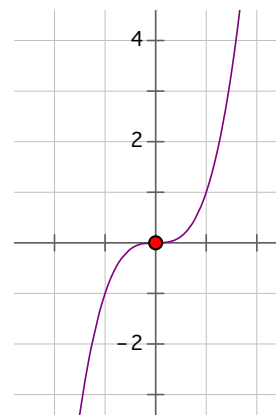
### Cubing Function

Domain  $(-\infty, \infty)$

Range  $(-\infty, \infty)$

$$f(x) = x^3$$

To graph let  $x = -1, 0, 1$



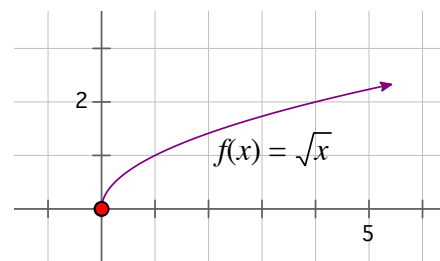
### Square Root Function

Domain  $[0, \infty)$

Range  $[0, \infty)$

$$f(x) = \sqrt{x}$$

To graph let the radicand,  $x = 0, 1, 4, 9$  because you know their square roots.



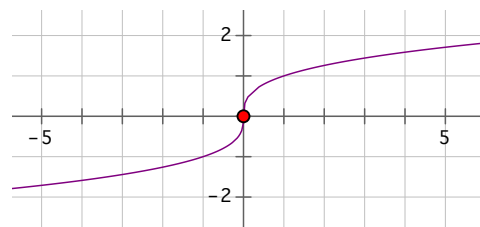
### Cube Root Function

Domain  $(-\infty, \infty)$

Range  $(-\infty, \infty)$

$$f(x) = \sqrt[3]{x}$$

To graph let the radicand,  $x = -8, -1, 0, 1, 8$   
Because you know their cube roots



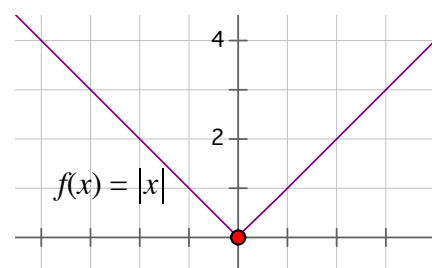
### Absolute Value Function

Domain  $(-\infty, \infty)$

Range  $[0, \infty)$

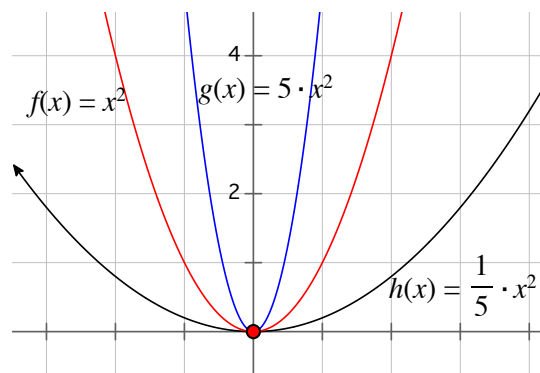
$$f(x) = |x|$$

To graph let  $x = -1, 0, 1$



## Stretching - Shrinking

I could also stretch or shrink those graphs by multiplying the function by numbers greater than one or by a fraction.



When I stretch or shrink a function, the graph gets steeper or flatter as shown above. If I compared the ordered pairs of the parent function and the function being stretched, we would see the domain (x-values) stay the same, the range changes by whatever number we multiplied.

Let's look at and compare the ordered pairs for  $y = x^2$  and  $y = 5x^2$ , as shown above and confirm that.

$$y = x^2$$

$$y = 5x^2$$

x	y	x	y
0	0	0	0
1	1	1	5
-1	1	-1	5

## Sec. 2 Translations

Besides being stretched, these graphs can be translated very easily on the rectangular coordinate system when we write the functions in certain forms.

The easiest way to graph is to look at the parent functions we have already graphed, stretch or shrink them, then translate them along the x and y axes.

In these equations, the  $a$  will be the stretch or shrink, the  $h$  is the horizontal shift (translation) and the  $k$  will be the vertical shift

Looking at the parent functions and adding translations, we have

$$f(x) = x^2 \rightarrow f(x) = a(x-h)^2 + k$$

$$f(x) = x^3 \rightarrow f(x) = a(x-h)^3 + k$$

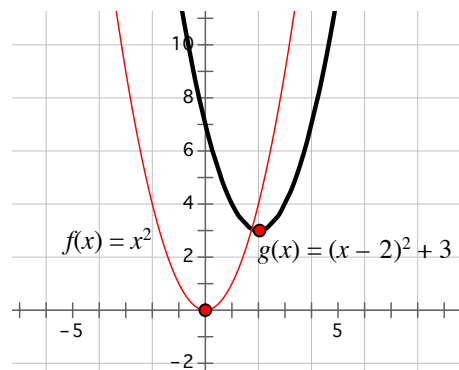
$$f(x) = \sqrt{x} \rightarrow f(x) = a\sqrt{x-h} + k$$

$$f(x) = \sqrt[3]{x} \rightarrow f(x) = a\sqrt[3]{x-h} + k$$

The  $a$  represents the stretch or shrink, the  $h$  is the horizontal translation and the  $k$  is the vertical translation.

For instance, in the squaring function;  $f(x) = x^2$ , looks like a bell shaped curve passing through the origin. If I were to add 3, that entire graph would be moved up the y-axis 3 units, we'll call that  $t(x) = x^2 + 3$ . Now, if I wanted to move the graph horizontally 2 units, I will take my original squaring function,  $f(x)$ , and have the graph moved 2 units to the right. We'll call the new rule  $h(x) = (x - 2)^2$ .

Combining those translations would result in a new rule  $g(x) = (x-2)^2 + 3$ , will move the original squaring function up three and over 2 on the coordinate system as shown to the right.



There was no stretch or shrink.

### Vertical Translations

If a function  $g$  is defined by  $f(x) + c$ , then for every point  $(x, y)$  on the graph of  $f(x)$ , there will be a corresponding point  $(x, y+c)$  on the graph  $g(x)$ .

### Horizontal Translations

If a function  $g$  is defined by  $g(x) = f(x - c)$ , then for every point  $(x, y)$  on the graph of  $f(x)$ , there will be a corresponding point  $(x + c, y)$  on the graph of  $g(x)$ .

\*\* What that means is this, when an  $h$  or  $k$  are in any type of grouping; parentheses, brackets, braces, under a vinculum (like a square root), you change the SIGN of the  $h$  or  $k$ .

You will see more of these types of translations when you work with the conic sections graphing parabolas, circles, ellipses, and hyperbolas.

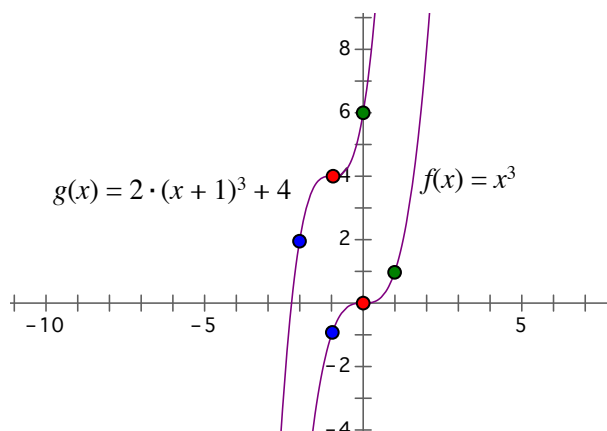
To graph these translations:

1. Use the parent function and multiply the range by  $a$  and rewrite the  $x$ - $y$  chart with the changes in the  $y$ 's.
2. With the new  $x$ - $y$  chart, add the opposite of  $h$  to the  $x$  values and  $k$  to the  $y$  values.
3. Plot the new points.

**Example 1**                      **Graph  $y = 2(x + 1)^3 + 4$**

The parent function is  $y = x^3$ , the values for the domain we will use for the parent function are  $-1, 0$  and  $1$  because they are convenient and easy.

Parent		Stretch (x2)		Translation $-1$ for $x$ 's and $+4$ for $y$ 's	
$x$	$y$	$x$	$y$	$x$	$y$
$-1$	$-1$	$-1$	$-2$	$-2$	$2$
$0$	$0$	$0$	$0$	$-1$	$4$
$1$	$1$	$1$	$2$	$0$	$6$



You could have done that translation without making new  $x$ - $y$  charts. To do that, graph the parent function with the stretch or shrink, then move those points horizontally and vertically.

Actually doing this is very easy. After you graph using the stretch, move that graph 1 unit left and 4 units up.

**Example 2** Graph by inspection

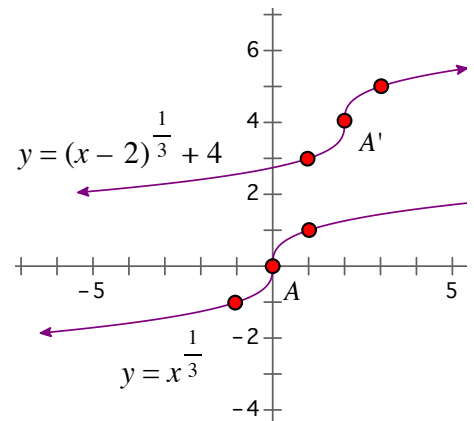
$$y = \sqrt[3]{x-2} + 4$$

First we will graph the parent function.

Since there is no stretch,  $a = 1$ , we will move each point of the graph 2 units to the right and up 4 units.

Look at points A and A'; over 2 up 4.

If I didn't know the parent function and I Graphed this manually, I would pick **convenient** values for x, then find the corresponding y-values.



What are convenient values when taking the cube root? Perfect cubes,  $-1, 0, 1, 8, \dots$ . So, I will let the radicand,  $(x-2)$  equal those perfect cubes to find x. Then use that x to find the corresponding y-values.

$$\begin{array}{llll} x - 2 = -1; & x - 2 = 0; & x - 2 = 1; & x - 2 = 8; \text{ etc} \\ x = 1; & x = 2; & x = 3; & x = 10 \end{array}$$

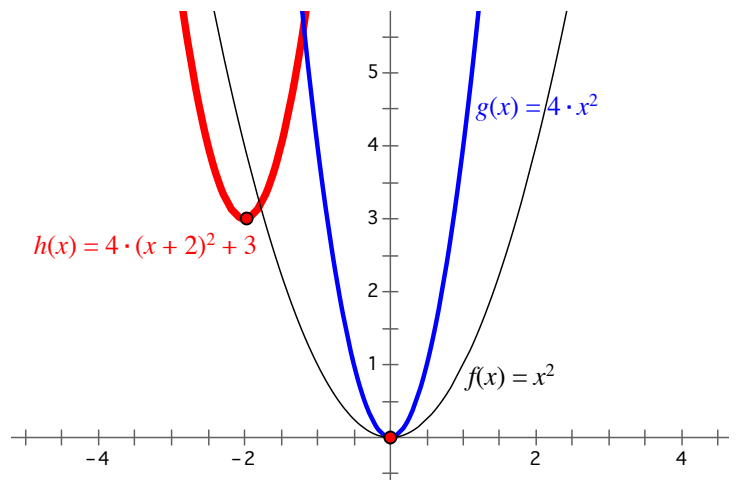
Substituting those values of x into the equation, gives you a perfect cube in the radicand which will allow you to find the y-values easily.

**Example 3** Graph by inspection

$$y = 4(x+2)^2 + 3$$

First, I graph the parent function, that's in black

Second, I will need to graph the stretch using the parent function which is in blue;  $a = 4$ .



After graphing the stretch, I will then move the graph 2 units to the left and up 3 which is in red.

## Sec. 3 Symmetry

To determine if a function  $P$  has symmetry, we have 3 tests:

- symmetric to y-axis**, if the replacement of  $x$  with  $-x$  results in an equivalent equation
- symmetric to x-axis**, if the replacement of  $y$  with  $-y$  results in an equivalent equation
- symmetric with origin**, the the replacement of  $x$  with  $-x$  and  $y$  with  $-y$  results in an equivalent equation

## Zeros

### Number of Zeros Theorem

A polynomial of degree  $n$  has at most  $n$  distinct zeros.

What this theorem means is an equation of degree 2 can have at most 2 zeros. An equation of degree 3 can have at most 3 zeros, etc. From a graphing standpoint, that means if I have an equation of degree 3, we could have 3 zeros, or the graph could cross the  $x$ -axis at most 3 times.

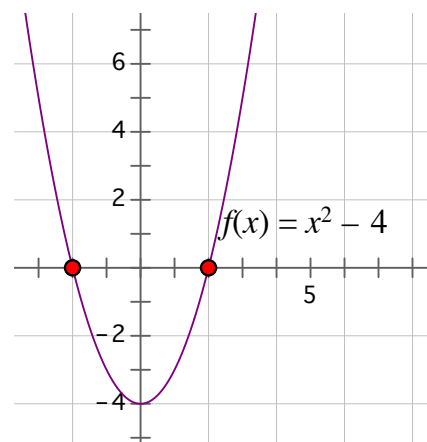
Let's look at a few graphs and see if we can determine what the might look like.

### Example 1

In the equation,  $f(x) = x^2 - 4$ , the polynomial is of degree 2, so at most the graph will intersect the  $x$ -axis twice.

### End Behaviors

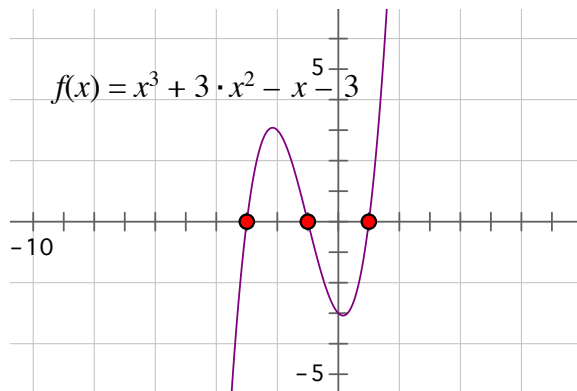
What happens for large vales of  $x$ ? So when  $x = 100$ , what happens to the graph? Substituting values into the equation, we can see  $f(x)$  get very large too. When  $x$  gets very small, what happens? So if  $x = -100$ , what happens? Again the values of  $f(x)$  get very large too. We can see the end behaviors on the graph are positive. So, the graph goes up to the right and up to the left.



## Example 2

In the equation,  $g(x) = x^3 + 3x^2 - x - 3$ , so let's check for x and y intercepts. The y intercept is easy to find by letting  $x = 0$ . The x intercepts require us to let  $y = 0$  and solve the equation.  $x^3 + 3x^2 - x - 3 = 0$ .

$x^2(x + 3) - 1(x + 3) = (x + 1)(x - 1)(x + 3) = 0$   
We have zeros at  $-3$ ,  $-1$ , and  $1$ . Now let's check end behaviors.

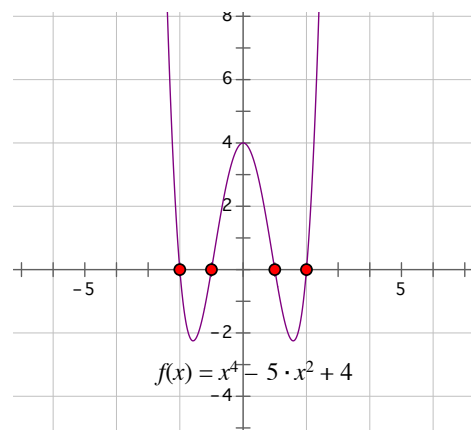


In this cubic equation, the cubic term dominates the equation for very large and small values of  $x$ . So, when  $x = 100$ ,  $f(x)$  will be a very large positive number. When  $x = -100$ ,  $f(x)$  will be very large negative number. We can see the end behaviors on the graph. For large vales of  $x$ , the graph stays positive. For large negative values, the graph stays negative. So the graph goes up the right and down to the left.

## Example 3

Let's look at a 4th degree equation,  
 $f(x) = x^4 - 5x^2 + 4$ .

At most the graph will intersect the x axis 4 times. The y-intercept is 4. The zeros occur at  $-1$ ,  $-2$ ,  $1$ , and  $2$ .



What happens for very large value of  $x$ , say  $x = 100$ , since  $x^4$  dominates the polynomial, we see when we substitute very large vales of  $x$ ,  $f(x)$  is positive. Checking large negative numbers,  $x = -100$ , the 4th degree term dominates, so  $f(x)$  has a positive end behavior that can be seen in the graph. So the graph goes up to the right an up to the left.

If we wanted, we could have substituted  $-x$  for  $x$  and found that we have symmetry with respect the y-axis as well.



## Intermediate Value Theorem

If  $f(x)$  defines a polynomial function with real coefficients, and if for real numbers  $a$  and  $b$ , the values of  $f(a)$  and  $f(b)$  have opposite signs, then there is at least one real zero between  $a$  and  $b$ .

What that tells us is the graph will cross the x-axis somewhere between those 2 points.

## Multiplicity Theorem

If  $(x - r)^m$  is a factor of a polynomial, then  $r$  is a zero of multiplicity  $m$  of  $f$ .

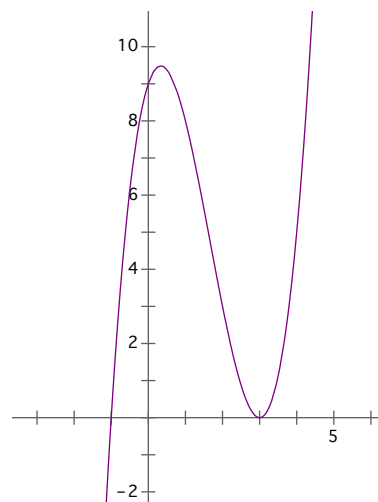
**Example 1**       $f(x) = 5(x - 2)(x + 1)^3(x - 3)^4$

2 is a zero of multiplicity 1  
-1 is a zero of multiplicity 3  
3 is a zero of multiplicity 4

When the multiplicity of a zero is even, the graph touches the x-axis at that point, but does not cross the x-axis. When the multiplicity of a zero is odd, the graph crosses the x-axis at that point.

**Example 2**      Graph  $f(x) = (x - 3)^2(x + 1)$

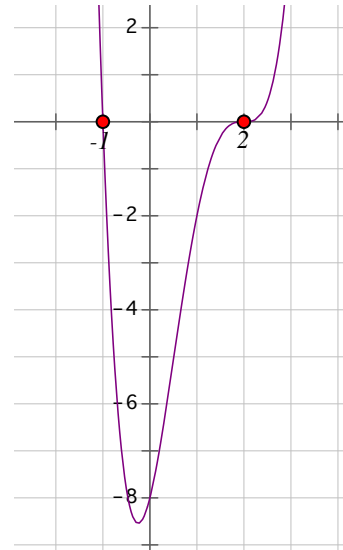
Notice the graph did not cross the x-axis at  $x = 3$



### Example 3

Graph  $h(x) = (x - 2)^3(x + 1)$

Notice the graph crossed the x-axis at  $x = 2$



So, to sketch a polynomial function of degree  $n$ , it helps to have a general idea of what the graph should look like, then follow the following steps:

1. Find the zeros; the x intercepts
2. Find  $f(0) = c$ , the y intercept
3. Use test points within intervals to determine the sign  $f(a)$ , above or below the x axis
4. Use end behaviors

## Graphing Rational Functions - Asymptotes

A **rational function** is a function defined by a rational expression.

A rational function is often defined mathematically:

$$f(x) = \frac{p(x)}{q(x)}$$

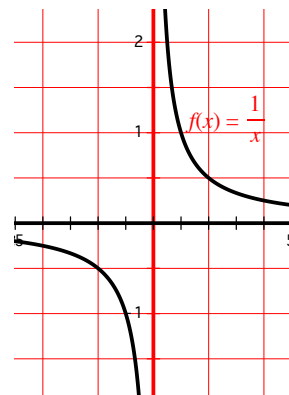
where  $p(x)/q(x)$  is in **simplest terms** polynomials and  $q(x) \neq 0$ .

Let's look at the graph of a simple rational function;

$$f(x) = 1/x$$

If we examine the domain, we realize quickly that we cannot have a zero in the denominator. But all the other values of  $x$  would work, so the domain is

$$(-\infty, 0) \cup (0, \infty)$$



Notice the graph is not continuous at  $x = 0$  because the expression will be undefined. In other words, the graph of  $f(x) = 1/x$  will never intersect the vertical line  $x = 0$ , the  $y$  axis. That line is called a **vertical asymptote**.

If we look at each portion of the graph and look at their end behaviors, we also see as  $x$  gets very large, the value of  $f(x)$  gets smaller and smaller and gets closer to zero but will never get there. That means the graph gets closer and closer to the  $x$  axis but will never touch the axis or cross it. That means the  $x$  axis,  $y = 0$ , is a **horizontal asymptote**.

Looking at positive values of  $x$  close to zero, we see the values of  $f(x)$  get very large the closer they get to the  $y$  axis.

The same arguments can be used for the left side of the graph.

So when graphing rational functions, an added dynamic is introduced. We have to check to see where the function is not defined, when the denominator is zero, because the graph will be discontinuous at those points, and we will see the vertical asymptotes divide the graph into sections. We then use our techniques for graphing in these intervals.

So, let's try to make some generalization for graphing rational functions. But let me introduce this arrow, " $\rightarrow$ ", mathematically that means "approaches".

### Asymptotes of $f(x)$ , a rational expression

1. If  $f(a)$  is undefined, then  $x = a$  is a vertical asymptote.
2. If  $f(b) \rightarrow c$  as  $x \rightarrow \infty$ , then  $y = c$  is a horizontal asymptote.

In the last example, we saw when  $x$  gets very large,  $x \rightarrow \infty$ , then  $f(x)$  got closer and closer to zero. So we said,  $y = 0$  is a horizontal asymptote.

## Finding Asymptotes

To find asymptotes of rational functions:

### Vertical Asymptotes

1. Find vertical asymptotes by setting the denominator equal to zero and solve for  $x$ . If  $a$  is a zero of the denominator, then the line  $x = a$  is a vertical asymptote.

### Horizontal Asymptotes

1. If the degree of the numerator is less than the degree of the denominator, then there is a horizontal asymptote at  $y = 0$ .
2. If the numerator and the denominator have the same degree, there is a horizontal asymptote determined by the fraction of the leading coefficients of the numerator and denominator.

### Oblique Asymptotes

3. If the degree of the numerator is exactly one more than the degree of the denominator, then there will be an oblique asymptote determined by dividing the numerator by the denominator and disregarding the remainder and setting that equal to  $y$ .

Now, let's put all this information into play and see how this helps us graph rational expressions.

## Graphing Rational Functions of the form $f(x) = \frac{p(x)}{q(x)}$

- Step 1.* Find all vertical asymptotes
- Step 2.* Find horizontal or oblique asymptotes
- Step 3.* Find the  $y$  intercepts, let  $x = 0$
- Step 4.* Find the  $x$  intercepts, let  $y = 0$
- Step 5.* Determine if the graph will intersect the horizontal or oblique asymptotes
- Step 6.* Plot selected points in each interval determined by the vertical asymptotes

### Example 1

Graph  $f(x) = \frac{x+1}{2x^2+5x-3}$

- 1) Following the steps above, find the vertical asymptotes by setting the denominator equal to zero.

$$2x^2 + 5x - 3 = (2x - 1)(x + 3)$$

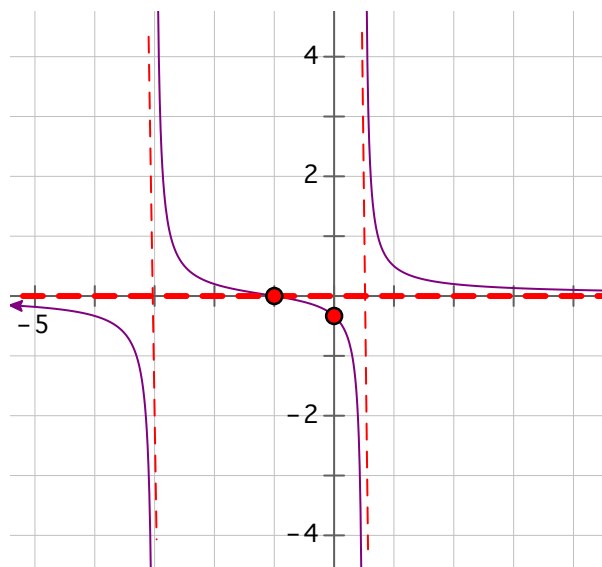
$(2x - 1)(x + 3) = 0$ , so  $x = -3$  and  $x = 1/2$  are vertical asymptotes.

- 2) Since the degree of the numerator is less than the denominator,  $y = 0$  is a horizontal asymptote.
- 3) The y intercept is  $-1/3$ , letting  $x = 0$
- 4) The x intercept by setting  $f(x) = 0$ ,  $x = -1$  is an x intercept
- 5) Set  $f(x) =$  horizontal asymptote,  $f(x) = 0$ , so the graph intersect the horizontal asymptote at  $(-1, 0)$
- 6) Check selected points and behaviors in each interval

The vertical asymptotes occurs at  $x = -3$  and  $x = 1/2$ , represented by the red dashed lines. So that divides the graphs into 3 intervals.

The horizontal asymptote occurs  $y = 0$ , so we can see what happens for large positive and negative values of  $x$ .

We found the x and y intercepts and we then check end behaviors in each interval defined by the vertical asymptotes.



Remember a graph can **never** cross a vertical asymptote. They can cross horizontal and oblique asymptotes.

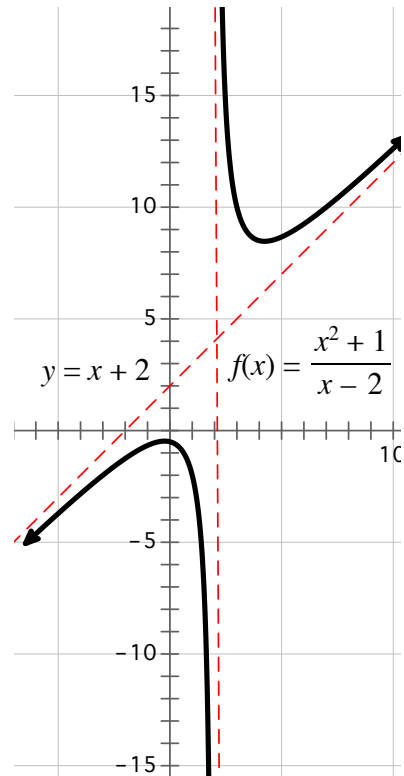
**Example 2** Graph  $f(x) = \frac{x^2 + 1}{x - 2}$

Using our steps, there is a vertical asymptote at  $x = 2$ . red dashed line. If I divide  $(x^2 + 1)$  by  $(x - 2)$ , we get our horizontal asymptote at  $y = x + 2$ .

The y intercept occurs at  $-1/2$ . There are no x intercepts because  $x^2 + 1$  has no real solutions.

And the graph does not intercept the oblique asymptotes because the rational function and the equation of the oblique asymptote has no solution. In other words,  $\frac{x^2 + 1}{x - 2} = x + 2$  does not have a solution.

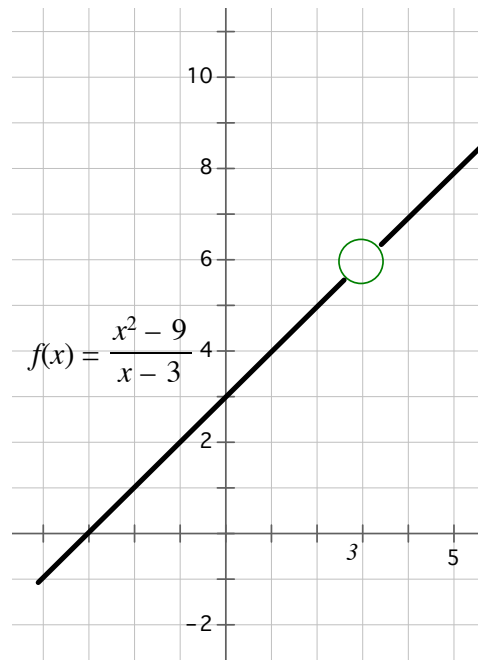
Finally, we check end behaviors around the vertical asymptotes and then for large positive and negative values of  $x$ .



**Example 3** Graph  $f(x) = \frac{x^2 - 9}{x - 3}$

The rational expression is not reduced. So simplifying the expression, we have

$$f(x) = x + 3, x \neq 3.$$

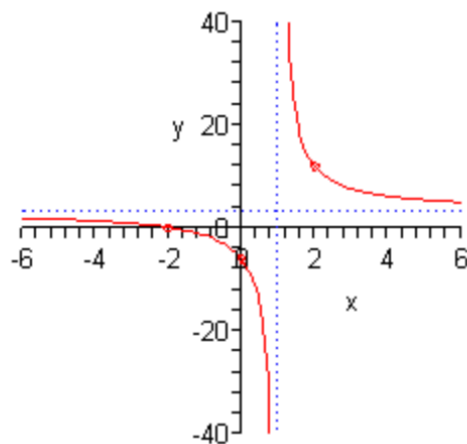


Graphing the line, we see the function is not continuous at  $x = 3$ , so there is an open circle there.

**Example 4** Graph  $f(x) = \frac{(3x + 6)}{(x - 1)}$

Let's quickly find the vertical asymptote(s) by setting the denominator equal to zero, so  $x = 1$ . And since the degree of the numerator and denominator are the same, the horizontal asymptote will be  $\frac{3}{1}$  or  $y = 3$ . Draw those two asymptotes with dashed lines.

Then we will pick value of  $x$  close to the vertical asymptote, find the corresponding  $y$ , and plot that point. (Finding zeros sometimes makes that easier.)



Now, pick another value of  $x$  **on the same side** of the vertical asymptote, find that corresponding value of  $y$ , and plot that point.

Now, use symmetry to graph on the other side of the vertical asymptote.

Like everything else you do in life, the more you do of these, they become almost automatic. While I can't make these more difficult, I can make them longer ;-)

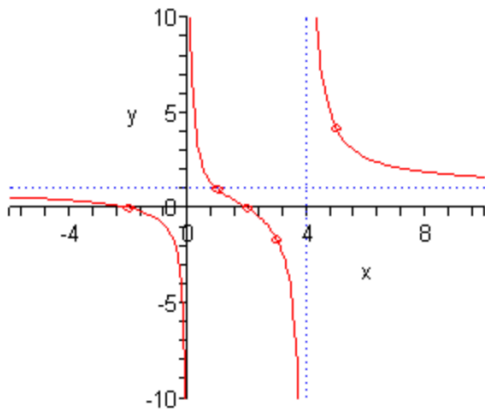
Let's look at another example.

**Example 5** Graph  $f(x) = \frac{(x^2 - 4)}{(x^2 - 4x)}$

Factoring, we have  $\frac{[(x + 2)(x - 2)]}{[x(x - 4)]}$ , we have vertical asymptotes at 0 and 4. And since the degrees of the numerator and denominator are the same, we have a horizontal asymptote at  $1/1$ , or  $y = 1$ . Draw those asymptotes.

The two vertical asymptotes divide the graph into three regions. That results in picking points in each region - yes, that's fun!

Here's a sketch of that graph.



Picking an  $x$  to the right of the vertical asymptote on the right, like 5, we get  $y$  equaling  $21/5$  or  $4 \frac{1}{5}$ . Graph that point, now pick another value of  $x$  on the same side of that vertical asymptote like  $x = 10$ , that results in  $y = 96/60$  or  $8/5$ . Connecting those, we get the graph on the far right using the asymptotes.

The graph on the far left can be done the same way - we could also use symmetry to make that a little easier.

And finally, we have to sketch the graph of the middle region, when  $x = 1$ ,  $y = 2$ . When  $x$  is 3,  $y$  is  $-5/3$ . Again, use the asymptotes

Kind of looks like Example 1.