Chapter 3 Equations of Lines

Sec 1. Slope

The idea of slope is used quite often in our lives, however outside of school, it goes by different names. People involved in home construction might talk about the pitch of a roof. If you were riding in your car, you might have seen a sign on the road indicating a grade of 6% up or down a hill. Both of those cases refer to what we call slope in mathematics.

Kids use slope on a regular basis without realizing it. A student buys a cold drink for $0.50, if two cold drinks were purchased, the student would have to pay $1.00. He's not thinking the rate of change for each additional purchase as slope.

I could describe that mathematically by using ordered pairs; (1, $0.50), (2, $1.00), (3, $1.50), and so on. The first element in the ordered pair represents the number of cold drinks, the second number represents the cost of those drinks. Easy enough, don't you think?

Now if I asked the student, how much more was charged for each additional cold drink, hopefully the student would answer $0.50. So, the difference in cost from one cold drink to adding another is $0.50. The cost would change by $0.50 for each additional cold drink. The change in price for each additional cold drink is $0.50. Another way to say that is the *rate of change* is $.50. - we call the rate of change—slope.

In math, the rate of change is called the slope and is often described by the ratio $\frac{rise}{run}$.

The rise represents the change (difference) in the vertical values (the y's), the run represents the change in the horizontal values, (the x's). Mathematically, we write

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Let's look at any two of those ordered pairs from buying cold drinks, (1,$0.50) and (3,$1.50) and find the slope. Substituting in the formula, we have:

$$m = \frac{1.50 - 0.50}{3 - 1} \rightarrow \frac{1.00}{2}$$

Simplifying, we find the slope is $0.50. The rate of change per drink is $0.50

Example 1 Find the slope of the line that connects the ordered pairs (3,5) and (7, 12)

To find the slope, I use $m = \frac{y_2 - y_1}{x_2 - x_1}$

Subtract the y values and place that result over the difference in the x values.

$\frac{12 - 5}{7 - 3} = \frac{7}{4}$ The slope is 7/4

Example 2 Find the slope of the line that connects the ordered pairs (7, 8) and (2, 3)

Using $m = \frac{y_2 - y_1}{x_2 - x_1}$, we have $\frac{8 - 3}{7 - 2} = \frac{5}{5} = 1$

Example 3 Find the slope of the line tat connects (2, – 3) and (–5, 8)

$$\frac{8 - (-3)}{-5 - 2} = \frac{11}{-7}$$

Find the slopes of the lines passing through the following points.

1.	(1, 7) (5, 15)	2,	(3, 1) (6, 13)	3.	(5, 8) (7, 2)
4.	(4, – 3) (6, 9)	5.	(–2, 5) (–8, 11)	6.	(–1, 3) (7, –4)

Sec. 2 Point – Slope Form of a Line

Slope has been defined as the rate of change and we have mathematically written it as:

$$m = \frac{\text{rise}}{\text{run}} \qquad\qquad m = \frac{y_2 - y_1}{x_2 - x_1}$$

If we were to rewrite the formula leaving out the subscript 2, we would have the

$$\frac{y - y_1}{x - x_1} = m$$

Now, if I played with that equation and multiplied both sides of the equation by denominator, $(x - x_1)$, I would have:

$$y - y_1 = m(x - x_1)$$

Turns out, that is an equation of a line with slope ***m*** passing through the point (x_1, y_1). It's called the **Point-Slope Form** of a line. The reason for that name is simple, you are given a point and a slope in the equation.

What's nice about the Point-Slope form of a line is if you are given information and asked to ***find an equation of a line***, then all you need to know is a point and the slope, then substitute that information into the formula

Point-Slope Equation of a Line

$$y - y_1 = m(x - x_1)$$

To find an equation of a line, you must know two pieces of information, a point and a slope.

Now, anytime someone asks you to find an equation of a line, you can use that formula.

Example 1 Find an equation of a line that passes through (1, 4) and has slope 5.

Using $y - y_1 = m(x - x_1)$, substitute values for x, y and slope

$$y - 4 = 5(x - 1)$$

That is an equation of a line passing through (1, 4) with slope 5.

If I solved that equation for y, we have $y - 4 = 5x - 5$

$$y = 5x - 1$$

Example 2 Find an equation of a line that passes through (5, – 2) and has slope 3.

Using $y - y_1 = m(x - x_1)$ and substituting values.

$$y - (-2) = 3(x - 5)$$

$$y + 2 = 3(x - 5)$$

That is an equation of a line passing through (5, – 2) with slope 3.

Solving for y, $y + 2 = 3(x - 5)$

$$y + 2 = 3x - 15$$

$$y = 3x - 17$$

Parallel & Perpendicular Lines

Remember, parallel lines have the same slope – perpendicular lines have negative reciprocal slopes. Sometimes you are asked to find an equation of a line and the slope is not explicitly given to you. If that occurs, you must first find the slope, then substitute the numbers into the equation.

What's nice about the Point-Slope Form of an Equation of a Line is that it is derived from the slope formula. That means if you happen to forget it, all you need do is write the equation for slope, then multiply both sides of the equation by the denominator.

To use the Pont Slope Form of an Equation, you must know at least one point and the slope. Up to this point, both were given to you explicitly. If they are not given explicitly, you have to be able to find a point or a slope to use the Point-Slope form of an equation of a line.

Example 3 Find an equation of a line that passes through the points (2, 5) and (4, 13).

First, we'll write out the formula for finding equations of lines.

$$y - y_1 = m\,(x - x_1)$$

Two points are given to us, so I know a point (x_1, y_1). The slope, m, was not given to me. Can I find the slope given the information? The answer is yes. The slope is the change in y over the change in x.

Using the two ordered pairs (2, 5) and (4, 13) given in the problem, we have:

$$m = \frac{13-5}{4-2}$$

$$= \frac{8}{2} = 4$$

Now that I know a point and I just found the slope the slope, I just substitute those into the Point Slope Equation.

It doesn't matter which point you use. Using either point, you will arrive at the same answer. I'll use (2, 5) only because the numbers are smaller.

$y - y_1 = m\,(x - x_1)$ Point Slope for of a Line

$$y - 5 = 4\,(x - 2)$$

$$y - 5 = 4x - 8$$

$$y = 4x - 3$$

***** If we looked at enough of these equations, you would begin to see the relationship between the slope and the coefficient of the x when the equation is solved for y. They are the same!**

Often times the slope is not just given to you explicitly, you need to find it as we just did. Things you need to remember are: **parallel lines have the same slope perpendicular lines have negative reciprocal slopes.**

As soon as someone says ***find or write an equation of a line***, no matter what comes next, I recommend writing the Point Slope Form of a Line, and substitute in the appropriate numbers.

Example 4. Find an equation of a line passing through (2, –3) that is parallel to $y = 4x + 5$

Using $y - y_1 = m(x - x_1)$

I have a point and the number in front of the coefficient is 4, the slope is 4. Substituting, we have

$$y + 3 = m(x - 2)$$

The slope of $y = 4x + 5$ is 4, so $y + 3 = 4(x - 2)$

$$y + 3 = 4x - 8$$

$$y = 4x - 11$$

Find an equation in Point Slope form, then solve the equation for y.

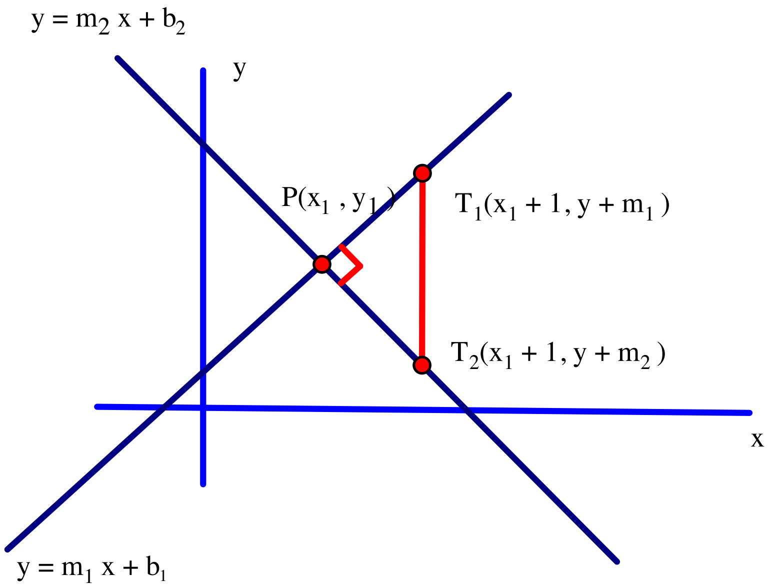
1. Find an equation of a line passing through (2, 3) and (3, 7).

2. Find an equation of a line passing through (4, 5) having a slope 3.

3. Find an equation of a line passing through (3, 1) that is parallel to $y = 2x - 5$.

4. Find an equation of a line passing through (2, 3) that is perpendicular $y = 4x + 1$

Why do perpendicular lines have negative reciprocal slopes? – Algebra Derivation



The diagram shows l_1 and l_2, with equations $y = m_1x + b_1$ and $y = m_2x + b_2$, intersecting at $P(x_1, y_1)$

Since $P(x_1, y_1)$ lies on both lines and using the slope, I can go up m_1 and over 1 on l_1 and using the same reasoning for l_2, go over 1 and up m_2. That results in two points; $T_1(x_1 + 1, y_1+ m_1)$ and $T_2(x_1 + 1, y_1 + m_2)$.

The points P, T_1 and T_2 are the vertices of a triangle. Since the lines are perpendicular, ΔPT_1T_2 is a right triangle. Because I have a right triangle, I can use the Pythagorean Theorem to determine the lengths of each side of the triangle.

$$[d(T_1, T_2)]^2 = [d(P, T_1)]^2 + [d(p, T_2)]^2$$
$$(m_1 - m_2)^2 = (1 + m_1^2) + (1 + m_2^2)$$
$$m_1^2 - 2m_1m_2 + m_2^2 = 2 + m_1^2 + m_2^2$$
$$-2m_1m_2 = 2$$
$$m_1m_2 = -1$$

Sec. 3 Graphing Linear Equations

If I solved the Point Slope form of a line for y, that results in the Slope Intercept form of a line – $y = mx + b$

In order to plot the graph of a linear equation, we solve the equation for y in terms of x, then we assign values for x and find the value of y that corresponds to that x. Each x and y, called an ordered pair (x, y), represents the coordinate of a point on the graph.

Example 1 Graph $3x + y = 2$

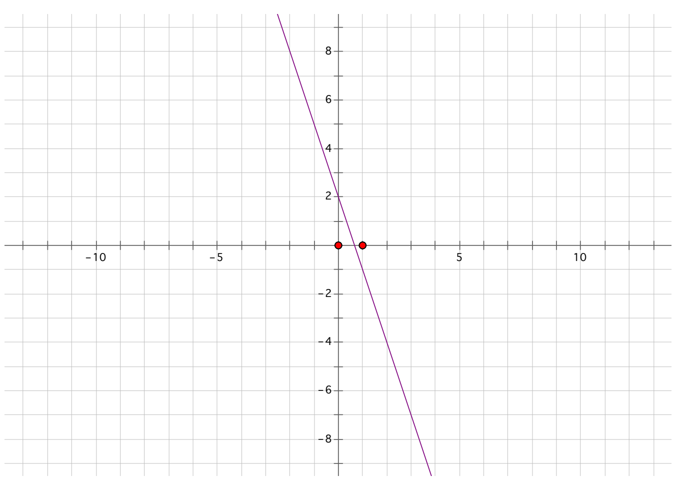
x	y
0	2
2	–4
–1	5

Solving for y, I subtract 3x from both sides. $y = 2 - 3x$.

Which I could rewrite, using the Commutative Property as,

$$y = -3x + 2$$

Always pick convenient values for x, then find the corresponding y value.



Rewriting as ordered pairs: (0, 2), (2, – 4), (–1, 5) as plotting the graph. When I connect those three points I get a straight line, called a LINEAR equation.

I could have chosen any values for x and found the corresponding values of y. However, it is easier to choose convenient numbers, like zero, one, or two. Choosing a number like 100 would make my graph a lot larger. Or, I could have chosen a fraction, but that can be messy.

Example 2 Graph $x - 3y = 8$

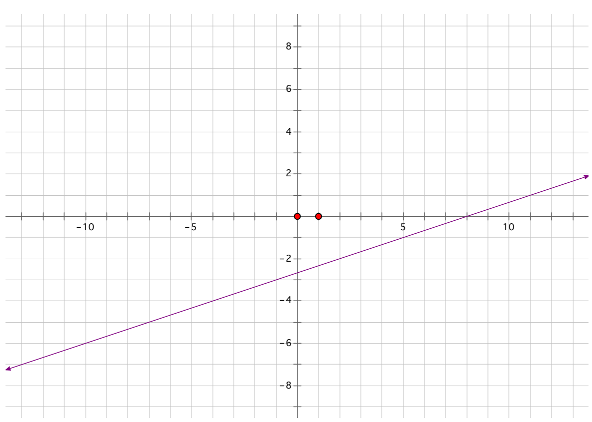
x	y
0	–8/3
8	0
5	–1

Again, I solve for y in terms of x

$$x - 8 = 3y \quad \text{or} \quad \frac{x-8}{3} = y$$

$$y = \frac{x}{3} - \frac{8}{3}$$

Assign values for x and find the corresponding y's. The ordered pairs $(0, -\frac{8}{3})$, (8, 0), and (5, –1) represent the points on the graph.



Graphing Equations of Lines

1. Solve the equation for y
2. Make an x-y chart
3. Pick convenient values of x and find corresponding y's
4. Write those as ordered pairs and plot on the coordinate plane
5. Draw line through the points

Example 3 Graph $2x - 5y = 10$ Solve for y

$$2x - 10 = 5y$$

$$y = \frac{2}{5}x - 2.$$

I'll pick x = 0, which results in (0, –2). Since the denominator is 5, to eliminate the fraction, I let x = 5 which results in (5, 0).

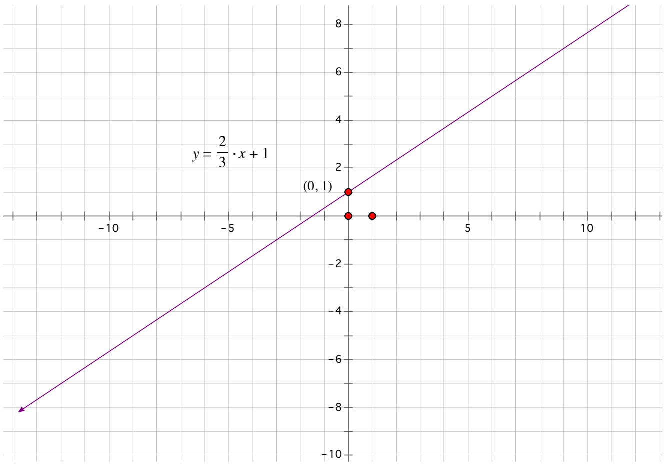
Now, we have already saw the pattern that suggested that when I solve an equation in Point Slope form for y, the coefficient of the x is the slope. That was both interesting and helpful for finding equations of lines.

Now, if I go back and look at the constant, that corresponds to where the graph crosses (intercepts) the y-axis. We also notice that occurs when the value of x is zero. This recognition is going to make our life so much easier.

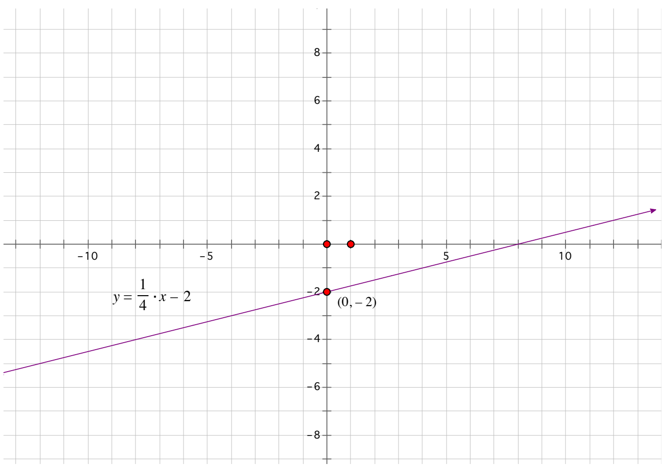
Sec. 4 Slope Intercept Form of a Line

If we did enough of graphing by solving for y and making an x-y chart, we are seeing a quicker way of graphing those same linear equations – by inspection.

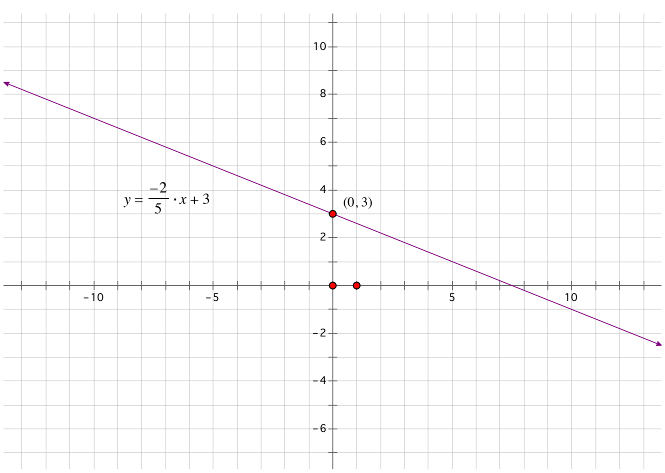
Finding a pattern to graph linear equations would save us a lot of manipulation and computation. So, let's look at the following graphs and look at their corresponding equations and see if you see any relationships.



$y = 2x/3 + 1$



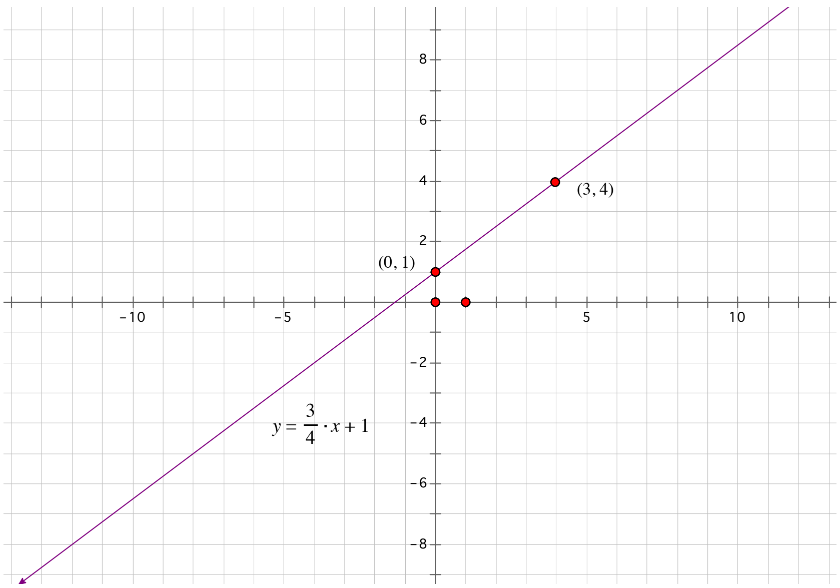
$y = \frac{1}{4}x - 2$



$y = \frac{-2}{5}x + 3$

First, all linear equations are graphs of lines, therefore all we need do is graph two points. Second, we might notice that the value of x when the graph crosses the y-axis is always zero. Look at the three previous examples, the graphs cross the y-axis at 1, –2 and 3 respectively and corresponds to the y intercept (b) in the Slope Intercept form of a line - y = mx + b

I'm going to plot and label two points to see if you see another relationship, between the graph and the equation that would work in all the examples. $y = \frac{3}{4}x + 1$



Notice, where the graph crosses the y-axis is also the number being added to the variable, b. That's called the y-intercept, where the graph crosses the y-axis. So, in the equation $y = \frac{3}{4}x + 1$, +1, the point (0, 1) is a point on the graph called the y-intercept.

Now look at the coefficient of the x, that always seems to be the slope. From the y-intercept, a second point was found by going up three and over four on the graph and results in a point (4,4). See if that works in the 3 other graphs.

Seeing that pattern will allow me to look at an equation and graph it. In other words, I won't have to solve the equation for y, make an x-y chart, pick values of x, find values of y, then graph.

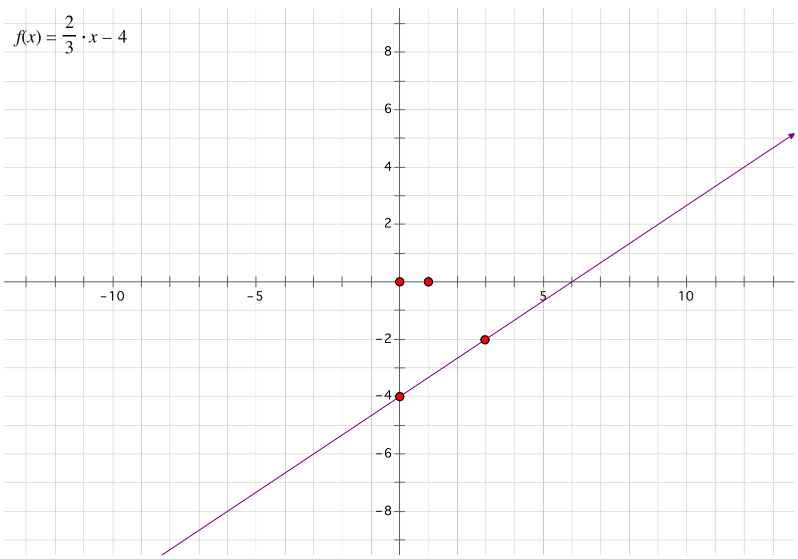
<u>Graph Slope Intercept by Inspection</u>

1. **Plot the y-intercept, b**
2. **From b, use *m* to find the second point**
3. **Draw a line through the 2 points**

- **if the slope is a whole number, write it as a fraction**

Example 1. Graph $y = \frac{2}{3}x - 4$

1) The y-intercept is –4, b is –4
2) From – 4 go up 2 and over 3
3) Draw a line through the 2 points

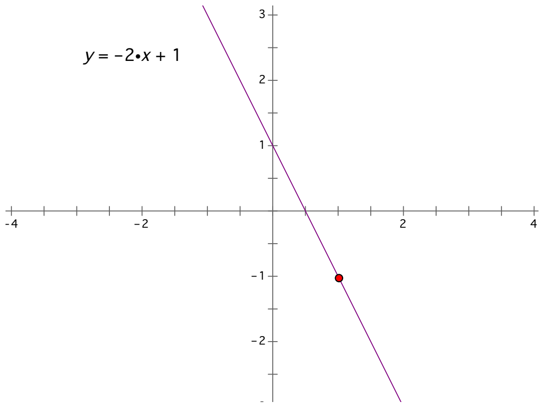


Graphing by inspection sure beats solving for y, picking convenient values of x, then finding corresponding values of y, then plotting and connecting points**.**

Example 2 Use the algorithm to graph y = –2x + 1 by inspection.

The y intercept is at (0,1), b = 1

The slope is –2, remember a slope of –2 is the same as –2/1 or 2/–1
Which is either down 2 and over 1 OR up 2 and over left 1



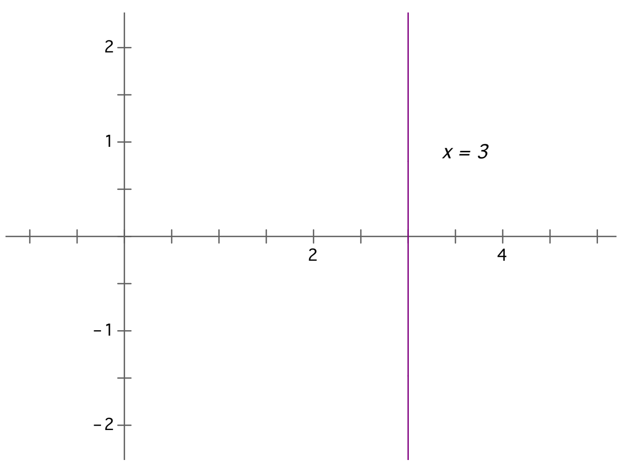
y = –2x + 1

The y-intercept is 1, the slope is –2, down 2 and over 1

Example 3 Graph x = 3

Sometimes an equation will be written with only one variable like x = 3. What does that mean and how is that graphed might be something one might ask. The answer is relatively straight forward.

When we have only one variable in an equation, it means there are no restrictions on the other variable. In other words, the other variable could be equal to any number. So, for our x = 3, y could be any number from negative infinity to positive infinity. Its graph would be a vertical line.

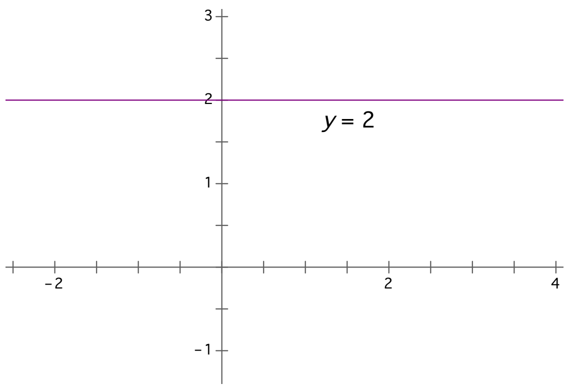


Examples of ordered pairs:
(3,0), (3,1), (3,10), (3,–1), (3,–2), (3,-20)

Now, what if someone asked about the slope of that vertical line. Well, we know the slope is defined as the change in y over the change in x, $\frac{\Delta y}{\Delta x}$. When we do that for any of our ordered pairs, the x's subtract out. That results in a denominator of zero – which is not allowed. Therefore, we say the slope is **undefined**.

Example 4 Graph y = 2

Using our same logic as in example 4, we know that x can take on any value as long as y = 2



Examples of ordered pairs:
(0,2), (1,2), (8,2) (–4,2), (–10,2)

What might be the slope of this line? Again, going back to the definition of slope, the difference of the y's over the difference of the x's. Subtracting any of those ordered pairs always results in the y's subtracting out. That means the numerator is always zero or the slope is zero. That makes sense since when we look at the graph of the line, there is no steepness. So, the slope is 0.

Graph the following by inspection. That is identify the y-intercept, the value of b, then the slope, the coefficient of the x.

1. $y = 3x + 2$
2. $y = \frac{1}{2}x + 5$
3. $y = 2x - 4$
4. $y = -3x + 2$
5. $y = -4x + 6$
6. $y = \frac{2}{5}x + 3$
7. $y = 4x$
8. $y = 3x - 2$
9. $y = 5$

Sec. 5 General Form of an Equation of a Line

Putting this into perspective is we see these equations of lines are connected. The slope was defined initially. From the slope we found the Point Slope form of a line by multiplying the slope formula by the denominator. When we solved the Point Slope equation for y, we found the Slope Intercept form of a line. In other words, if you know the formula for slope, we can derive (find) all the other equations.

Instead of solving for y as we did initially when we found the Point Slope, and having an equation like $2x - y = 6$ transform into $y = 2x - 6$, if we decided to leave the x's and y's on the same side there are some observations we can make that will make our work even easier.

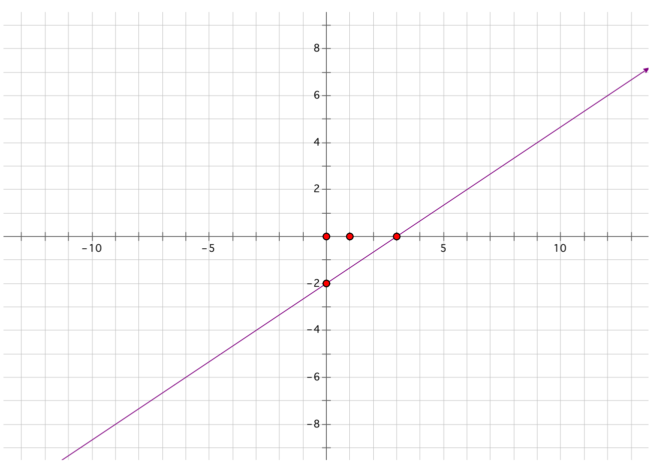
When we studied enough graphs, we have already seen that the graph crosses the y-axis when $x = 0$. If we looked at enough graphs, we would also see the graph crosses the x-axis when y is 0. That observation is important.

In other words, in the equation $2x - 3y = 6$, we can see by letting $x = 0$, the y intercept is -2. If we let $y = 0$, the x intercept is 3.

If we connect the x and y intercepts, we have graphed the equation.

Find the x and y intercepts of the following equations by letting $x = 0$, then $y = 0$.

Example 1 Graph by finding the x and y intercepts of $2x - 3y = 6$



$2x - 3y = 6$

let $x = 0$, $2(0) - 3y = 6$

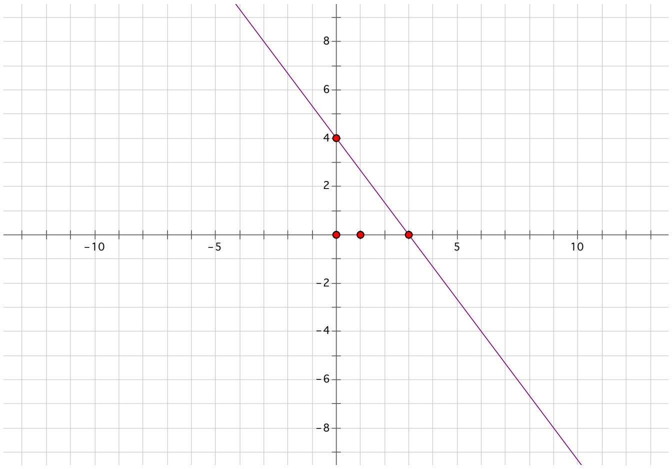
$y_{int} = -2$

$2x - 3y = 6$

let $y = 0$, $2x - 3(0) = 6$

$x_{int} = 3$

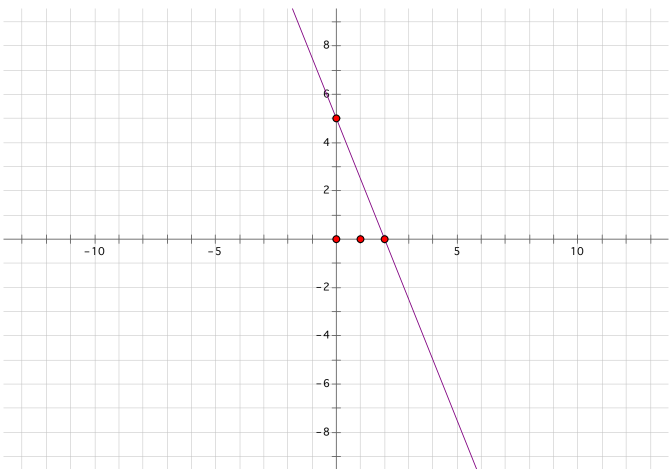
Example 2 Graph 4x + 3y = 12 by finding the x and y intercepts.



4x + 3y = 12
let x = 0, 4(0) + 3y = 12
3y = 12
$y_{int} = 4$

4x + 3y = 12
let y = 0, 4x + 3(0) = 12
4x = 12
$x_{int} = 3$

Example 3 Graph 5x + 2y = 10



5x + 2y = 10
let x = 0, 5(0) + 2y = 10
2y = 10
$y_{int} = 5$

5x + 2y = 10
let y = 0, 5x + 2(0) = 10
5x = 10
$x_{int} = 2$

These observations lead us to be able to find two points (the x and y intercepts) by just looking at an equation, which allows us to graph equations of lines written in the General Form, Ax + By = C by inspection.

$$\text{Ax + By = C, if x = 0, then By = C} \quad \rightarrow \quad y_{\text{int}} = \frac{C}{B}$$

$$\text{Ax + By = C, if y = 0, then Ax = C} \quad \rightarrow \quad x_{\text{int}} = \frac{C}{A}$$

Upon further inspection, we would find the slope of a line in General Form can be found by –A/B by solving for y and noticing the coefficient of the x is –A/B.

Just as importantly, it should be noted that the Point-Slope Form of a Line came directly from the definition of slope. The Slope-Intercept was derived from the Point-Slope, and the General Equation came from the Slope-Intercept. They are all linked.

Graphing the General Form of the Equation of a Line – By Inspection

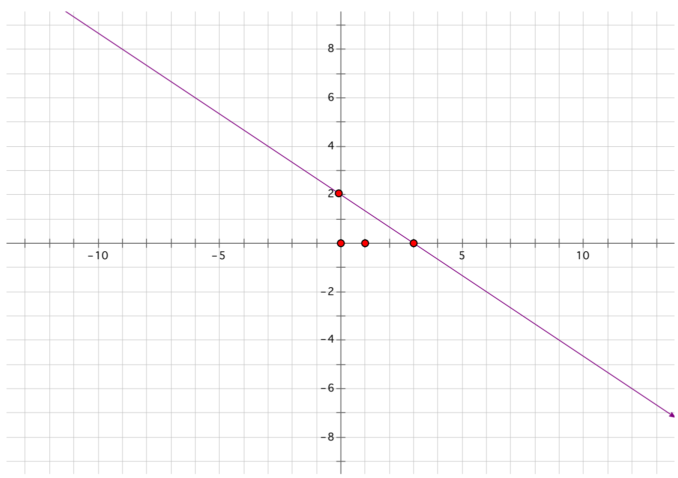
1. Find x-intercept, let $y = 0$
2. Find y-intercept, let $x = 0$
3. Draw line to connect points

Example 4: Graph $2x + 3y = 6$

When $y = 0$, $2x = 6$ therefore the x-int. = 3

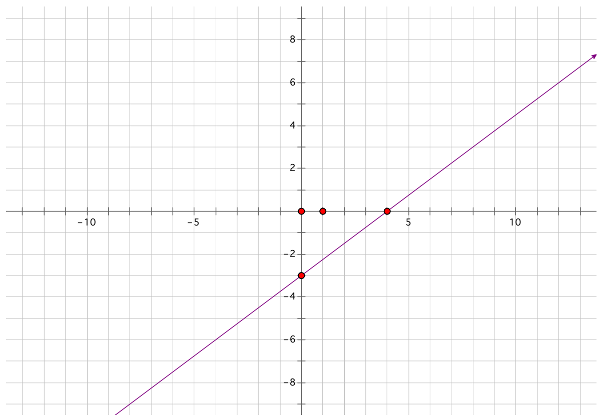
When $x = 0$, $3y = 6$, therefore y-int. = 2

Plot (3, 0) and (0, 2) and you're done. That beats solving for y and plugging in values for x.



Example 5: Graph $3x - 4y = 12$

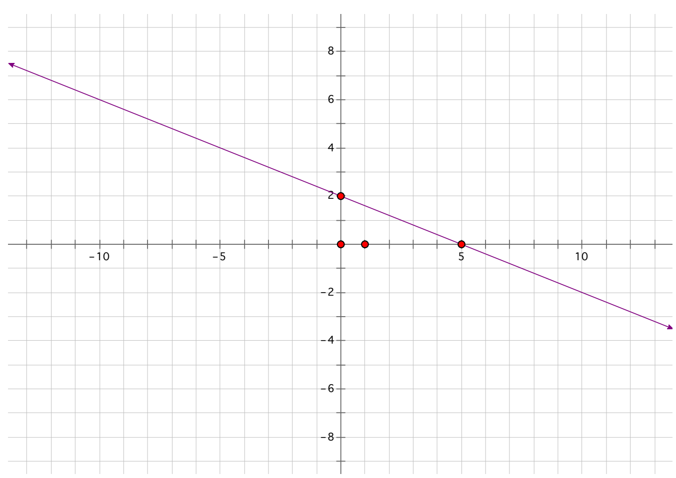
When $x = 0$, the $3x$ falls out of the problem, so $-4y = 12$ or $y = -3$. When $y = 0$, the $-4y$ falls out, so we get $3x = 12$, or $x = 4$. Notice, (0. –3) and (4 , 0) were ordered pairs doing the problem the other way. Look on the chart. All we need do is plot the point just like before and we're done.



Graphing by inspection takes a lot less time than plotting points. However, if you don't take the time to learn the different terms of a linear equation, you will have no other choice but to solve for y and plug in x's that takes time & space.

Example 6: Graph $2x + 5y = 10$

Again, finding the x and y intercepts by letting x and y be zero. We found the x intercept is 5, when $y = 0$, and the y intercept is 2 when $x = 0$. The ordered pairs are (5, 0) and (0, 2). Plotting & Connecting the points, we have



The form of the equation $2x + 5y = 10$ is called the General Form of Equation of a line. It is written $Ax + By = C$.

Recognizing patterns is important in mathematics – it means the difference between doing problems quickly without arithmetic mistakes or trudging through.

It's also important to know vocabulary. We've discussed where the graph of the line crosses the x-axis, we call that the x-intercept. Where the graph crosses the y-axis is called the y-intercept. The x-intercept occurs when y is zero. The y-intercept occurs when x is zero.

Graph the following equations by finding the x and y intercepts.

1) $2x + 3y = 6$

2) $3x + 4y = 12$

3) $5x - 2y = 10$

4) $6x - 2y = 6$

5) $x + y = 1$

6) $x - y = 1$

Putting this into perspective. By knowing how slope was defined, I was able to find the Point Slope form of an equation of a line. I typically use that when someone asks me to "find" or "write" an equation of a line.

By solving the Point slope form of an equation of a line for y, we found that the coefficient of the x-term was the slope and the constant (b) was the y intercept and the y intercept occurred when $x = 0$.

That led us to notice that the x-intercept occurred when $y = 0$. And that when graphs were written in General Form, I could graph them by inspection by just knowing the x and y intercepts. In short, those patterns made finding equations and graphing a lot easier and much quicker to do.

So, in a nutshell, all these equations initially came from knowing how slope was defined mathematically.

Sec. 6 Graphing Linear Inequalities

Knowing how to graph linear equations in 2 variables translates to graphing linear inequalities. We start the same way, hopefully, graphing by inspection.

That is, we plot the boundary lines of the inequality using either

Slope Intercept; **$y = mx + b$** or **General Form** **$Ax + By = C$**

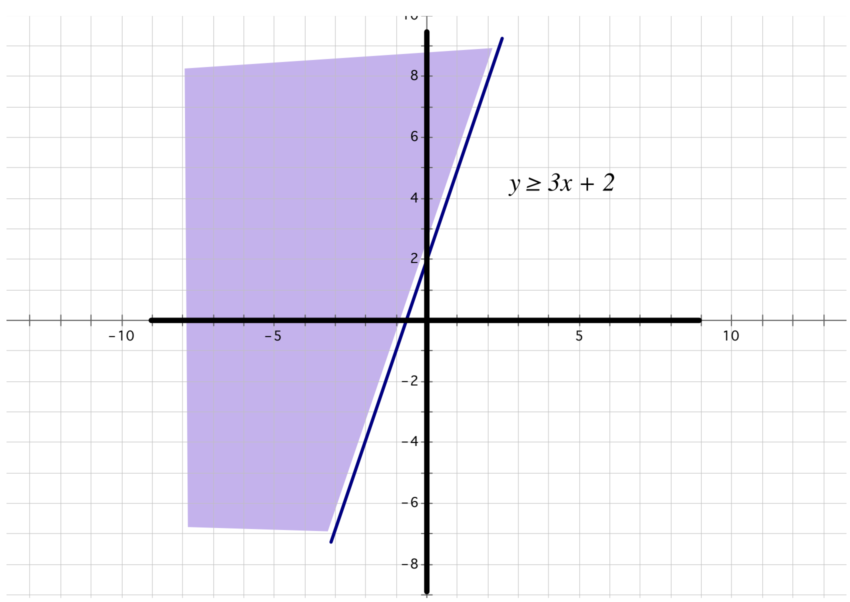
Slope Intercept	General Form
1. Plot the y_{int}, (b)	1. Let $x = 0$, find y_{int}
2. From b, use m to find 2^{nd} pt	2 Let $y = 0$, find x_{int}
3. Connect pts	3. Connect intercepts

If the inequality sign contains an equality (≥,) graph a solid line. If the inequality does **not** contain an equality (>), graph a dotted line.

After graphing the boundary line, pick a convenient point on either side of the line (like the origin) and substitute those values into the inequality. If that ordered pair makes the statement true, shade in that side of the line. If it does not, shade in the other side of the line.

Example 1 Graph $y \geq 3x + 2$

Graph the boundary, $y = 3x + 2$ - **solid line**

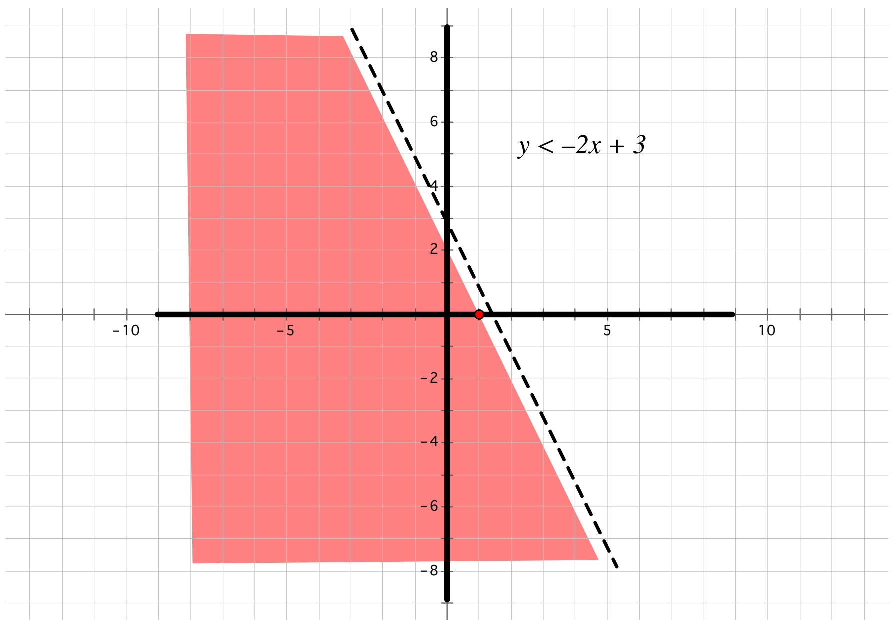


Choose a convenient point, (0, 0) and substitute into the inequality $y \geq 3x + 2$.

Is $0 \geq 3(0) + 2$? The answer is no, so shade the other side of the boundary line.

Example 2 Graph $y < -2x + 3$

Graph the boundary line - **dotted line**



Choose a convenient point (0, 0) and substitute into the inequality $y < -2x + 3$.

Is $0 < -2(0) + 3$? The answer is yes, shade in that side of boundary.

Now, If I did enough of these, I would see a pattern develop that would eliminate me having to choose a convenient point on one side of the line. I would notice when + y is positive and greater than (>), we always shade in above the line. If + y is less than (<), we shade in below the line.

Graph the following inequalities.

1) $y \geq 3x - 1$

2) $y < x + 2$

3) $y \leq 2x - 3$

4) $x + y < 1$

5) $3x + 4y \geq 12$

6) $x - y \leq 1$

7) $5x - 2y > 10$

8) $y \leq x$

9) By inspection, determine if you shade above or below the boundary line.

II Test Eqns of Lines Name________________________________
Date____________________________

****

1. ***Define 'Slope"

2. ***Write the Point Slope form of a line and its use

3. ***Write the procedure for graphing using the Slope Intercept form of a line.

4. ***In the General Form of a line, Ax + By = C, write the procedure for graphing.

5. ***Write the procedure for graphing an inequality

6. ***What theorem was used to determine perpendicular lines have negative reciprocal slopes?

7. **Find the slope of a line that passes through (2, –4) and (5, 9)

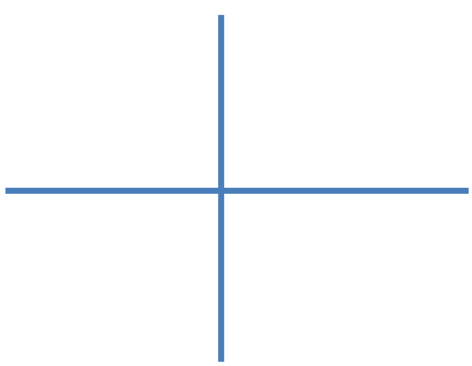
8. ** Find an equation of a line that passes through (2, 5) and has slope 3.

9. ** Find an equation of a line that passes through (1, 3) and (5, 11).

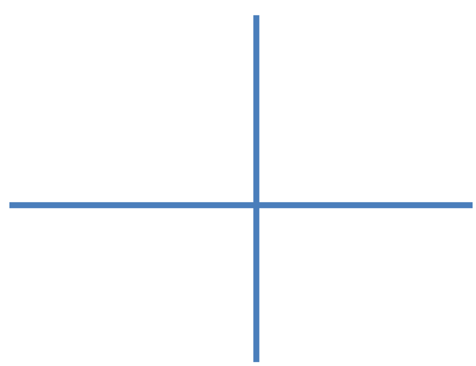
10. **Find an equation of a line that passes through (3, –1) and is parallel to $y = 4x–2$

11. **Find an equation of a line that passes through (1, –1) and is perpendicular to $4x – y = 8$

12. ** Graph $y = 3x – 4$



13. ** Graph $y = 3/5\ x + 1$



14 ** Graph $4x - 3y = 12$

15. **Graph $5x + 2y = 10$

16.** Graph $y \leq 4x - 3$

17. Graph $4x - 3y > 12$

18. ** Graph $y = |x| + 1$

!9. ** Graph

$$y = \begin{cases} 2x + 1; & x > 0 \\ -1/3\ x + 2; & x \leq -1 \end{cases}$$

20. ***Provide contact information for a parent/guardian, home or cell phone, email address or address (CHP)