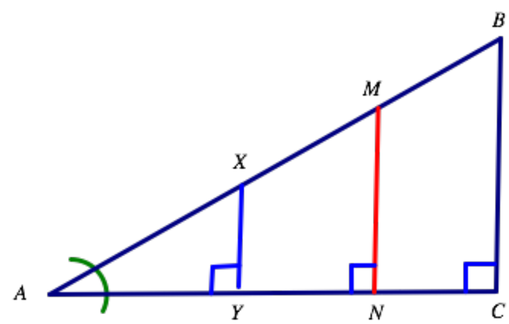
Ch. Y Intro Right Triangle Trig

In our work with similar polygons, we learned that, by definition, the angles of similar polygons were congruent and their sides were in proportion - which means their ratios are the same. We also learned the Angle-Angle Postulate. That is, if two angles of one triangle were congruent to two angles of another triangle, the triangles were similar.

As we look at the ratios of the sides of similar triangles, we are going to use that information and name some of those ratios - an introduction to right triangle trigonometry. Look at the three triangles below - ΔABC, ΔAMN and ΔAXY. They are similar because of the AA Postulate. Each triangle has a right angle and each triangle contains ∠A.



That means BC:MN have the same ratio as MN:XY. In fact, I could continue the relationships by saying BC:AB as MN:AM as XY:AX.

Let me write some of those relationships in words: the side opposite ∠A to the hypotenuse of each of the triangles, ΔABC , ΔAMN and ΔAXY have the same ratio.

$$\frac{BC}{AB} = \frac{MN}{AM} = \frac{XY}{AX}$$

This is kind of important, so we are going to name that ratio. The side opposite an angle over the hypotenuse will be called the sine ratio, abbreviated sin, of that angle.

So, sin A is equal to any of those ratios; $\frac{BC}{AB} = \frac{MN}{AM} = \frac{XY}{AX}$

If we were to look at other ratios based on the above triangles, we would also realize because of similar triangles, these ratios are also equal; $\frac{AC}{AB} = \frac{AN}{AM} = \frac{AY}{AX}$

Putting that ratio in words, the side adjacent to the angle over the hypotenuse will be called the cosine, abbreviated cos, of that angle.

So, the cos A is equal to any of those ratios; $\frac{AC}{AB} = \frac{AN}{AM} = \frac{AY}{AX}$

And one more set of ratios that would be equal are the ratio of the side opposite that angle over that adjacent side. That ratio is called the tangent, abbreviated tan.

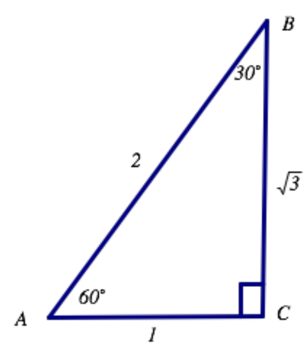
Summarizing those names, we have

$\sin A = \frac{opposite}{hypotenuse}$ $\qquad$ $\cos A = \frac{adjacent}{hupotenuse}$ $\qquad$ $\tan A = \frac{opposite}{adjacent}$

A popular way to remember those ratios is by memorizing **SOHCAHTOA**.

The S stands for sine, O for opposite side, H for hypotenuse, C for cosine, A for adjacent side and T for tangent.

Example 1 Using a 30-60-90˚ special right triangle, find the sine, cosine and tangent of 30˚



Using SOHCAHTOA

$\sin 30^\circ = \frac{opposite}{hypotenuse} = \frac{1}{2}$

$\cos 30^\circ = \frac{adjacent}{hypotenuse} = \frac{\sqrt{3}}{2}$

$\tan 30^\circ = \frac{opposite}{adjacent} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Example 2 Using the same 30-60-90˚ special right triangle, find the tan 60˚ and the cos 60˚.

Using SOHCAHTOA

$$\tan 60^\circ = \frac{opposite}{adjacent} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\cos 60^\circ = \frac{adjacent}{hypotenuse} = \frac{1}{2}$$

Using your knowledge of special right triangles, you should be able to find the sine, cosine and tangent ratios for 30˚, 45˚ and 60˚ angles. The ratios for other angles would have to be looked up in your book or found on your calculator.

We just said, based on our knowledge of special right triangles, the cos 60˚ = 1/2. If you were to use a table or your calculator, you would find the cos 60˚ = .500.

To find the tan 42˚, use either your calculator or book and you will find that ratio of the opposite side to the adjacent side is always 0.9004. Because of similar triangles and the sides being in proportion, the ratio of the opposite side : adjacent side is called the tangent and the tangent of 42˚ will always be equal to 0.9004.

Example 3 Use the tables in your calculator or in your book to find the sine, cosine and tangent of the following angles.

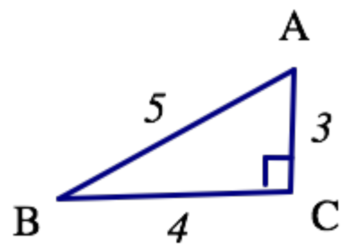
1. 30˚
2. 60˚
3. 40˚
4. 63˚

Find the measure of the angle with the following values (ratios).

5. sin A = .5150
6. cos B = .7071

7, tan C = 1.3270

Example 4 Let's find the sin B, cos B, and the tan B



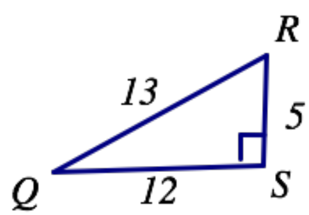
Remember SOHCAHTOA.

sin B = 3/5 cos B = 4/5 tan B = 3/4

Was that hard? Of course not. Using the same triangle, find the sin A, cos A, and tan A.

sin A = 4/5 cos A = 3/5 tan A = 4/3

Example 5 Find the sin, cos, and tan for angles Q and R.



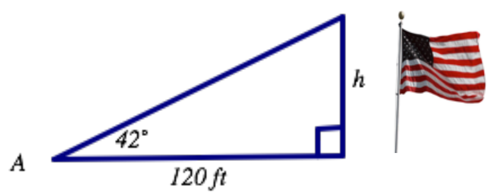
Using SOHCAHTOA

sin Q = 5/13 sin R = 12/13
cos Q = 12/13 cos R = 5/13
tan Q = 5/12 tan R = 12/5

In these examples, we have just asked you to find the ratios given a specific angle. To solve problems, you will have to determine which trig ratio is most appropriate based on the information provided.

Example 6 A boy standing on level ground notices (or can measure) that the angle he has to look up (angle of elevation) to see the top of a flagpole is 42 degrees. He can also measure the distance he is from the pole and finds it to be 120 feet. How high is the flagpole?

We'll call the height of the flagpole "h." Let's look at a picture and fill in any information we have



Now, how can we determine the height? Well, since we are studying trig, let's hope we use one of the trig ratios. Which one do we use? That's the question.

The sine is the opposite over the hypotenuse; we don't know the opposite or the hypotenuse. So, we won't use the sine. Cosine is the adjacent over the hypotenuse; that does not give the height. So, I won't use that. Tangent is the opposite, which is h, over the adjacent. There, I know the adjacent is 120 and I'm looking for the opposite (height). Therefore, I will use the tangent ratio.

I will have to look up the tan 42 on my trig table.
The tan 42˚ = .9004

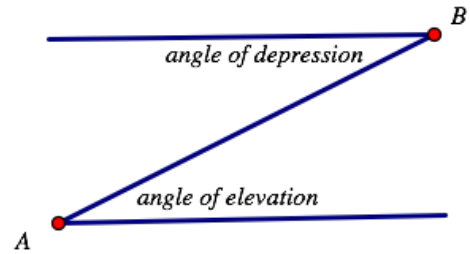
$$\tan 42^\circ = \frac{opposite}{adjacent} = \frac{h}{120}$$

$$.9004 = \frac{h}{120}$$

$$108 \approx h$$

So, the height of the flagpole is approximately 108 feet.

In a number of applications, professionals use terms like “angle of elevation” or “angle of depression”

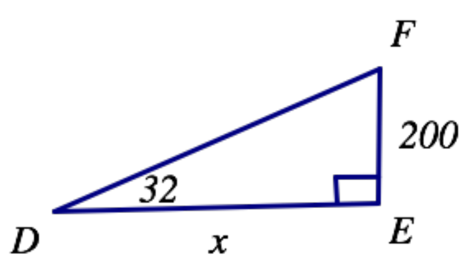


Line segment AB is the line of sight, the other two lines are horizontals. Notice how those two angles are defined with the horizontals. If you are looking up, that’s an angle of elevation. Looking down is an angle of depression.

To solve problems involving the trigonometric ratios, it’s always wise to think first, then set up a ratio. As we have said before, the choices we make can either make math easier or more cumbersome.

So, we have to ask, given certain information, am I going to use the sine cosine or tangent ratios and from what angle? So, thinking first is important.

Example 7 Find the measure of DE in the diagram.



Using the trig ratios, we are working with 2 sides and an angle. To solve problems, we need to know 2 of those 3 and find the unknown.

In this problem, we have 2 sides and an angle. Notice that the hypotenuse is NOT one of the sides given. For me, that eliminates using the sine or cosine ratios.

So, that leaves me with the tangent ratio, do I want to use angle D or F? Remember, if ∠D = 32˚, then ∠F = 58˚. The sum must be 90˚.

So now decide, which is an easier equation to solve, A or B?

$$A. \quad \tan 32^\circ = \frac{200}{x} \qquad or \qquad B. \quad \tan 58^\circ = \frac{x}{200}$$

In either case, I have to use a table or calculator to find the respective tangent values. However, since x is in the numerator of the B, I would choose to multiply, rather than divide.

$$\tan 58^\circ = \frac{x}{200}$$

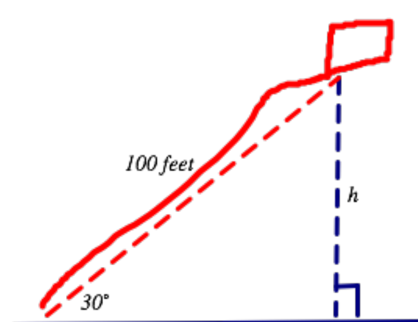
tan 58˚ ≈ 1.6003 (using a calculator)

$$1.6003 = \frac{x}{200}$$

$$x \approx 200(1.6003) \approx 320$$

Example 8 The length of a kite string is 100 feet long. How high is the kite if the kite make a 30˚ angle with the round?

The first best thing to do is draw a picture and label what we can.



Drawing the picture, you can see I straightened out the string and the height of anything is always denoted by perpendicular lines to form a right triangle.

Now, the question is; which trig ratio is best to use? To solve a problem with a trig ratio, I need to know 2 of 3 things; sides and angles.

In this case, the sin 30˚ is something I know from applications of special triangles or I could look it up.

$$\sin 30^\circ = \frac{opposite}{hypotenuse}$$

Substituting sin 30˚ = 1/2 and the hypotenuse is 100 ft.

$$\frac{1}{2} = \frac{h}{100}$$

$$2h = 100$$
$$h = 50$$

The tip of the kite string is 50 feet above the ground.

Let's examine what we know. All trig is in the study of the ratios of the sides of the right triangle. Those ratios are given names; sine, cosine, tangent, cosecant, secant, and cotangent. To know which ratio we are talking about, we stand on the vertex of the angle and use the acronym SOHCAHTOA. By looking at different right triangles and knowing the ratios are the same for equal angles (using SIMILAR TRIANGLES) we are able to create trig tables for all the angles that will give us the ratio of those sides. Those ratios are written in decimal form.

Don't you just love this stuff? You are probably thinking "trig is my life." Try these next two problems. In the first problem, the angle of depression is given. You need to know how to draw that - see page 6. We should not need tables because we are working with a 60˚ angle - use the special right triangle relationships.

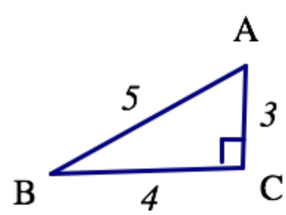
1. From the top of a building 300 feet high, you see a parked car on a bridge, if the angle of depression is 60˚, how far is the car from the building?

In this problem, we don't know the ratio of sides for an angle measuring 20˚, so we will have to use the book or calculator to determine that value based on the trig ratio you chose.

2. A pilot reads the angle of depression from his position 5000 feet in the air is 20˚, how far is the plane from the airport?

Sec. 2 Angle Relationships

In the previous section, we identified the trig values of angles when solving problems. In example 4, we had a right triangle shown below with the following relationships.



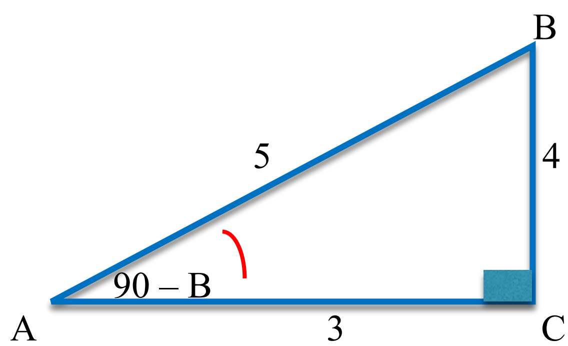
sin B = 3/5 cos B = 4/5 tan B = 3/4

sin A = 4/5 cos A = 3/5 tanA = 4/3

Notice the color coding. The sin A = 4/5 and the cos B = 4/5 and the sin B = 3/5 and the cos A also equals 3/5. Interesting?

Also notice, since ΔACB is a right triangle, then ∠A and ∠B are complementary – their sum is 90˚.

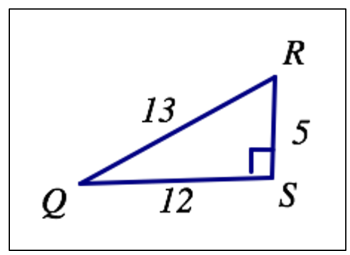
That means if we know one of the acute angles is 50˚, then the other angle, the cofunction, must be 90 – 50 or 40˚.



Now, let's combine those two ideas. sin B = 3/5 and the cos A = 3/5. Using substitution and letting ∠A = 90 – B, we have sin B = cos (90 – B).

Using the same reasoning, we have sin A = cos (90 – A).

From Example 5, we had the following relationships:



sin Q = 5/13	sin R = 12/13
cos Q = 12/13	cos R = 5/13
tan Q = 5/12	tan R = 12/5

Generalizing those **Cofunction Relationships**, we have

sin Q = cos (90 – Q)	cos Q = sin (90 – Q)
tan Q = cot (90 – Q)	cot Q = tan (90 – Q)
sec Q = csc (90 – Q)	csc Q = sec (90 – Q)

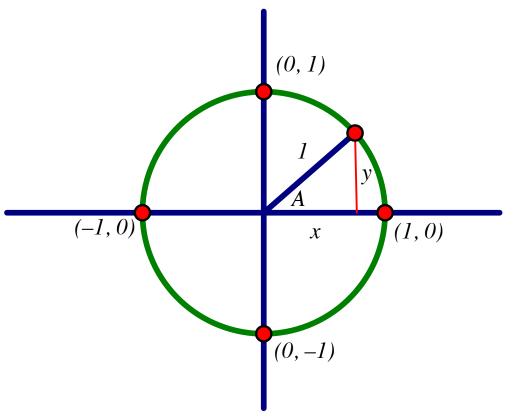
If we looked for further relationships, we would also see other patterns. For instance,
if we placed the value of the sine over the value of the cosine, Quotient Relationships, we get the value of the tangent. Let's look.

$\tan B = \frac{3}{4}$, placing the $\frac{\sin B}{\cos B} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$, the same as the tan B. Therefore, we can have the identity:

$$\tan B = \frac{\sin B}{\cos B}$$

Using the same thinking, the **cot** $B = \frac{\cos B}{\sin B}$

Let's extend these trig ratios to the rectangular coordinate system and draw a circle of radius 1 around the origin. This will lead us to a very important identity.



If I pick a point on the circle, (x, y) in the first quadrant and draw a line straight down to the x-axis, a right triangle is formed.

The length of the side opposite ∠A is y, adjacent side has measure x.

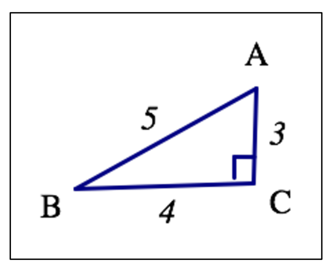
$\sin A = \frac{y}{1}$ or $y = \sin A$ and
$\cos A = \frac{x}{1}$ or $x = \cos A$

Using the Pythagorean Theorem, we know in the right triangle, $x^2 + y^2 = 1$

Substituting $x = \cos A$ and $y = \sin A$, we have $\cos^2 A + \sin^2 A = 1$

If we continued looking at the other ratios from previous examples, we would see the same pattern. This is a **Pythagorean Relationship.**

Using the triangle on the right, we said the $\sin B = \frac{3}{5}$ and the $\cos B = \frac{4}{5}$. If I find the sum of those trig ratios squared, we have

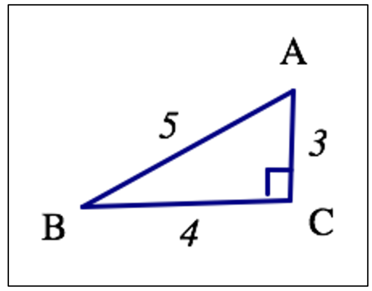


$$\sin^2 B + \cos^2 B = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1.$$

$$\cos^2 x + \sin^2 x = 1$$

Moving on, we will find more patterns – **Reciprocal Relationships**

And, using the same triangle we have been using, we can see there are a total of 6 ratios that can be made formed by the sides. The second set of three ratios are the reciprocals of the first three.



$\frac{3}{5}$ $\frac{3}{5}$ $\frac{3}{4}$

$\frac{5}{3}$ $\frac{5}{3}$ $\frac{4}{3}$

As we defined the sine, cosine and tangent ratios, we will define the reciprocals as the cosecant, secant and cotangent, respectively.

In other words,

The cosecant x, abbreviated csc x, is equal to the reciprocal of the sine x.

The secant x, abbreviated sec x, is equal to the reciprocal of the cosine x.

The cotangent x, abbreviated cot x, is equal to the reciprocal of the tangent.

Mathematically, we write

$\csc x = \frac{1}{\sin x}$ $\sec x = \frac{1}{\cos x}$ $\cot x = \frac{1}{\tan x}$

So, if the sin 30° = ½, then the csc 30° = $\frac{2}{1}$ or 2. Piece of cake, right?

Now, using the Pythagorean and the **Quotient Relationships**,
$\sin^2 x + \cos^2 x = 1$, dividing both sides by $\cos^2 x$, we have

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

And, if I divided both sides by $\sin^2 x$, we'd have

$$1 + \cot^2 x = \csc^2 x$$

These three formulas are called **Pythagorean Relationships**:

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

So, in a nutshell, all we are doing is looking at similar triangles and noticing some patterns based on how we name angles – SOHCAHTOA.

$\sin A = \frac{opposite}{hypotenuse}$ $\cos A = \frac{adjacent}{hupotenuse}$ $\tan A = \frac{opposite}{adjacent}$

From there, we saw some patterns for Cofunctions

sin Q = cos (90 – Q)
tan Q = cot (90 – Q)
sec Q = csc (90 – Q)

cos Q = sin (90 – Q)
cot Q = tan (90 – Q)
csc Q = sec (90 – Q)

then Reciprocal Relations

$\csc x = \frac{1}{\sin x}$ $\sec x = \frac{1}{\cos x}$ $\cot x = \frac{1}{\tan x}$

And we continued and "discovered" Quotient Relationships

$\tan B = \frac{\sin B}{\cos B}$ $\cot B = \frac{\cos B}{\sin B}$

which all led to the Pythagorean Relationships

$\sin^2 x + \cos^2 x = 1$ $\tan^2 x + 1 = \sec^2 x$ $1 + \cot^2 x = \csc^2 x$