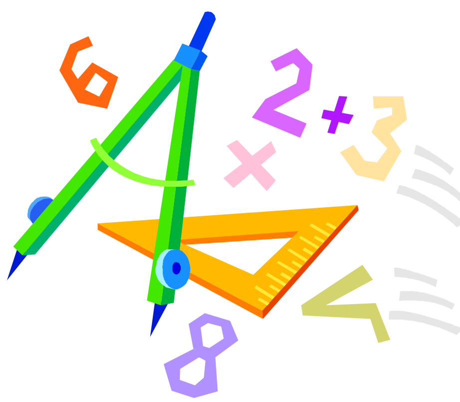
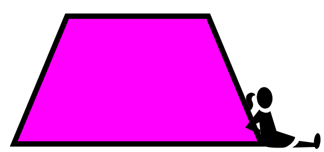
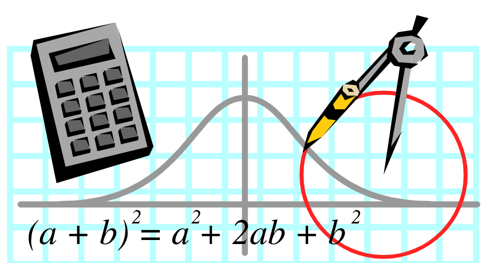
HANLONMATH

Geo Proofs







by Bill Hanlon

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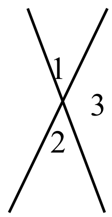
Future Reference – To prove congruence, it is important that you remember not only your congruence theorems, (SSS, SAS, ASA, AAS, HL, HA, HL, LA) but know the relationships with angles formed by intersecting lines, parallel lines, right angles, angles bisectors, medians and theorems concerning triangles. Quite often you will need to use construction to create triangles that will allow you to do proofs or solve problems.

Thm - Vertical angles are congruent



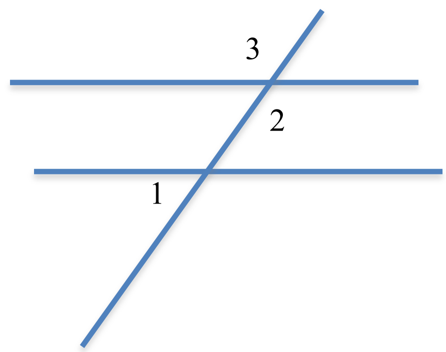
Given: $\angle 1$ and $\angle 2$ are vertical angles

Prove: $\angle 1 \cong \angle 2$

Strategy: Knowing angles 1 and 3 are two angles whose ext. sides form a straight line as do angles 2 and 3. Those angles form sup $\angle$s , whose sum is 180˚

Statements	Reasons
1. $\angle 1$ and $\angle 2$ are vert $\angle$'s	Given
2. $\angle 1$ and $\angle 3$ are supp $\angle$'s	Ext sides, 2 adj $\angle$'s in a line
3. $\angle 2$ and $\angle 3$ are supp $\angle$'s	Same as #2
4. $\angle 1 + \angle 3 = 180^\circ$ $\angle 2 + \angle 3 = 180^\circ$	Def of Supp $\angle$s
5. $\angle 1 + \angle 3 = \angle 2 + \angle 3$	Sub
6. $\angle 1 = \angle 2$	Sub Prop of Equality
7. $\angle 1 \cong \angle 2$	Def of Congruence

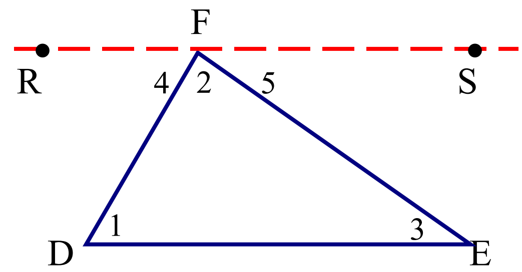
Thm If 2 parallel lines are cut by a transversal, the alternate interior angles are congruent



Strategy: Since we know by postulate that $\angle 1$ and $\angle$ 3 are equal by corresponding $\angle$s and $\angle 2$ and $\angle 3$ are equal because of vertical, we can make those equations and substitute

	Statements	Reasons
1.	l ll m $\angle 1$ and $\angle 2$ are alt int $\angle$'s	Given
2.	$\angle 1$ and $\angle 3$ are corr. $\angle$s	Def of corr. $\angle$s
3.	$\angle 1 \cong \angle 3$	Two ll lines, cut by t, corr. $\angle$'s $\cong$
4.	$\angle 3 \cong \angle 2$	Vert $\angle$'s
5.	$\angle 1 \cong \angle 2$	Transitive Prop

Thm: The sum of the measures of the angles of a triangle is 180º



G: Δ DEF

P: $\angle 1 + \angle 2 + \angle 3 = 180^\circ$

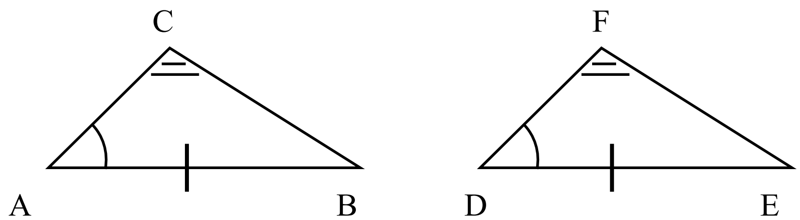
Strategy: Using our knowledge of ll lines being cut by a transversal, we will construct a line thru F ll to line segment DE, then segments DF and EF are transversals and the alt int $\angle$ s are

	Statements	Reasons
1.	Draw $\overline{RS}$ ll $\overline{DE}$	Construction
2.	$\angle 4 \wedge \angle DFS$ are supp	Ext sides of 2 adj 2's
3.	$\angle 4 + \angle DFS = 180$	Def Supp $\angle$'s
4.	$\angle DFS = \angle 2 + \angle 5$	Angle Add Post
5.	$\angle 4 + \angle 2 + \angle 5 = 180$	Sub
6.	$\angle 1 = \angle 4$ $\angle 3 = \angle 5$	2 $\parallel$ lines cut by t, alt int $\angle$'s =
7.	$\angle 1 + \angle 2 + \angle 3 = 180$	Sub into line 5

Congruence Postulates

SSS **SAS** **ASA**

Thm If two angles and the non included side of one triangle are congruent to the corresponding parts of another triangle, the triangles are congruent.



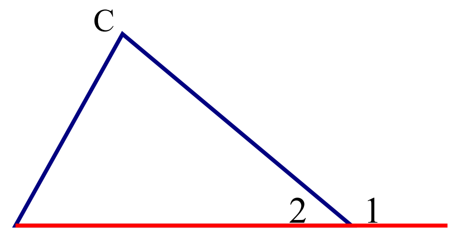
Given : $\angle A \cong \angle D$, $\angle C \cong \angle F$, $\overline{AB} \cong \overline{DE}$

Prove: $\Delta ABC \cong \Delta DEF$

Strategy: Knowing 2 $\angle$s of one triangle are equal to two angles of another triangle, the third angles must be equal. That allows me to them use one of the other congruence postulates to complete the proof.

Statements		**Reasons**
1.	$\angle A \cong \angle D$ $\angle C \cong \angle F$ $\overline{AB} \cong \overline{DE}$	Given
2.	$\angle B \cong \angle E$	2$\angle$'s of a Δ congruent 2 's of another Δ, 3rd 's congruent
3.	$\Delta ABC \cong \Delta DEF$	ASA

Thm. The exterior $\angle$ of a Δ is equal to the sum of the 2 remote interior $\angle$'s



G: ΔABC

P: $\angle 1 = \angle A + \angle C$

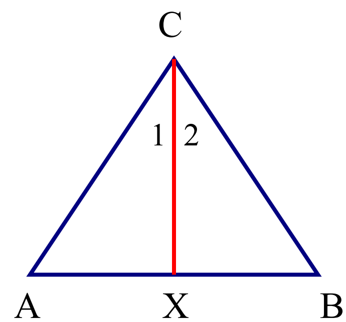
Strategy: We know the sum of the interior angles of a triangle is 180, we know that the ext sides of angles 1 and 2 lie in a line, hence equal 180, we set them equal and solve

	Statements	Reasons
1.	$\angle A + \angle C + \angle 2 = 180$	Int $\angle$'s of $\Delta = 180$
2.	$\angle 1 \wedge \angle 2$ are supp	Ext sides 2 adj $\angle$'s
3.	$\angle 1 + \angle 2 = 180$	Def Supp $\angle$'s
4.	$\angle A + \angle C + \angle 2 = \angle 1 + \angle 2$	Sub
5.	$\angle A + \angle C = \angle 1$	Sub Prop Equality

Thm. If 2 $\angle$'s of Δ are $\cong$, the sides opposite those $\angle$'s are $\cong$

G: Δ ABC
$\angle A \cong \angle B$

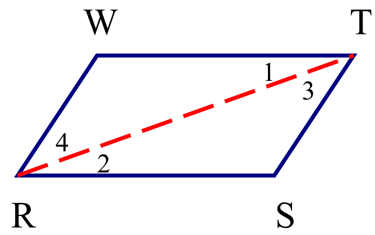
P: $\overline{AC} \cong \overline{BC}$



Strategy: To prove congruence, we need two triangles, we only have one. Constructing an angle bisector results in two triangles being formed which allows me to use the congruence theorems to prove triangles congruent, then use cpctc

Statements	Reasons
1. Draw $\angle$ bisector CX	Construction
2. $\angle 1 \cong \angle 2$	Def $\angle$ bisector
3. $\angle A \cong \angle B$	Given
4. $\overline{CX} \cong \overline{CX}$	Reflexive Prop
5. $\Delta CAX \cong \Delta CBX$	AAS
6. $\overline{AC} \cong \overline{BC}$	cpctc

Thm. A diagonal of a ||ogram separates the ||ogram into 2 ≅ Δ's



G: ▱RSTW

P: ΔRST ≅ ΔTWR

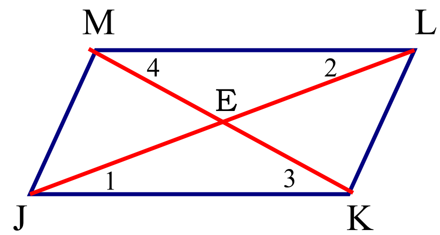
Strategy: Use the congruence theorems to prove the triangles congruent

Statements	Reasons
1. RSTW is a ‖ogram	Given
2. $\overline{RS} \parallel \overline{WT}$	Def - ‖ogram
3. $\angle 1 \cong \angle 2$	2 ‖ lines cut by t, alt int ∠'s ≅
4. $\overline{RT} \cong \overline{RT}$	Reflexive
5. $\overline{RW} \parallel \overline{ST}$	Def - ‖ogram
6. $\angle 3 \cong \angle 4$	2 ‖ lines cut by t, alt int ∠'s ≅
7. ΔRST ≅ ΔTWR	ASA

Thm: The diagonals of a ||ogram bisect each other.

G: ▱ JKLM

P: $\overline{LE} \cong \overline{JE}$; $\overline{KE} \cong \overline{ME}$



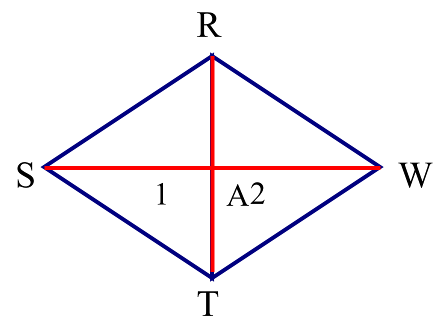
Strategy: Drawing the diagonals and using the theorems we have about parallel lines being cut by a transversal, we are able to label angles, show triangles congruent, then use cpctc

	Statements	Reasons
1.	JKLM is ||ogram	Give
2.	$\overline{JK} \parallel \overline{ML}$	Def - ||ogram
3.	$\angle 1 \cong \angle 2$	Alt int $\angle$'s
4.	$\overline{JK} \cong \overline{ML}$	opposite sides ||ogram ≅
5.	$\angle 3 \cong \angle 4$	Alt int $\angle$'s
6.	$\Delta JEK \cong \Delta LEM$	ASA
7.	$\overline{JE} \cong \overline{LE}$ $\overline{KE} \cong \overline{ME}$	cpctc

Thm: The diagonals of a rhombus are ⊥

G: Rhombus RSTW

P: RT ⊥ SW



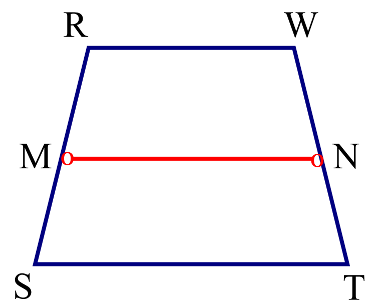
Strategy: Constructing the diagonals and using the def of a rhombus and parallelogram theorems, we have to show congruent adjacent angles to prove lines are perpendicular. So by proving triangles are congruent, we can use cpctc

	Statements	Reasons
1.	RSTW – Rhombus	Give
2.	$\overline{RS} \cong \overline{SW}$	Def – Rhombus
3.	$\overline{SA} \cong \overline{WA}$	Diagonals ||ogram bisect each other
4.	$\overline{RA} \cong \overline{RA}$	Reflexive
5.	ΔRSA ≅ ΔRWA	SSS
6.	∠1 ≅ ∠2	cpctc
7.	RT ⊥ SW	2 lines form ≅ adj ∠'s

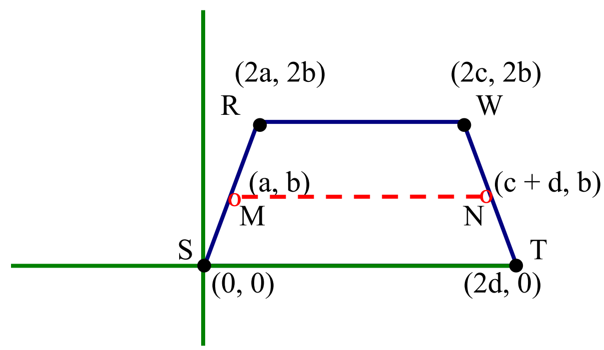
Thm: The median of a trapezoid is || to the bases and is equal to half the sum of the bases.

G: RSTW – trap

P: $\overline{MN} \parallel \overline{ST}$
$\overline{MN} \parallel \overline{RW}$
$MN = \frac{1}{2}(ST + RW)$



Strategy: Use Coordinate Geometry. Place trap on coordinate axes, label pts. carefully keeping relationships. Find slopes, || lines have = slopes. Find distances.



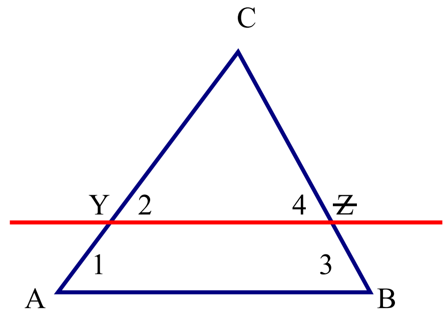
Since MN, RW, and ST have the same slope, the lines are ||

$MN = c + d - a$
$RW = 2c - 2a$
$ST = 2d$

$RW + ST = 2c - 2a + 2d$
$\quad = 2(c - a + d)$
$MN \quad = c + d - a$

$$MN = \frac{1}{2}(STW + RW)$$

Thm: If a line is || to one side of a Δ and intersects the other 2 sides, it divides them proportionally.



G: ΔABC, $\overline{YZ} \parallel \overline{AB}$

P: $\dfrac{AY}{YC} = \dfrac{BZ}{ZC}$

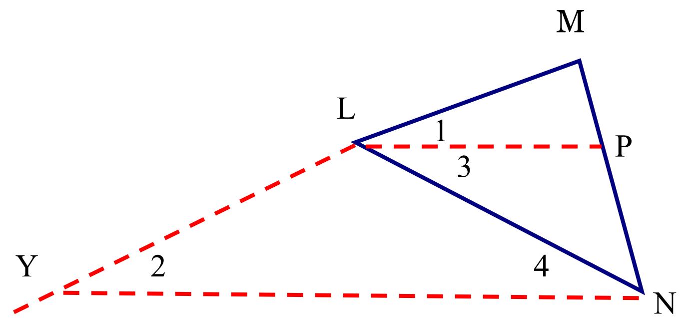
Strategy: Show 2 triangles formed are similar to write proportion, then use the Segment Add Post to make substitutions

Statements	Reason
1. $\overline{YZ} \parallel \overline{AB}$	Given
2. $\angle 1 \cong \angle 2$; $\angle 3 \cong \angle 4$	2 || lines cut by t, c ∠'s ≅
3. $\Delta ACB \sim \Delta YCZ$	Angle Angle Post
4. $\dfrac{AC}{YC} = \dfrac{BC}{ZC}$	~ Δ sides in proportion
5. $\dfrac{AC - YC}{YC} = \dfrac{BC - ZC}{ZC}$	Prop of proportions
6. $AY + YC = AC$ $BZ + ZC = BC$	Segment Add Post
7. $AY = AC - YC$ $BZ = BC - ZC$	Sub Prop =
8. $\dfrac{AY}{YC} = \dfrac{BZ}{ZC}$	Sub. In line 5

Thm: If a ray bisects an $\angle$ of Δ, it divides the opposite side into segments whose lengths are proportional to the lengths of the other 2 sides.

G: Δ LMN
$\overline{LP}$ bisects $\angle$MLN

P: $\frac{MP}{PN} = \frac{LM}{LN}$

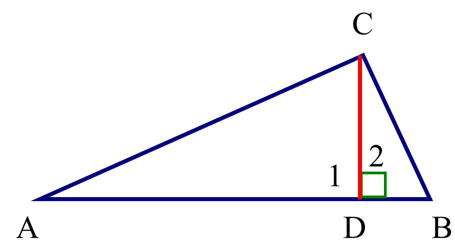


Statements	Reasons
1. Extend ML, YN || LP	Construction
2. $\frac{MP}{PN} = \frac{LM}{LY}$ for ΔMYN	line ||, divides Δ pro
3. $\angle 1 \cong \angle 2$	Corr $\angle$'s
4. $\angle 1 \cong \angle 3$	Def $\angle$ bisector
5. $\angle 3 \cong \angle 4$	Alt int $\angle$'s
6. $\angle 2 \cong \angle 4$	Sub
7. $\overline{LY} \cong \overline{LN}$	Base $\angle$'s $\cong$, sides $\cong$
8. $\frac{MP}{PN} = \frac{LM}{LN}$	Sub

Thm: If the altitude is drawn to the hypotenuse of a right Δ, the 2 Δ's formed are similar to the given Δ and each other.

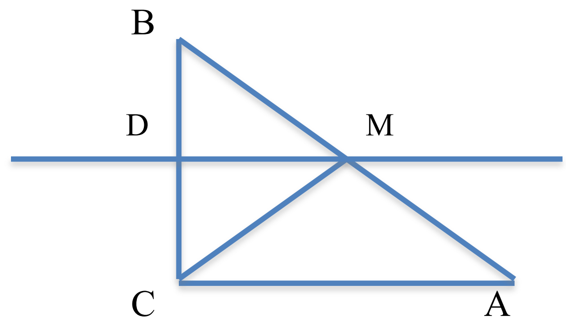
G: Rt Δ ABC, $\overline{CD} \perp \overline{AB}$

P: ΔADC ~ ΔACB
ΔCDB ~ ΔACB
ΔADC ~ ΔCDB



Statements		Reasons
1.	∠ACB rt ∠, $\overline{CD} \perp \overline{AB}$	Give
2.	∠1, ∠2 are rt ∠'s	⊥ lines form rt ∠'s
3.	∠A ≅ ∠A; ΔACB & ΔADC	Reflexive
4.	ΔADC ~ ΔACB	AA Postulate
5.	∠B ≅ ∠B, ΔACB & ΔCDB	Reflexive
6.	ΔCDB ~ ΔACB	AA Postulate
7.	ΔADC ~ ΔCDB	Transitive; 4 & 6

Thm: The length of the median to the hypotenuse of a right triangle is equal to one-half the length of the hypotenuse



Given: ΔABC, C is the right angle

$\overline{CM}$ is the median

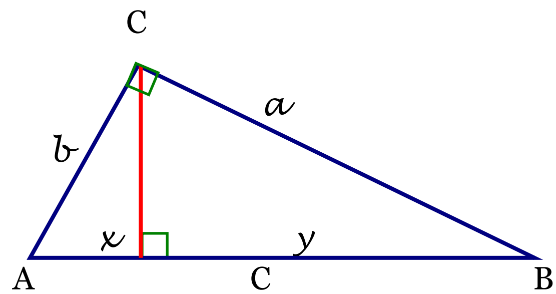
Prove: CM = ½ (AB)

1.	ΔABC is rt Δ	Given
	$\overline{CM}$ is the median	
2.	Construct $\overline{DM}$ thru M	Construction
	parallel to $\overline{AC}$	
3.	$\angle$MDC is rt angle	Corresponding angles
4.	BM = AM	Def of median
5.	BD = DC	|| lines, congruent segments
6.	$\Delta MBD \cong \Delta MCD$	HL Thm
7.	CM $\cong$ MB	cpctc
8.	BM + MA = AB	Segment Add Thm
9.	BM + BM = AB	Sub
10.	2 BM = AB	Dist Prop
11.	2 CM = AB	Sub
12.	CM = ½ (AB)	Div Prop Eq

Thm: In any rt Δ the square of the hypotenuse is equal to the sum of the squares of the legs.

G: Rt ΔACB

P: $c^2 = a^2 + b^2$

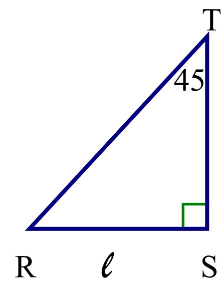


Statements	Reasons
1. Draw $\perp$ from c to AB	Construction
2. $\frac{c}{a} = \frac{a}{y}$; $\frac{c}{b} = \frac{b}{x}$	The length of a leg of a rt Δ is the geo mean….
3. $c = x + y$	Segment Addition Post
4. $cy = a^2$; $cx = b^2$	Prop of Proportion
5. $cy + cx = a^2 + b^2$	APE
6. $c(y + x) = a^2 + b^2$	D – Prop
7. $c \bullet c = a^2 + b^2$	Sub
8. $c^2 = a^2 + b^2$	Exponents

Thm: In a $45 - 45 - 90^o$ Δ, the hypotenuse is $\sqrt{2}$ times the leg.

G: Rt ΔRST
$\angle R = \angle T = 45^o$

P: $RT = \ell\sqrt{2}$

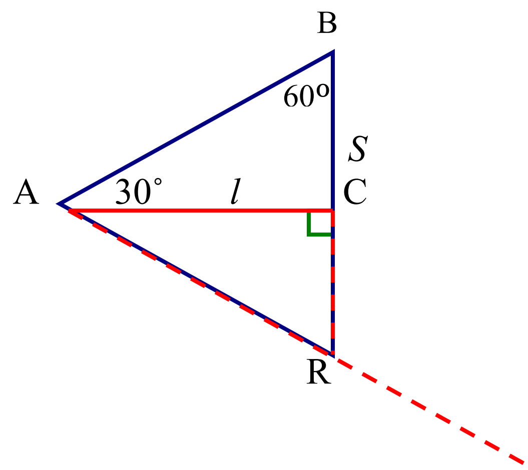


Statements	Reasons
1. Rt ΔRST $\angle R = \angle T$	Give
2. RS = TS	legs opp. base $\angle$'s isos. Δ
3. $(RT)^2 = \ell^2 + \ell^2$	Pythagorean Thm
4. $(RT)^2 = 2\ell^2$	D – Prop
5. $RT = \sqrt{2}\ \ell$	Exponents

Thm: In a 30 – 60 – 90 Δ, the hypotenuses is twice the shorter leg and the longer leg is $\sqrt{3}$ times the shorter leg.

G: Rt ΔACB
$\angle B = 60^\circ$
$\angle BAC = 30$
$BC = S$
$AC = l$

P: $AB = 2S$
$l = S\sqrt{3}$

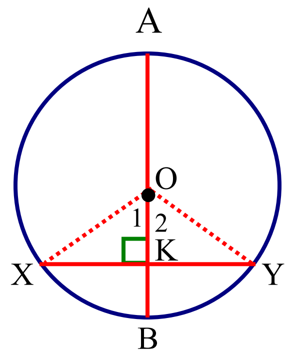


Statements	Reasons
1. Draw $\overrightarrow{BC}$, & $\overrightarrow{AX}$ so that $\angle BAX = 60$	Construction
2. ΔBAR is equilateral	equiangular Δ's are equilateral
3. AB = BR	Def equilateral Δ
4. BR = 2S = AB	Altitude of Δ bisects opp side
5. $s^2 + l^2 = (AB)^2$	Pyth Thm
6. $s^2 + l^2 = (2S)^2$	Sub
7. $s^2 + l^2 = 4S^2$	Exp
8. $l^2 = 3s^2$	SPE
9. $l = \sqrt{3}\ s$	

Thm: A diameter that is $\perp$ to a chord bisects the chord and its 2 arcs

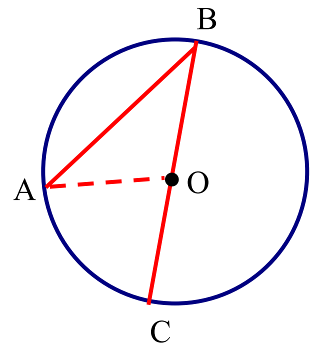
G: $\odot$ O
$\overline{AB} \perp \overline{XY}$

P: $\overline{XK} \cong \overline{YK}$
$\overset{\frown}{XB} \cong \overset{\frown}{YB}$



Statements	Reasons
1. Draw $\overline{OX}$ and $\overline{OY}$	Construction
2. $\overline{AB} \perp \overline{XY}$	Give
3. $\overline{OK} \cong \overline{OK}$	Reflexive
4. $\overline{OY} \cong \overline{OX}$	Radii of $\odot$ are $\cong$
5. Δ OKX $\cong$ ΔOKY	HL
6. $\overline{XK} \cong \overline{YK}$ $\angle 1 \cong \angle 2$	cpctc
7. $\overset{\frown}{XB} \cong \overset{\frown}{YB}$	2 central $\angle$'s are $\cong$, the arcs are $\cong$

Thm: The measure of an inscribed $\angle$ is half the measure of the intercepted arc.



Give: $\angle ABC$ inscribed

Prove: $\angle ABC = \frac{1}{2}\,\overset{\frown}{AC}$

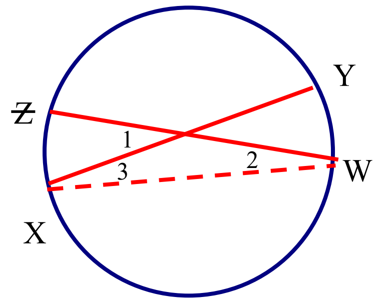
Strategy: Go back to triangle theorems

Statements	Reasons
1. Draw OA	Construction
2. $\overline{OB} \cong \overline{OA}$	Radii
3. $\angle A \cong \angle B$	2 sides of Δ are $\cong$, $\angle$'s opposite are $\cong$
4. $\angle AOC = \angle A + \angle B$	Ext $\angle$ = 2 remote int $\angle$'s
5. $\angle AOC = \angle B + \angle B$	Sub
6. $\angle AOC = 2\angle B$	D-Prop
7. $\frac{1}{2}\angle AOC = \angle B$	Div Prop Eq
8. $\angle AOC \cong \overset{\frown}{AC}$	Def Central $\angle$, arc
9. $\frac{1}{2}\overset{\frown}{AC} = \angle B$	Sub

Thm: When 2 secants intersect in a circle, the $\angle$ formed is $=$ to $\frac{1}{2}$ the sum of the arcs formed by the vertical $\angle$.

G: XY & ZW intersect

P: $\angle 1 = \frac{1}{2}(\overset{\frown}{XZ} + \overset{\frown}{YW})$



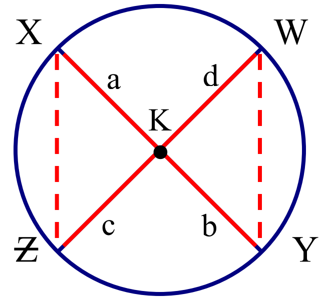
Strategy: Construct triangles and use their relationships with the theorems on central angles and inscribed angles

	Statements	Reasons
1.	Draw $\overline{XW}$	Construction
2.	$\angle 1 = \angle 2 + \angle 3$	Ext $\angle$ of Δ = 2 remote int $\angle$'s
3.	$\angle 2 = \frac{1}{2}\overset{\frown}{XZ}$ $\angle 3 = \frac{1}{2}\overset{\frown}{YW}$	Inscribed $\angle = \frac{1}{2}$ intercepted arc
4.	$\angle 1 = \frac{1}{2}\overset{\frown}{XZ} + \frac{1}{2}\overset{\frown}{YW}$	Sub
5.	$\angle 1 = \frac{1}{2}(\overset{\frown}{XZ} + \overset{\frown}{YW})$	D-Prop

Thm: When 2 chords intersect, the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other.

G: Chords $\overline{XY}$ & $\overline{ZW}$

P: $a \bullet b = c \bullet d$

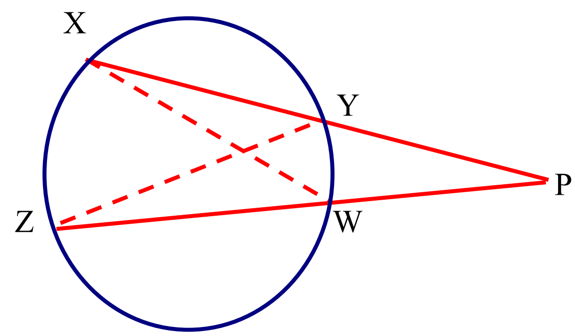


Statements	Reasons
1. Draw $\overline{XZ}$ and $\overline{WY}$	Construction
2. $\angle X \cong \angle W$ $\angle Z \cong \angle Y$	Inscribed $\angle$'s intercept same arc
3. $\Delta XKZ \sim \Delta WKY$	AA Postulate
4. $\frac{a}{d} = \frac{c}{b}$	$\sim \Delta$'s proportic
5. $ab = cd$	Prop of Proportion

Thm: If 2 secants are drawn to a circle from an exterior pt, the product of the lengths of one secant segment and its external segment is equal to the product of the other secant and its external segment.

Given: secants $\overline{PX}$ and $\overline{PY}$

Prove: $PX \cdot PY = PZ \cdot PW$



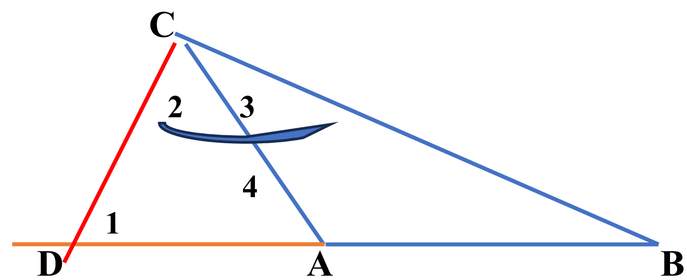
	Statements	Reasons
1.	Draw $\overline{XW}$ and $\overline{ZY}$	Construction
2.	$\angle X \cong \angle Z$	Inscribed $\angle$, same arcs
3.	$\angle P \cong \angle P$	Reflexive
4.	$\Delta\, XPW \sim \Delta\, ZPY$	AA Postulate
5.	$\dfrac{PX}{PZ} = \dfrac{PW}{PY}$	$\sim \Delta$'s, sides in proportion
6.	$PX \cdot PY = PZ \cdot PW$	Prop of Proportion

Triangle Inequality

Thm. The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Given: ΔABC

Prove: AC + AB > CB



Statements	Reasons
1. On BA take D so DA + AC	On a ray, exactly on point =
2. $m\angle 1 = m\angle 2$	Isosceles Δ
3. $m\angle 4 = m\angle 2 + \angle 3$	Angle Add Postulate
4. $m\angle 4 > m\angle 2$	If $a = b + c$, $c>0$, $a > b$
5. $m\angle 4 > m\angle 1$	Substitution
6. DB > CB	One ∠ greater then another ∠
7. DB = DA + AB	Def of Betweeness
8. DA + AB > CB	Substitution, from 6 & 7
9. AC + AB > CB	Substitution, from 1 & 8