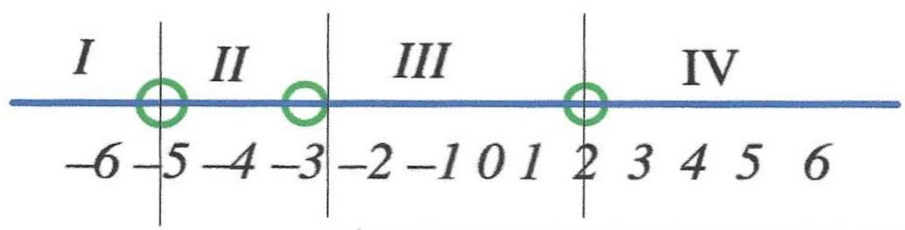
Graphing Higher Degree Equations
One Variable

The algorithm for solving quadratic and higher degree inequalities

1. **Find the critical points; solutions to equation and where denominator equals zero**
2. **Graph the critical points on the number line**
3. **Check the intervals to determine if the inequality is true**

Find the solution set. (x–2)(x+3)(x+5) = 0



Graph $f(x) = x^2 + x - 12$

$= (x + 4)(x - 3)$

Graph $h(x) = x^3 + 3x^2 - 6x - 8$

$= (x - 2)(x + 4)(x + 1)$

Graph $p(x) = x^3 - 3x^2 - 13x + 15$

$= (x - 1)(x + 3)(x - 5)$

Graph $y = (x - 6)(x + 1)(x - 3)(x + 7)$

Graph $y = (x - 3)^2 (x + 1)$

Graph $y = (x - 2)^3(x + 1)$

Finding Asymptotes

Finding vertical, horizontal, and oblique asymptotes are important when graphing rational functions. These asymptotes determine the end behavior of graphs under consideration.

To find asymptotes of a rational function defined by a rational expression in lowest terms, use the following:

1. **Vertical Asymptotes**
   Find vertical asymptotes by setting the denominator equal to zero and soling for x. If a is a zero of the denominator, then the line $x = a$ is a vertical asymptote.

2. **Horizontal Asymptotes**
   Case A – If the numerator has a lower degree than the denominator, then there is a horizontal asymptote at $y = 0$, the x-axis.

   Case B – If the numerator and the denominator have the same degree, then the horizontal asymptote is $y = a/b$, where a and b are the leading coefficients.

3. **Oblique Asymptotes (slanted)**
   If the numerator is of degree exactly one more than the denominator, then there will be an oblique asymptote. To find it divide the numerator by the denominator and disregard the remainder.

Example Find the asymptotes of the graph $f(x) = \dfrac{2x+1}{x-3}$

Use Rule 1 above to find the vertical asymptote(s). Set the denominator equal to zero, $x - 3 = 0$, So $x = 3$ is a vertical asymptote.

Use Rule 2, Case B to find the horizontal asymptote(s) since the numerator and denominator have the same degree. The leading coefficients of the numerator and denominator are 2 and 1 respectively. So the horizontal asymptote is $y = 2/1$ or $y = 2$.

You will use these asymptotes to graph rational functions, so it is important to know the rules for finding vertical, horizontal, and oblique asymptotes and when to use them.

Graphing a Rational Function

To graph rational functions, you will need to recall the rules for finding asymptotes and the following guide.

Let $f(x)=\frac{p(x)}{q(x)}$ define a function where p(x) and q(x) are polynomials and the rational expression is written in simplest form. To sketch the graph, follow the following steps.

1. **Find any vertical asymptotes.**
2. **Find any horizontal or oblique asymptotes**
3. **Find the y-intercept by evaluating f(0)**
4. **Find the x-intercepts, if any, by solving f(x) = 0. {These are the zeros of of the numerator - p(x)}**
5. **Determine whether the graph will interest its non-vertical asymptotes y = b or y = mx + b by solving f(x) = b or f(x) = mx + b**
6. **Plot selected points, as necessary. Choose an x in each domain interval determined by the vertical asymptotes and x-intercepts.**
7. **Complete the sketch.**

Example Graph $f(x)=\dfrac{3x^2-3x-6}{x^2+8x+16}$

Going through each step from above.

1. Find the vertical asymptote(s) ~ the denominator is $(x+4)^2$, so $x = -4$ is the only vertical asymptote.
2. Use Rule 2, Case B for finding horizontal asymptotes. $y = 3/1$ or $y = 3$
3. The y-intercept is $f(0) = -6/16$ or $-3/8$
4. The x-intercepts, $f(x) = 0$, set the numerator equal to zero and solve; $3x^2 - 3x - 6 = 0$. The x-intercepts are $x = 2$ or $x = -1$
5. Setting $f(x) = 3$ to find where the may intersect the horizontal asymptote that results in $x = -2$, the point of intersection is $(-2,3)$
6. Pick some values of x in each interval to plot some points
7. Sketch the graph

1. Graph $y = \frac{-2}{x}$ and identify the vertical and horizontal asymptotes.

2. Graph $f(x) = \frac{3}{x}$ and identify the vertical and horizontal asymptotes.

3. Graph $f(x) = \frac{-1}{x} + 2$ and identify the vertical and horizontal asymptotes.

4. Graph

$f(x) = \frac{2}{x} - 3$ and identify the vertical and horizontal asymptotes.

5. Graph $y = \frac{1}{x-2}$ and identify the vertical and horizontal asymptotes.

6. Graph $f(x) = \frac{-2}{x+3}$ and identify the vertical and horizontal asymptotes.

7. Graph $f(x) = \frac{2}{x+1} - 1$ and identify the vertical and horizontal asymptotes.

8. Graph $y = \dfrac{-3}{x-4} + 2$ and identify the vertical and horizontal asymptotes.

9. Graph $f(x) = \dfrac{x}{x+3}$ and identify the vertical and horizontal asymptotes.

10. Graph $y = \dfrac{3x}{x-2}$ and identify the vertical and horizontal asymptotes.

11. Graph $f(x) = \dfrac{-2x}{x+1}$ and identify the vertical and horizontal asymptotes.

12. Graph $y = \dfrac{-x}{x-3}$ and identify the vertical and horizontal asymptotes.

13. Grap $f(x) = \dfrac{2x-3}{x+2}$ and identify the vertical and horizontal asymptotes.

14. Graph $f(x) = \dfrac{-x+5}{x-1}$ and identify the vertical and horizontal asymptotes.

15. Graph $y = \dfrac{2x^2}{x^2-4}$ and identify the vertical and horizontal asymptotes.

16. Graph $f(x) = \dfrac{x^2}{x^2+4}$ and identify the vertical and horizontal asymptotes.

17. Graph $y = \dfrac{x^2+2}{x}$ and identify the vertical and slant asymptotes.

18. Give the end behavior of $f(x) = \dfrac{3}{x+2}$ as x approaches ∞.

19. Give the end behavior of $f(x) = \dfrac{-5}{x-3}$ as x approaches ∞.

20. Give the end behavior of $f(x) = \dfrac{2x+1}{x-3}$ as x approaches ∞.

21. Give the end behavior of $f(x) = \dfrac{x+3}{3x+1}$ as x approaches ∞.

22. Give the end behavior of $f(x) = \dfrac{-x+2}{2x+1}$ as x approaches ∞.

23. Give the end behavior of $f(x) = \dfrac{3x-1}{-5x+2}$ as x approaches ∞.

Graphing Rational Functions of the form $f(x) = \frac{p(x)}{q(x)}$

Step 1. Find all vertical asymptotes
Step 2. Find horizontal or oblique asymptotes
Step 3. Find the y intercepts, let x = 0
Step 4. Find the x intercepts, let y = 0
Step 5. Determine if the graph will intersect the horizontal or oblique asymptotes, set eqn = to HA or oblique asymptote
Step 6. Plot selected points in each interval determined by the vertical asymptotes

Graph the following:

1. $f(x) = \frac{2x-6}{x+2}$ VA = x = –2, HA = 2/1 = 2, $y_{int} = -6/2 = -3$, $x_{int} = 3$; No intersection of f(x) = HA - does not cross HA

2. $h(x) = \frac{x^2-4}{x-2}$

3. $g(x) = \frac{x+6}{x^2+9x+18}$

4. $q(x) = \frac{2x^2-x-1}{x^2-x-12}$

5. $t(x) = \frac{x^3+3x^2}{x^2+2x-3}$

6. $m(x) = \frac{x-3}{x^2-3x}$

7. $n(x) = \frac{x^2+5x-6}{2x-2}$

8. $p(x) = \frac{(x+8)(x-4)}{(x+8)(x+5)}$

Understanding Math

Graphing; 2 Variables, Asymptotic Lines

HANLONMATH

When students learn how to graph rational functions, *asymptotes* are certain to be a point of discussion. Different types of asymptotes are explored along with their relevant characteristics. Unfortunately, there are some aspects of asymptotes that our students fail to grasp. We will examine those details in the paragraphs below.

Any discussion must begin with the definition of asymptote: *An **asymptote** is a line related to a given curve such that the distance between the line and a point on the curve approaches zero as the distance of the point from the origin increases without bound.* More informally, as we "travel" along the curve, the curve gets closer and closer to the line but does not touch it. We then classify asymptotes as vertical, horizontal, or slant/oblique, with each having its own characteristics.

The first thing that students don't always get is that slant asymptotes are very similar to horizontal asymptotes. With horizontal asymptotes we examine the end behavior of the function; what it does as *x* gets "very large" in either direction, that is, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. For example, if $f(x)=\dfrac{2x+1}{x}$, then $\lim_{x\to\infty}\dfrac{2x+1}{x}=2$ and $\lim_{x\to-\infty}\dfrac{2x+1}{x}=2$, thus $f(x)=\dfrac{2x+1}{x}$ has a horizontal asymptote of $y=2$. On the other hand, if $f(x)=\dfrac{2x^2+1}{x}$, then $\lim_{x\to\infty}\dfrac{2x^2+1}{x}=\infty$ and $\lim_{x\to-\infty}\dfrac{2x^2+1}{x}=\infty$, thus $f(x)=\dfrac{2x^2+1}{x}$ has no horizontal asymptotes. However, it *does* have a *slant* asymptote. As *x* gets "very large", $f(x)$ is just a tiny bit more than 2*x*, with that "tiny bit" decreasing as *x* increases.. (See Figure 1.)

$f(x)=\dfrac{2x^2+1}{x}$ therefore has a slant asymptote of $y=2x$.

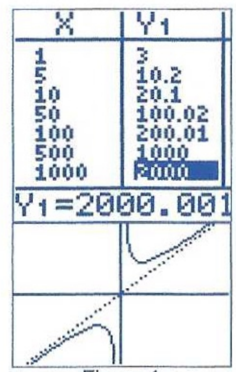


Figure 1

The next detail that students miss is that a function may have different horizontal asymptotes for positive and negative values of *x*. Consider the function $f(x)=\dfrac{|2x+1|}{x}$: $\lim_{x\to\infty}\dfrac{|2x+1|}{x}=2$ and $\lim_{x\to-\infty}\dfrac{|2x+1|}{x}=-2$. Thus, $f(x)$ has a horizontal asymptote of $y=2$ when $x>0$ and $y=-2$ when $x<0$. (See Figure 2.)

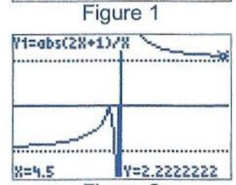


Figure 2

The last point students don't get is the misconception that a graph can't cross an asymptote. For vertical asymptotes, this is true, as the function is undefined where they occur. However, a function can definitely cross a horizontal asymptote. Take a look at Figures 3 and 4, of $f(x)=\dfrac{2x^2+2x+1}{x^2-1}$ along with its horizontal asymptote $y=2$. We can see that function crosses the horizontal asymptote at $x=-1.5$. How can this be? The definition of asymptote does not allow the curve to touch the asymptote, not to mention crossing it!

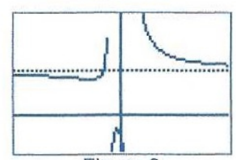


Figure 3

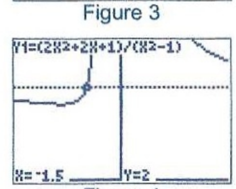


Figure 4

There is no contradiction here. Horizontal asymptotes are important as we consider values of *x* that are "very large," again, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. We really don't care about the horizontal asymptotes when *x* is "small." This characteristic of curves crossing horizontal asymptotes also applies to slant asymptotes, that is, the curve may "break through" the asymptote from one side and approach it from the other as *x* increases without bound.

One final note is about how we ask students to identify special points on curves—intercepts, extrema, etc. In the example above, there is a local minimum at $\left(\frac{-3-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$ that students should identify. (See Figure 5.)

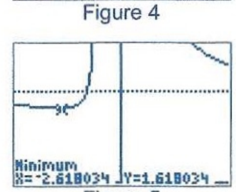


Figure 5