RATIO & PROPORTION

Sec 1. Defining Ratio & Proportion

A **RATIO** is a comparison between two quantities.

We use ratios every day and never think of it as math; one Pepsi costs 50 cents describes a ratio. On a map, the legend might tell us one inch is equivalent to 50 miles or we might notice one hand has five fingers. Those are all examples of comparisons – ratios.

3 Ways to Write Ratios

A ratio can be written three different ways. If we wanted to show the comparison of one inch representing 50 miles on a map, we could write that as; *inches to miles*

using "to"	1 to 50	or
using a colon	1 : 50	or
using a fraction bar	$\frac{1}{50}$	

No matter which way you write it, the way you say it is *1 to 50.*

Because we are going to learn to solve problems, it's easier to solve problems using the fraction bar. If we looked at the ratio of one inch representing 50 miles on a map, $\frac{1}{50}$, we might determine 2 inches represents 100 miles, 3 inches represents 150 miles by using equivalent ratios.

That just seems to make sense. Look at that from a mathematical standpoint, it appears that we might also be able to simplify ratios $\frac{3}{150}$ to $\frac{1}{50}$.

Does $\frac{3}{150}$ represent the same comparison as $\frac{1}{50}$?

The answer is yes – and if we looked at other ratios, we would see that reducing/simplifying ratios does not affect those comparisons. Those are equivalent ratios. As an example, travelling 300 miles in 5 hours could be described as travelling 60 miles per hour.

Now, we have some good news. We not only discovered how to write ratios, we also learned they can be simplified. And the really good news is we simplify them exactly the same way we simplified fractions.

Now back to the example using a map. We noticed that $\frac{3}{150}$, 3 inches represents 150 miles, could be reduced to $\frac{1}{50}$ meaning 1 inch represents 50 miles.

Mathematically, by setting the ratios equal, we could write

$$\frac{1}{50} = \frac{3}{150}$$

Simplifying Ratios

You simplify ratios the same way you simplify fractions. That is, you find the GCF and simplify.

Example 1 if 7 bars of candy cost \$1.40, we write that using fractional notation and simplify. *Candy bars to cost.*

$$\frac{7 \text{ bars}}{\$1.40} = \frac{1 \text{ bar}}{\$0.20}; \text{ common factor of 7}$$

Example 2 If seven inches on a map represented 175 miles, what would one inch represent?

$$\frac{inches}{miles}; \quad \frac{7}{175} = \frac{1}{25},$$

so inch represents 25 miles on a map.

To write ratios, it is very important that the ratio describes what is being asked. So, if I say the ratio of boys to girls is 10 to 9 in a sentence, the ratio should be written as boys to girls and you should write the words; $\frac{boys}{girls}$, so you know how to write the ratio $\frac{10}{9}$.

If you were told there were 7 adult tickets for every 12 tickets sold, the ratio in words would be $\frac{\text{adult tickets}}{\text{tickets sold}} = \frac{7}{12}$.

I can't make writing ratios difficult. You write the word description, then using that, write the numbers in that ratio.

Example 2

If you were given the following pictographs; ☆☆☆♡♡♡♡♡◇◇◇◇, I could ask for a number of different ratios; stars to hearts, stars to diamonds, hearts to diamonds, diamonds to hearts, stars to total number of figures. To do those, write the words to the ratios before filling in the numbers in the ratio.

Notice or count, there are 3 stars, there are 5 hearts, and 4 diamonds. That's important!

So, the ratio of stars to hearts; $\frac{\text{stars}}{\text{hearts}} = \frac{3}{5}$

The ratio of stars to diamonds; $\frac{\text{stars}}{\text{diamonds}} = \frac{3}{4}$

The ratio of hearts to diamonds; $\frac{\text{hearts}}{\text{diamonds}} = \frac{5}{4}$

Diamonds to hearts; $\frac{\text{diamonds}}{\text{hearts}} = \frac{4}{5}$

Notice the hearts to diamonds ratio is **different** from the diamonds to hearts. It is important that you write the words in a ratio before you write the ratio.

The ratio of stars to total figures; $\frac{\text{stars}}{\text{total figures}} = \frac{3}{12}$ which can be simplified to $\frac{1}{4}$.

Write a ratio and simplify the following ratios:

1. 12 boys for every 18 girls
2. 24 for every 6 dollars
3. 72 hours for every 3 days
4. 18 drinks cost $27

5. 7 inches is 175 miles

6. Express each ratio as a fraction.

a. 3 to 4 b. 8 to 5 c. 9:13 d. 15:7

7. Express each ratio in simplest form.

a. 12 to 10 b. 24:36 c. $\frac{15}{18}$

8. A certain math test has 50 questions. The first 10 are true – false and the rest are matching. Find:

a. The ratio of true-false questions to matching questions.

b. The ratio of true-false questions to the total number of questions.

c. The ratio of the total number of questions to matching questions.

But before we move on, let me make a point. While ratios can be written like fractions and simplified like fractions, a ratio is NOT a fraction! A fraction is part of a unit. A ratio represents a comparison.

For instance;

$$\frac{1}{3}+\frac{1}{2}=\frac{5}{6} \text{ as a fraction}$$

$$\frac{1}{3}+\frac{1}{2}=\frac{2}{5} \text{ as a ratio}$$

What??? On the first example we had fractions, we found the common denominator, made equal fractions, added the numerators and brought down the denominator because you were determining what part of a unit we had.

In the second example, let's say you got 1 hit out of 3 times at bat last week and 1 hit out of 2 times at bat this week. That results in 2 hits out of 5 times at bat. What we want clear here is a ratio is a comparison between hits and times at bat. We cannot get one-third a hit. We either get a hit or we don't. Clearly using the fraction bar; "–" , for both fractions and ratios, the notation, can be misleading to students first learning about ratios.

Now that we cleared that up, let's get back to simplifying these ratios.

If we know that we can simplify ratios such as: $\frac{3}{12} = \frac{1}{4}$, that does mimics what we can do with simplifying fractions.

Another way to look at the equivalence is to recognize it as an equation. Since we notice the equality, that leads us to a new definition:

Proportion

A ***PROPORTION*** is a statement of equality between 2 ratios.

Looking at a proportion like $\frac{1}{2} = \frac{3}{6}$, we might see some other relationships that exist if we take time and manipulate the numbers.

For instance, what would happen if we tipped both ratios ***up-side down*** in the proportion $\frac{1}{2} = \frac{3}{6}$? In other words, if the ratio given was 3 boys for every 6 girls which resulted in a simplified ratio 1 to 2, could we say that 6 girls for every 3 boys would simplify to 2 girls for every 1 boy and that proportion stull be true?

$\frac{2}{1}$ and $\frac{6}{3}$, notice they are also equal, so $\frac{2}{1} = \frac{6}{3}$

Noticing that is true is interesting, because it leads us to something we might do in our daily lives, like checking prices. In math, we call it unit ratios or unit rates.

Sec. 2 Unit Ratios – Unit Rates

Unit ratios are ratios that are expressed as a quantity of one, such as 2 feet per second, 7 miles per hour, $5.00 per case; where the number written on the bottom describes ONE unit, hence the word unit. Unit ratios are often expressed using the word "per".

So, from **Example 1**, if 7 bars of candy cost $1.40, we wrote that using fractional notation and simplified.

$$\frac{bars}{\$};\ \frac{7}{1.40} = \frac{1}{.20}$$ one bar for 20 cents.

And that was great. But we just discovered that we can tip ratios upside down and maintain equalities.

That is

$$\frac{\$}{bars};\ \frac{1.40}{7} = \frac{.20}{1}$$

is the same as the first proportion we wrote. And, that allows us to write the ratio as a **unit ratio**, the denominator being ONE. The cost is $0.20 per candy bar.

Note, that when working with unit ratios, dollars is typically written in the numerator.

Not all ratios can be reduced to a unit ratio, so there are times we might use an alternative method for finding unit ratios.

2 Ways to Determine Unit Ratios

1. **Equivalent fractions**
2. **Dividing**

Example 3. 24 cases of pens sell for $16, find the unit price.

Writing that as a proportion, I want to know the cost of one case, so I am going to write the ratio with cases in the denominator. $16 for 24 cases.

Setting up the ratio and proportion, we have $\frac{\$}{cases}$; $\frac{16}{24} = \frac{2}{3}$

As we can see, that does not simplify to having one in the denominator using equivalent fractions. The best answer we get by simplifying the ratio is $2 for every 3 cases. That is not a unit ratio.

Another way, and in this case, a better way to handle unit rates is to simply divide the numerator by the denominator. In other words, once we decide how to write the ratio to get the unit we are looking for in the denominator – just divide.

So, in that problem, the unit ratio would be 16/24 or 2/3 simplified, then divide.

$$24\overline{)16} \text{ or } 3\overline{)2}$$

Either way, the answer is $0.6666 or $0.67 per case – a unit ratio

Usually, we write the ratio the way it was given to us, and simplify it. To find unit pricing, the best approach is to determine how you want to write the ratio first by asking yourself how you want to compare the quantities so you will have the unit written in the denominator.

Find the following unit rates – round your answers to the nearest hundredths.

1. $14 for 4 books.
2. 13 chocolate bars for $18.
3. 95 miles on 9 gallons of gas.
4. 7 movie tickets cost $40.
5. 5 calculators cost $140.
6. Mowed 5 lawns for $35.

Sec 3. Properties of Proportion

Using a little math, we can manipulate proportions.

Upside Down

Now, let's go back and look at these proportions. We found that we could write the $\frac{1}{2} = \frac{3}{6}$ proportions upside down and maintain the equality.

$$\frac{1}{2} = \frac{3}{6} \quad or \quad \frac{2}{1} = \frac{6}{3}$$

Sideways

How about writing the original proportion ***sideways***, will we maintain the equality?

$$If \ \frac{1}{2} = \frac{3}{6}, will \ \frac{1}{3} = \frac{2}{6} \text{ still work?}$$

Yes, we can see the are still equal.

Cross Multiply

If we continued looking at the original proportion $\frac{1}{2} = \frac{3}{6}$, we might also notice we **could cross multiply** and retain an equality.

In other words, 1x6 = 2x3. This idea of manipulating numbers is pretty interesting stuff, don't you think?

Makes you wonder whether tipping ratios up-side down, writing them sideways or cross multiplying only works for our original proportion?

Well, to make that determination, we would have to play with some more proportions. Try some, if our observation holds up, we'll be able to generalize what we saw.

Let's try these observations with the proportion $\frac{2}{3} = \frac{4}{6}$ Can I tip them **upside down** and still retain an equality? In other words, does $\frac{3}{2} = \frac{6}{4}$?

How about writing them **sideways**, does $\frac{2}{4} = \frac{3}{6}$

How about **cross-multiplying** in the original proportion, does 2x6 = 3x4

The answer to all three questions is yes.

Since everything seems to be working, we will generalize our observations using letters instead of numbers.

If $\frac{a}{b} = \frac{c}{d}$, then

$\frac{b}{a} = \frac{d}{c}$ $\frac{a}{c} = \frac{b}{d}$ **and** $ad = bc$

Those 3 observations are referred to as ***Properties of Proportions***.

Those properties can and will be used to help us solve problems. While we made these properties by observation, we could also have shown them to be true by using the Properties of Real Numbers. In other words, if the original proportion is true, $\frac{a}{b} = \frac{c}{d}$ is true, I could multiply both sides of the equation by the common denominator, ***bd,*** that would have resulted in property 3 above (cross-multiplying). Then, I could have further manipulated to get the relationships in properties 1 and 2. In other words, we could prove these relationships exist for all proportions.

So, let's start with a simple proof.

1. Theorem 1. If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$

1. $bd\frac{a}{b} = bd\frac{c}{d}$	Mult both sides by CD
2. $da = bc$	Mult Inverse
3. $ad = bc$	Comm. Prop.

2. Theorem 2. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$

1. $ad = bc$	Theorem 1
2. $\frac{ad}{ac} = \frac{bc}{ac}$	Div Prop of Equality
3. $\frac{d}{c} = \frac{b}{a}$	Mult Inverse
4. $\frac{b}{a} = \frac{d}{c}$	Reflexive Prop

** Notice, since I needed an *a* and a *c* in the denominator, I divided both sides by *ac.*

Sec. 4 Ways to Solve a Proportion

2 Ways of Solving Proportions

1. **Equivalent Fractions**
2. **Cross Multiplying**

Since we have seen that ratios can be simplified like fractions, that will help us solve proportions very quickly.

If the numbers work, the easiest way to solve for a variable in a proportion is by using what we learned in fractions. That is, make equivalent fractions.

Example 4 Find the value of x

$$\frac{2}{5} = \frac{x}{30}$$

Using equivalent fractions, multiply both numerator and denominator by 6. That results in x = 12

Example 5 Find the value of y . $\frac{7}{10} = \frac{56}{y}$

Using equivalent fractions, multiply the numerator and denominator by 8. y = 80

If the numbers are not convenient, the other method to use is cross multiplying.

Let's redo **Example 4** by cross multiplying. Find the value of x.

$\frac{2}{5} = \frac{x}{30}$, that results in 5x = 60 or x = 12 just like before.

This problem could have been done by either equivalent fractions or by cross multiplying.

Some problems, equivalent fractions just don't work well. So, when that occurs, use cross multiplying.

Example 6 Find the value of n.

$$\frac{3}{5} = \frac{n}{22}$$

We notice immediately that 5 does not go into 22, so using equivalent fractions is not the way to solve the problem. That leaves us with cross multiplying.

5n = 3(22)
5n = 66
$n = \frac{66}{5}$ or $13\frac{1}{5}$

In many instances, it might appear at first glance that using equivalent fractions might not work, so we go immediately to cross multiplying. That's okay, but … . It might be advantageous to simplify the ratio first for two reasons; 1) you then might be able to solve the proportion by equivalent fractions, and 2) you will end up the smaller numbers if you decide to cross multiply.

In exercises 1-8, find the value of the variable that makes the proportion true.

1. $\frac{10}{18} = \frac{x}{45}$

2. $\frac{5}{8} = \frac{25}{y}$

3. $\frac{120}{200} = \frac{36}{w}$

4. $\frac{1}{8} = \frac{t}{6}$

5. $\frac{x}{10} = \frac{200}{30}$

6. $\frac{500}{25} = \frac{100}{y}$

7. $\frac{7}{5} = \frac{P}{100}$

8. $\frac{A}{75} = \frac{15}{100}$

Note: If you simplified the ratios, two things could have happened. 1) you may have been able to do the problem by equivalent fractions and 2) the numbers would have been smaller which might make the arithmetic much easier.

Redo number 1 in the problem set by simplifying the ratio first.

Sec 5. Solving Problems Using Proportions

To solve problems, most people use either equivalent fractions or cross multiplying to solve proportions. Either way, start by writing the ratio in words, then fill in the numbers using the ratio to solve the proportions

Example If a turtle travels 5 inches every 10 seconds, how far will it travel in 50 seconds?

What we are going to do is set up a proportion. The way we'll do this is to identify the comparison we are making. In this case we are saying 5 inches every 10 seconds. Therefore, and this is very important, we are going to set up our proportion by saying inches is to seconds $\frac{inches}{seconds}$.

On one side, we have $\frac{5}{10}$ describing inches to seconds. On the other side we have to again use the same comparison, inches to seconds. We don't know the inches, so we'll call it "n". Where will the 50 go in the ratio, top or bottom? Bottom, because it describes seconds – good deal. So now we have.

$$\frac{5}{10} = \frac{n}{50}$$

Now, we can find n by equivalent fractions or we could use property 3 and cross multiply.

$$\frac{5}{10} = \frac{n}{50}$$

$10n = 5 \text{ x } 50$

$10n = 250$

$n = 25$ The turtle will travel 25 inches in 50 seconds

It makes it easier to understand initially to write the same comparisons on both sides of the equal signs. In other words, if we had a ratio on one side comparing inches to seconds, then we write inches to seconds on the other side.

If we compared the number of boys to girls on one side, we would have to write the same comparison on the other side, boys to girls. We could also write it as girls to boys on one side as long as we wrote girls to boys on the other side. The first Property of Proportion, tipping the ratios upside down, permits this to happen.

Solve these problems by setting up a proportion.

1. If there were 7 males for every 12 females at the dance, how many females were there if there were 21 males at the dance?
   Ask yourself; is there a ratio, a comparison in that problem? What's being compared?
2. David read 40 pages of a book in 5 minutes. How many pages will he read in 80 minutes if he reads at a constant rate?
3. On a map, one inch represents 150 miles. If Las Vegas and Reno are five inches apart on the map, what is the actual distance between them?
4. Bob had 21 problems correct on a math test that had a total of 25 questions, what percent grade did he earn? (In other words, how many questions would we expect him to get correct if there were 100 questions on the test?)
5. If there should be three calculators for every 4 students in an elementary school, how many calculators should be in a classroom that has 44 students? If a new school is scheduled to open with 600 students, how many calculators should be ordered?
6. If your car can go 350 miles on 20 gallons of gas, at that rate, how much gas would you have to purchase to take a cross country trip that was 3000 miles long?

All the proportion problems up to this point provided a ratio, then more information was given in terms of those ratios. That doesn't always happen.

Sec 6. Information Not Given in Terms of Ratio

The ratio and proportions problems we have done up to this point have expressed a ratio, then given you more information in terms of the ratio previously expressed.

In other words, if the ratio expressed was male to female, then more information was given to you in terms of males or females and you set up the proportion. Piece of cake, right?

Well, what happens if you were given a ratio, like males to females, but then the additional information you received was not given in the terms of the original ratio, males or females? Maybe the additional information told you how many students were in a class altogether?

You wouldn't be able to set up a proportion based on what we know now.

But, don't you love it when somebody says but? It normally means more is to come. So, hold on to your chair to contain the excitement, we get to learn more by seeing patterns develop.

Up to this point, we know if 2 ratios are equal, then I can tip them upside down, write them sideways and cross multiply and we will continue to have an equality.

If $\frac{a}{b} = \frac{c}{d}$, **then**

1. $\frac{b}{a} = \frac{d}{c}$ ***2.*** $\frac{a}{c} = \frac{b}{d}$ ***and*** ***3.*** $ad = bc$

Remember, the way we developed those properties was by looking at equal fractions and looking for patterns. If we continued to look at equal ratios, we might come up with even more patterns.

Let's look, we know $\frac{1}{3} = \frac{2}{6}$. Now if we played with these proportion, then looked to see if the same things we noticed other equalities, then we might be able to make some generalizations.

For instance, if I were to keep the numerator 1 on the left side, and add the numerator and denominator together to make a new denominator on the left side, would I still have an equality if I did the same thing to the right side. Let's peek.

We have $\frac{1}{3} = \frac{2}{6}$, keeping the same numerators, then adding the numerator and denominator together for a new denominator, we get

$\frac{1}{1+3}$ and $\frac{2}{2+6}$. Are they equal? Does $\frac{1}{4} = \frac{2}{8}$?

Oh boy, the answer is yes. Don't you wonder who stays up at night to play with patterns like this? If we looked at other equal ratios, we would find this seems to be true. So, what we do is generalize this.

To show this works, we generalize using variables, by adding **one** to both sides of the original proportion in the form of b/b and d/d.

$$If\ \frac{a}{b} = \frac{c}{d}, then\ \frac{a+b}{b} = \frac{c+d}{d}$$

$\frac{a}{b} = \frac{c}{d}$	*Given*
$\frac{a}{b} + \frac{b}{b} = \frac{c}{d} + \frac{d}{d}$	*Addition Property of Equality*
$\frac{\boldsymbol{a+b}}{\boldsymbol{b}} = \frac{\boldsymbol{c+d}}{\boldsymbol{d}}$	*Simplifying Fractions*

Other patterns we might see by looking at the fraction $\frac{1}{3} = \frac{2}{6}$ include adding the numerators and writing those over the sum of the denominators that would be equal to either of the original fractions.

In other words, if $\frac{1}{3} = \frac{2}{6}, does\ \frac{1+2}{3+6} = \frac{1}{3}\ and\ \frac{2}{6}$

And of course, since this also seems to work with a number of different equal fractions, we again make a generalization.

$$\textbf{If}\ \frac{a}{b} = \frac{c}{d},\ \text{then}\ \frac{\text{a}+\text{c}}{\text{b}+\text{d}} = \frac{a}{b} = \frac{c}{d}$$

So now we have discovered two more patterns for a total of 5 Properties of Proportion.

Now, going back to the problem we were describing earlier that gave a ratio, then gave additional information that was not in terms of the ratio, we'll be able to manipulate the properties of proportion to solve additional problems. Say yes to math, this is really great stuff.

Example If there are 3 boys for every 7 girls at school, how many boys attend the school if the total student enrollment consists of 440 students?

The first comparison given is boys to girls. Knowing this, we would like to set up a proportion that looks like this:

$\frac{boys}{girls} = \frac{boys}{girls}$ just as we have done before.

The problem that we encounter is while we can put the 3 and 7 on the left side to represent boys and girls, we have to ask ourselves, where does 440 go? It does not represent just boys or just girls, so we can't put it in either position on the right side. 440 represents the total number of boys and girls.

Remembering what we just did, see there was a reason for looking for more patterns, we noticed if we have a proportion like

If $\frac{b}{g} = \frac{b}{g}$, then $\frac{b}{b+g} = \frac{b}{b+g}$ would be true.

From our problem, we now can see $b + g$ would represent the total of the boys and girls.

This leads us to the ratio of boys to total; $\frac{boys}{total}$

$\frac{b}{g} = \frac{b}{g}$, then $\frac{b}{b+g} = \frac{b}{b+g}$ $\qquad \frac{3}{3+7} = \frac{b}{440} \quad \rightarrow \quad \frac{3}{10} = \frac{b}{440}$

$$\begin{aligned} \text{Solving; } 10b &= 3 \text{ x } 440 \\ 10b &= 1320 \\ b &= 132 \end{aligned}$$

There would be 132 boys; to find the number of girls we could subtract 132 from 440.

The point being, if you were given a problem being described by a ratio, then additional information was given to you **not** using the descriptors in the original ratio, you could manipulate the information using one of the properties of proportion.

Sec 7. Using Equations to Solve Proportions

To be quite frank, that is not the way I would usually attack that sort of problem. What I would prefer doing is setting up the problem algebraically.

Let's step back and see how ratios work. Remember, we said you could reduce ratios. In other words, if I had the ratio of $\frac{3}{150}$, I could reduce it to $\frac{1}{50}$.

Visually, the way you reduce is by dividing out a common factor. To reduce $\frac{3}{150}$, we could rewrite it as $\frac{1 \times 3}{50 \times 3}$. Dividing out the 3's, we would have $\frac{1}{50}$.

Now going back to the previous example, we had 3 boys for every 7 girls, with a total enrollment of 440. Doing this algebraically, we still have the same ratio, boys to girls. But, again, we realize the additional information is not given in terms of boys or girls. We just did this problem by playing with the properties of proportion.

Using algebra, the ratio of boys to girls, $\frac{b}{g}$ is $\frac{3}{7}$.

Does that mean we have exactly three boys and seven girls? No, that ratio comes from reducing the actual number in the boys to girls ratio. Since we don't know the common factor we would have divided out, we'll call it X. Unbelievable concept.

Using the ratio; $\frac{b}{g}$ or $\frac{3}{7}$, we now know we have 3X boys and 7X girls. So, the ratio of $\frac{b}{g}$ looks like this; $\frac{3X}{7X}$.

But notice the sum of 3X and 7X would be the total number of students
The total number of students is 440, therefore we have

$$\begin{aligned} \text{boys} + \text{girls} &= 440 \\ 3X + 7X &= 440 \\ 10X &= 440 \\ X &= 44 \end{aligned}$$

The ratio of boys to girls $\frac{3X}{7X}$, that means there are $3X$ boys or 3 (44) which is 132 boys.

That manipulation allowed us to solve a proportion problem where more information was not given in terms of the first ratio.

Example If the ratios of the sides of a triangle are 4:5:6 and the perimeter is 75 inches, how long is each side?

Immediately notice we are given ratios in terms of sides and more information is given in terms of perimeter. To me, that suggests doing this problem algebraically.

The sides therefore are 4x, 5x, and 6x. The perimeter is the sum of the sides of a polygon, in this case 75 inches.

So,

$$\begin{aligned} 4x + 5x + 6x &= 75 \\ 4x + 5x + 6x &= 75 \\ 15x &= 75 \\ x &= 5 \end{aligned}$$

The sides were 4x, 5x and 6x, substituting, so the sides are 20, 25, and 30 inches.

From my standpoint, there are two types of **<u>Ratio & Proportion</u>** problems.

Type 1 A ratio is given, and then more information is given <u>in</u> terms of the descriptors of the first ratio. For this type of problem, you set the two ratios equal and solve by equivalent fractions or cross multiplying

Type II A ratio is given, and then more information is given to you that is <u>not</u> using the descriptors in the original ratio. This type of problem can be done algebraically as we did in the last example or we can manipulate the ratios to fit the problem.

So, like all of math, we have choices, decisions to make. On how to attack ratio and proportion problems. We can use the 5 properties identified below or we can try using algebra.

The 5 Properties of Proportion

If $\frac{a}{b} = \frac{c}{d}$, then the following 5 properties are true

1. $\frac{b}{a} = \frac{d}{c}$ **(upside down)**

2. $\frac{a}{c} = \frac{b}{d}$ **(sideways)**

3. $ad = bc$ **(cross multiply)**

4. $\frac{a+b}{b} = \frac{c+d}{d}$ **(add numerator and denominator over denominator)**

5. $\frac{a+c}{b+d} = \frac{a}{b} = \frac{c}{d}$ **(add numerators, add denominators)**

As you try some of these problems, first determine if they are Type I or Type II, then use the appropriate problem solving strategy.

1. There are 5 boys for every 3 girls in Biology, Out of the 56 students in the class, how many are girls?

2. Tom drove 320 miles in 5 hours. At this rate, how long would it take him to travel 600 miles?

3. If a 15 lb. ham is enough to serve 20 people, how many lbs. of ham would be needed to serve 50 people?

4. A 30 pound moonling weighs 180 pounds on the earth. How much does a 300 pound Earthling weigh on the moon?

5. The ratio of the sides of a certain triangle is 2:7:8. If the longest side of the triangle is 40 cm, how long are the other two sides?

6. If a quarterback completes 20 out of 45 passes in his first game, how many passes do you expect him to complete in his second game if he only throws 18 passes?

7. On a trip across the country Joe used 20 gallons of gas to go 300 miles. At this rate, how much gas must he use to go 3500 miles?

8. On a map 3 inches represents 10 miles. How many miles do 16 inches represent?

9. If our class is representative of the university and there are 2 males for every 12 females. How many men attend the university if the female population totals 15,000?

10. The ratio of length to width of a rectangle is 8:3, find the dimensions of the rectangle if the perimeter is 88 in.

Sec 8. Direct Variation; y = kx

Earlier, we defined a ratio as a comparison between two numbers. That is, there are three boys for every 4 girls, resulting in a ratio of boys to girls is 3:4. Someone else may have described that relationship as girls to boys or 4:3. Whichever way we set up the proportion, the value of the ratio was equal – constant.

Oftentimes in using ratios and proportions, it's important to note how one thing effects another. In the problems we have solved up to this point, we didn't typically have a cause and an effect relationship. In other words, whether we wrote our ratio as boys to girls or girls to boys, it didn't matter as long as we described the ratio the same way on both sides.

There are some problems in which there are cause and effects, that the value of one is dependent on the value of something else – like pay is to hours worked. The value of pay is dependent on the number of hours worked. That ratio would tell us how much a person makes an hour. It's a constant. We will call it the ***Constant of Variation***.

In algebra, we want that better defined so proportions can be seen as a function. So, we will say ***y varies directly as x***. OR ***y is directly proportional to x.*** That translates mathematically to: $\frac{y}{x} = k$, where k is the constant of variation OR the constant of proportionality

So x is the independent variable and y is the dependent variable and can also be written as ***y = kx***

From that rule, it is easy to see in a direct variation problem, as x increases, y increases or as x gets smaller, y gets smaller. That's very easily seen if we graph the equation.

Now the good news, we have a choice, we can solve these problems as proportions, like we have been doing or we can solve them algebraically.

If we solved them as proportions, we'd have the following proportion:

$$\frac{y_1}{x_1} = \frac{y_2}{x_2}$$

In the problem below, we previously solved that by setting the ratios equal, then using equivalent ratios or cross multiplying to solve for the amount of gas. In that problem, we knew the ratios were equal, a constant of variation existed, but we didn't care what it was. Other than setting the ratios equal, we didn't care and really did not need to know what that constant was. Let's do that same problem using direct variation - algebra; **y = kx**.

Why is that important? In previous problems, like the boy-girl problem, we could solve problems in terms of the ratio of boys to girls or girls to boys. The constant of variation, would be different depending upon how you chose to set up the ratio. In other words, it is not clear which would be the independent vs dependent variable and importantly, we don't care.

We have already solved the following problem using ratio and proportion, let's see how we can solve that same problem using algebra.

Let's look at a simple example:

Example 1 Given that ***m* varies directly as *t*** and the value of m is 4 when the value of t is 8, find the constant of variation and the value of m when $t = 30$

Using the word translation, $m = kt$
Substituting, $4 = k(8)$
Solving, $\frac{1}{2} = k$, the constant of variation

Using $k = \frac{1}{2}$ in the equation, $m = \frac{1}{2} t$
Substitute $t = 30$ $m = \frac{1}{2}(30)$
Solving, $m = 15$

To solve using algebra, we first must find the value of k.

Example 2 Your car can travel 350 miles on 20 gallons of gas. Using **y = kx**, the miles (y) your travel depends on the amount of gas (x) you have in the car. So, how many miles per gallon does your car get – that would be the constant of variation.

We could find that 2 ways 350/20, setting up the ratio as we did before, $\frac{y}{x} = k$ or we could substitute those values into **y = kx**.

$$y = kx$$
$$350 = k(20)$$
$$350/20 = k$$
$$17.5 = k$$

Either way, we would get 17.5.

k, the constant of variation is 17.5. That means you get 17.5 miles per gallon.

Now, to find out how much gas you would need to travel 3000 miles, you substitute the values of k and y into

$$\mathbf{y = kx}$$
$$3000 = 17.5\,x$$
$$171.42 = x$$, we'd need 171.42 gallons.

When you read **y = kx** mathematically, we say **y varies directly as x**.

Also, while we call **k** the constant of variation, k is also the **slope** of a line, the rate of change – normally written and recognized in algebra by using ***m***.

Below is the algorithm for how we solved that problem algebraically.

Algorithm

Solving Direct Variation Problems & Finding the Constant of Variation

1. Write the equation y = **k**x
2. Find the value of k by substituting the given values for x and y
3. Rewrite the equation y = kx
4. Substitute the value of k and the other value given to find the missing variable.

Example 3 The number of students passing the driving test varies directly with the number of students who enrolled in driving school. Of the 400 students enrolled, 80 passed. If 500 students passed the test, how many would have enrolled in the driving school?

The number of students passing (y) depends on the number of students enrolled (x).

1. Write $y = kx$,
2. $80 = k(400)$; therefore $\frac{80}{400} = \frac{1}{5} = k$
3. $y = kx$
4. $y = \frac{1}{5}(500)$, therefore $y = 100$

In these problems, the constant of variation provides us the relationship that exists between the ratios. In the problem above, rather than using algebra, I could have just as easily written the ratio; $\frac{passing}{enrolled} = \frac{80}{400} = \frac{1}{5}$.

Now, someone might ask, if we can solve these problems by proportion, as we did before, why are we solving them algebraically? There are a few reasons, we can now graph the relationships. $y = kx$ can be written as $y = mx$, an equation of a line that passes through the origin with slope k.

We can also see a set of ordered pairs that belong to that function, if the vales of $\frac{y}{x}$ are a constant, then we know that the relationship being described is not only linear, but is proportional.

As we have previously noted in our studies, math content is linked. The greatest differences occur in *language, notation and pattern development* because the math is being used in different contexts.

We have been working with proportions, we set the ratio on the left equal to the ratio on the right and solved.

Then we saw some of these relationships had a cause effect relationship. So instead of writing our ratios as $\frac{y_1}{x_1} = \frac{y_2}{x_2}$. We write the same problem as $\frac{y}{x} = k$, then convert that to $y = kx$ and solve algebraically.

Now, look at the original proportion; $\frac{y_1}{x_1} = \frac{y_2}{x_2}$. If I crossed multiplied, we get this relationship:

$$x_1y_2 = x_2y_1$$

That is just another way to describe a direct variation that some authors use. My point is if you know how to solve proportions, you have choices. My point is there are different ways to look at these and we don't need to get confused by them – always go back to the definition, then you can see where these different approaches come from.

And the best reason to use algebra to solve proportions is when problems get longer – not harder – algebra simplifies the problems. Up to this point, we have used simple proportions. But, in science, we see ***joint*** or ***combined*** relationships that are dependent on more than one variable. So, we could have y varies directly with *a, b*, and *c*. Algebraically we would write y = k*abc.*

As always, I could make them even longer by saying y varies directly with *a, b*, and c and inverely with d^2. That would translate to $y = \frac{kabc}{d^2}$.

As an example, the electrical resistance ***R*** of a wire varies directly as the length *l* of the wire and inversely as the square of the distance *d*. An equation expressing that is $R = \frac{kl}{d^2}$

The next section deals with indirect variations, that will explain why the d^2 was in the denominator.

Sec. 9 Indirect Variation; xy = k

An indirect variation is a relationship between 2 variables in which the product is a constant. In an indirect variation, when the value of one of the variables increases, the other decreases in proportion so that the product does not change. We say y varies indirectly as x; xy = k. Remember, in a direct variation, when one variable increases, the other variable also increases.

Let's examine a simple illustration like going on a trip. Rate varies indirectly with time. In other words, the faster you go, the less time it takes to get somewhere.

If you want to go 40 miles/hour for 3 hours, the constant of proportionality, k is 40(3) = 120. If the rate increases to 60 miles/hour multiplied by the time should also be 120. So, the time would be 2 hours.

Setting the problem up like we did for a direct variation, we now have

xy = k
40(3) = k, so k = 120
60(x) = 120
so x = 2

From the definition, xy = k, we can solve for y, $y = \frac{k}{x}$ and graph it. That turns out to be a hyperbola and we can very quickly see when x increases the value of y decreases.

Ratio & Proportion

Name
Date

Definitions

1***. Ratio

2.*** Proportion

3.*** Unit ratio

4.*** Write the 5 Properties of Proportion

5.*** Write 2 methods of solving proportional problems

6.** Express the ratio 5 to 7 as a fraction.

7. ** Express each ratio in simplest form.

a. 6 to 8

b. 24 to 32

8.** Bob's club has 50 members. Ten are male and the remainders are females. Find:

a. The ratio of males to females.

b. The ratio of males to the total number of members in the club.

c. The ratio of the total number of people in the club to the number of females.

9.** Solve for x. $\frac{3}{4} = \frac{21}{x}$

10.** Solve for x. $\frac{15}{24} = \frac{75}{x}$

11.** Solve for x. $\frac{2}{3} = \frac{15}{x}$

12.** If there are 12 inches in one foot, how many inches are there in five feet?

13. ** The legend on a map indicates one inch equals 35 miles. If the distance traveled on the map is four inches, how many miles would have to be traveled?

14.**. If a basketball player makes 25 out of 45 baskets in his first game, how many baskets would you expect him to make in the second game if he attempted 18 shots?

15.** On a trip across country Bob used 25 gallons of gas to travel 300 miles. At this rate, how much gas will he use to travel 3000 miles?

16.** If three pieces of candy cost 10¢, how much will 16 pieces cost?

17.** If the ratio of boys to girls in art class is 5 to 7, how many girls are there in class if there are 60 students enrolled?

18.* SBAC The ratio of length to width of a rectangle is 8 : 3. Find the dimensions of the rectangle if the perimeter is 44 inches.

19.* SBAC Bob saves $16 in 16 days. His sister Sarah saves $49 in 7 weeks, are these rates equivalent. Explain your answer.

20.* SBAC A candle is 30 inches long. After burning 12 minutes, the candle is 25 inches long. How long would it take the whole candle to burn?

21.*** Provide parent/guardian contact information; phone, cell, email, etc. (CHP)

