Ch. 10 Graphing Parabola

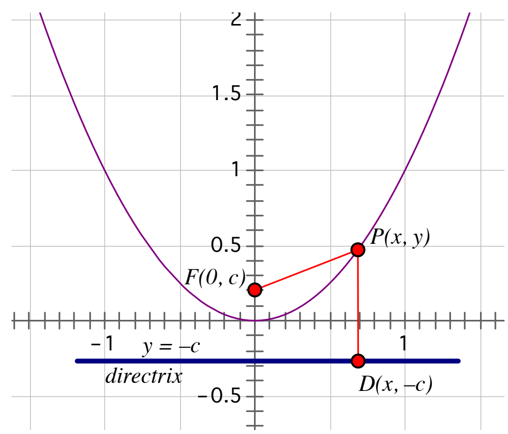
When we first learned to graph, we plotted points. We made x-y charts, picked convenient values of x, found corresponding values of y, plotted points, then connected points. We eventually found easier ways to graph equations of lines by identifying patterns. That's our intention in this chapter. To be frank, if you tried to graph higher degree equations by plotting points - you'd never get that done accurately. So, let's get started.

Parabolas

A parabola is a set of points P whose distance from a fixed point F, called the *focus*, is equal to the perpendicular distance from P to a line, called the *directrix.*

Since this curve is being defined by distances – we need to know the distance formula.

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$



By the definition of a parabola, we know FP = PD. Substituting those coordinates into the distance formula, we have

$$FP = PD$$

$$\sqrt{(x-0)^2+(y-c)^2}=\sqrt{(x-x)^2+(y+c)^2}$$

Squaring, $x^2 + (y - c)^2 = 0^2 + (y + c)^2$

Expanding $x^2 + y^2 - 2yc + c^2 = y^2 + 2yc + c^2$

Subtracting c^2 & y^2 $x^2 - 2yc = 2yc$

$$x^2 = 4yc$$

$$\frac{1}{4c}x^2=y$$

so,

This is an equation of a parabola with vertex at the origin and *c* being the distance between the Focus, F, and the origin and the origin and the directrix.

Mathematically, we write {(x, y)/ $y = \frac{1}{4c}x^2$ } is the graph of a parabola with focus F(0, C) and directrix with equation y = – c.

For our purposes, to make things easier, since we are just graphing equations in the form $y = x^2$, we will replace $\frac{1}{4c}$ with a different variable; ***a, and let a = 1.*** so our new equation is: $y = x^2$. We'll call that our Parent Function.

One way to graph functions is by plotting points - just like what was done originally when graphing equations of lines. We found better ways to graph those equations using either the Point-Slope Form of a Line or the General Form of a Line. Using either one of those two methods allowed us to graph linear equations quickly by inspection.

We'd like to do the same type activity with quadratic equations, in this case, equations of parabolas. So, the first thing we need to recognize an equation in two variables with only one of the variables squared will be a parabola. Then, as always, we look for patterns.

General Form of an Equation of a Parabola

$$y = ax^2 + bx + c$$

4 Methods of Graphing Parabolas

1. **Plotting points - not recommended**
2. **Vertex Form of Parabola with $y = x^2$, using translations**
3. **Converting General to Vertex Form - not recommended**
4. **Using General Form**

To begin with, let me say that if the equation to be graphed is already in Vertex Form, use that. If the equation is written in General Form, don't convert it to Vertex Form, just use –b/2a to find the vertex,

Method 1. Plotting points

While this worked for graphing linear equations, it won't necessarily work for quadratic or higher degree equations. If you didn't have an idea what the graph of a higher degree equation looks like, then plotting points will almost never get you there and won't give you important parts of a graph. For that reason, we don't use just plotting points.

Just as we did with absolute value graphs, we will move our parent function for a parabola, $y = x^2$, and move the graph along the coordinate systems by translation as we did with $y = |x|$. Absolute Value: **$y = a|x - b| + c$. You can see the similarity in equations.**

Method 2 Vertex Form - Parent Function using translations

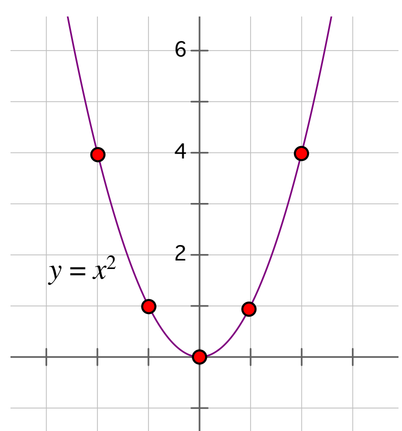
$$y = a(x-h)^2 + k, \quad \text{vertex } (h, k)$$

Writing an equation in, what we call, **vertex form** allows us to immediately identify the vertex, then move the graph around the coordinate system using translations. This will be done just like we graphed absolute value equation above. So first, let's graph the parent function, $y = x^2$ for a parabola.

Example 1 **Graph $y = x^2$**

We can make an xy-chart and can use any values of x and find the corresponding y values, then plot points.

x	y
0	0
1	1
2	4
–1	1
–2	4



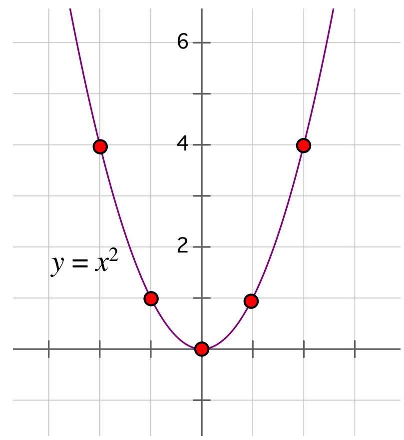
The point (0,0) is the vertex. I'll write the vertex in red on the chart above. I could then plot the other points.

Now, rather than plot *any* points, I will pick convenient values of x on **both** sides of the vertex; then find their corresponding y values. I'm going to do that because an equation in the form of $y = x^2$ is the parent function of a parabola and it is symmetric to the y-axis.

So, let's redo Example 1, picking the same points, but listing them in order from least to greatest so I have points on both sides of the vertex.

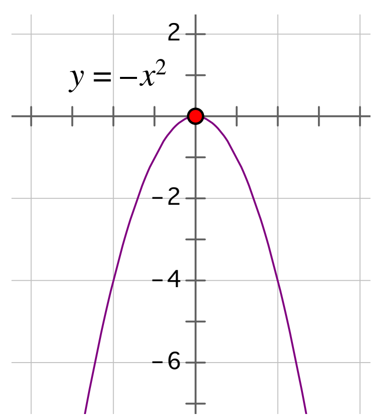
Example 1a. **Graph $y = x^2$**

x	*y*
–2	**4**
–1	**1**
0	**0**
1	**1**
2	**4**



Those are the ordered pairs when graphing $y = x^2$, the parent function of a parabola. Notice the x's go in order, and to find the corresponding y values, I just substituted the x values into the equation. See the chart.

Example 2 Graph $y = -x^2$



I'm going to pick my x's like I just did, –2, –1, 0, 1, 2. The only difference in the y values is the NEGATIVE in front of the x^2 , that results in all the y-values being negative. Other than that, the two charts match. LOOK at them!

x	y
–2	–4
–1	–1
0	0

1	–1
–2	–4

I plot those points, the graph looks the same but is reflected across the x-axis. Tipped upside down

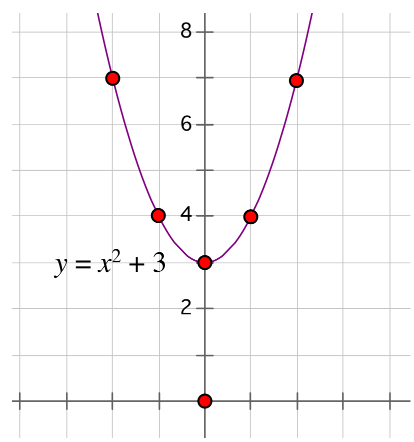
Now, let's look to see what happens if I add 3 to the equation as shown in the next example.

Again, I will use –2, –1, 0, 1, 2 for my x-values. Using the chart - the translation, all my y coordinates increase by 3 from the parent function. Or I could have substituted the values of x into the equation and found the same thing. Let's see.

Example 3 Graph $y = x^2 + 3$

Again, we can make an xy-chart on the left or plot points

Parent Fct		$y = x^2 + 3$	
x	y	x	y+3
–2	4	–2	7
–1	1	–1	4
0	0	0	3
1	1	1	4
2	4	2	7



If we did enough of these graphs, we would notice a few things: First, anytime we have an equation with only one variable squared, we always get this bell shaped curve, a parabola. The very top or bottom of the parabola is called the vertex. The maximum or minimum points of the graph.

So, in Examples 1, 1a and 2 , the vertex was at (0, 0). In the third example above, the vertex was at (0, 3), it was moved up 3 units. The graph is also symmetric with respect the to y-axis ($x = 0$). It appears when I add a constant to the quadratic term, that moves the graph up and down.

So, to generalize, an equation in the form $y = x^2 + \boldsymbol{k}$, the $\boldsymbol{k}$ moves the SAME graph vertically $\boldsymbol{k}$ units as we can see in the last example, $y = x^2 + 3$. Notice in the chart above all my y valuers were increased by 3 from the parent function.

In another equation such as, $y = x^2 – 5$, I have the SAME graph but all the points would be moved down 5 spaces from the vertex (origin)

If the coefficient of the quadratic term is negative, $y = – x^2$, then we get the same curve as $y = x^2$, except its reflected across the x axis.

Let's look at what happens when we have an equation $\mathbf{y = (x – h)^2}$**.** Again, we use our agreed upon x-values based on the parent function.

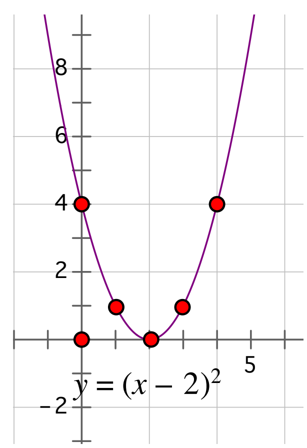
If you look at the following graph, it looks exactly like the graph of $y = x^2$, except it has been moved (translated) over 2 units to the right. The vertex is at (2, 0). Again, if I do enough of these graphs, I might see a pattern. The graph of an equation $y = (x – \boldsymbol{h})^2$, moves the graph horizontally $\boldsymbol{h}$ units.

Let's look at the graph, using our convenient values of x from the chart and find the values of y. Notice, I add 2 to my x-coordinates

Example 4 Graph $y = (x – 2)^2$

By making an xy-chart, we can plot points.

Parent Fct		$y = (x-2)^2$	
x	**y**	**x**	**y**
–2	4	0	4
–1	1	1	1
0	0	2	0
1	1	3	1
2	4	4	4



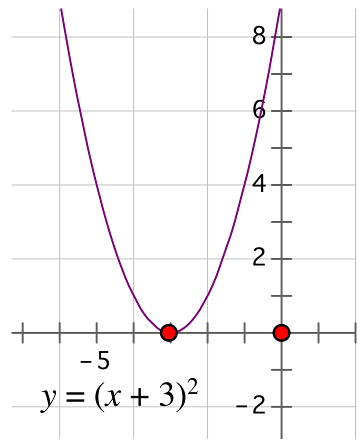
The vertex is over 2 units to the right of the parent function. Interesting, it appears that if we add 2 to all the x values and keep the corresponding y-values, just as if we substituted those values - we get the ordered pairs.

This is great because if we recognize the equation as a parabola and know the graph of the parent function, then we can quickly sketch the graph of longer equations.

Example 5 Sketch graph $y = (x + 3)^2$

I could make an xy-chart, but I'm going to use the pattern of translating the graph 3 units to the left. The vertex is at (– 3, 0)

Notice in this problem, we had an $(x + 3)^2$, and the vertex was moved 3 units to the left of the parent function.



All the points on this graph are moved to the left 3 units.

If we looked at 4 or 5 of these equations and their graphs, we might see a pattern that would allow us to graph longer equations by inspection.

Let's pull this together:

Here's what we need to recognize:

$y = a(x – h)^2 + k$ is a parabola in Vertex Form

1. The graph of an equation with only one variable squared is a parabola
2. The vertex can be found using translations from the parent function, $y = x^2$, with vertex (0,0), by moving the graph vertically using ***k*** and horizontally using ***h***. The new vertex is at (h, k)
3. The graph is symmetric with respect to the line x = ***h.*** The line passes through the vertex and is called axis or line of symmetry.

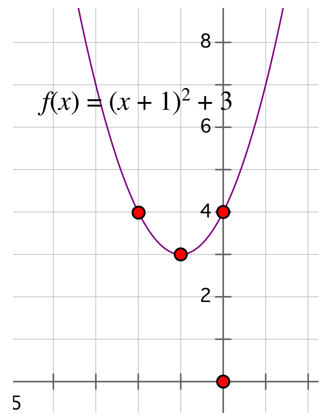
$y = a(x – h)^2 + k$, has a vertex at (h, k).

Recognizing the equation as a parabola and finding the vertex makes graphing a lot easier.

Let's make it easy by recognizing a pattern in this next example.

Example 6 **Sketch the** graph **y =** $(x + 1)^2 + 3$

By inspection, the graph is a parabola with vertex (– 1, 3). Which means the graph of the parent function, $y = x^2$ is translated over one unit to the left and up 3 units. It opens up because the a > 0.



If I pick convenient values of x to the left or right of the vertex, like x = 0, then y = 4 resulting in the point (0, 4). I can then use symmetry to find more points to sketch the graph : (–2, 4).

x = – 1 is the line of symmetry.

In Example 6, we graphed **y =** $(x + 1)^2 + 3$ and found the vertex was at (–1, 3) and line of symmetry at x = –1.

The equation is in **vertex form**. As can be readily seen, an equation in Vertex Form is written as a perfect square.

If we were asked to graph and plot points, I could have taken my convenient numbers and subtracted 1 from all my x-coordinates and added 3 to all my y-coordinates using translations. Or I could have simply substituted numbers into the equation. Since my vertex is at (–1, 3)

Parent Fct		New equation	
x	y	x – 1	y+3
–2	4	–3	7
–1	1	–2	4
0	0	–1	3
1	1	0	4
2	4	1	7

The x-coordinates in the chart were **not** just randomly chosen. We used the parent function to find the vertex at the origin, then picked two convenient values of x to the left of the origin and two numbers to the right. That gave us the x and y coordinates in the chart. Once we had those and determined the Vertex Form of a Parabola, I just moved the graph over (h, k) units from the origin.

Example 7 Identify the vertex and axis of symmetry of $y = (x - 2)^2 - 3$

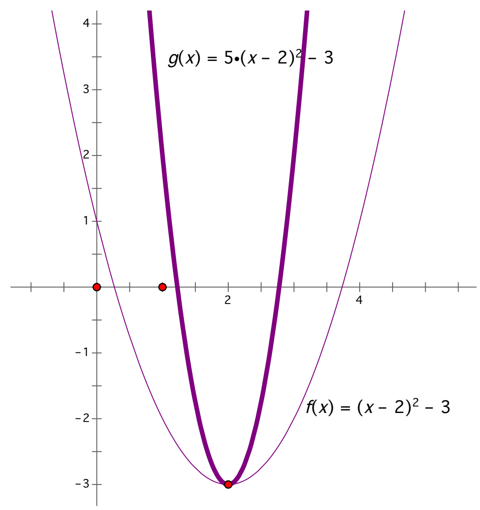
Immediately, we can see the vertex is at (2, –3), the axis of symmetry is x = 2. *We're done!*

To graph that, I would graph the vertex, then pick a couple of convenient points to the right of the vertex and graph those. Then use symmetry to graph 2 points to the left of the vertex.

Up till now, we have let ***a = 1***. Let's see how that affects the equation by redoing example 7 with ***a = 5***.

Example 8 Graph on the same coordinate axes: $f(x) = (x - 2)^2 - 3$

$$g(x) = 5(x - 2)^2 - 3$$



Notice the vertex is the same, the 5 stretched the graph vertically. So, to the graph g(x), I would graph the vertex, then use a couple of convenient values of x to find points, and symmetry to find other points.

As always in math, I can't make math any more difficult - only longer.

If $0 < a < 1$, a fraction, the graph of the parabola would be horizontally stretched - a wider bowl.

What's important is you can immediately recognize an equation in Vertex Form as a parabola, you can find its vertex by inspection and sketch a graph by picking convenient points and use the line of symmetry to find other points.

What if you were given an equation that wasn't in Vertex Form? Well, in the past, we changed what we did not know how to do into a problem we did. That just doesn't change.

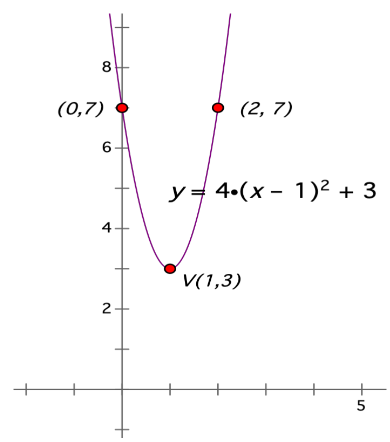
Let's look at this as a procedure:

Graphing Parabolas – Vertex Form

$y = a(x - h)^2 + k$, vertex (h, k)

Procedure

1. **Identify the vertex as (h, k) and plot point**
2. **Pick a convenient value of x and fund y-coordinate**
3. **Use symmetry to find a 3rd point to plot**
4. **Sketch the graph**



Example **Graph $y = 4(x - 1)^2 + 3$**

1. **Vertex is at (1, 3)**
2. **Let x = 0, then y = 7, (0, 7)**
3. **Use symmetry, 3rd point is (2, 7)**

From the vertex, we went over 1 to the left and up 4, so by using symmetry, we go over 1 to the right and up 4

Method 3 Converting General Form to Vertex Form

$y = ax^2 + bx + c$, Vertex (–b/2a, substitute)

What would happen if we were given an equation such as $y = x^2 + 6x - 1$ and asked to graph it? We could change the equation to Vertex Form by Completing the Square. I would not necessarily recommend that. But, let's look.

The first thing we should realize is only one of the variables is squared, so the graph of the equation is a parabola. If that's the case, how do we find the vertex? That's right, we have to change the equation into Vertex Form by completing the square. Let's do one.

$y = x^2 + 6x - 1$ Given

$y = x^2 + 6x + \_\_ -1 + \_\_$ Complete Square

3 (1/2 linear term)

$y = x^2 + 6x + 9 - 1 + (-9)$ Square, then Add & Subtract 9

$y = (x + 3)^2 - 10$ Factor/combine terms

The graph is a parabola that opens up with vertex (–3, –10).

Graphing Parabolas – General Form

$y = ax^2 + bx + c$

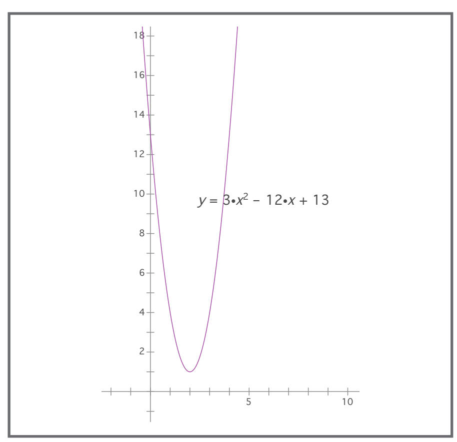
Remembering the Quadratic Formula; the –b/2a is the midpoint of the x-intercepts - the line of symmetry. So rather than going through the process of Completing the Square to find the vertex, all we need to do is use –b/2a as the x-coordinate of the vertex and substitute that value in the equation to find the y-coordinate of the vertex. From there, pick a convenient point, like zero, to find a second point. Then use symmetry to find a third point.

Strategy – In the equation $y = ax^2 + bx + c$: find the vertex using –b/2a, pick a convenient point and then use symmetry symmetry to graph.

Procedure

1. **Find the vertex letting x = –b/2a**
2. **Find y-coordinate by substituting that value into the equation**
3. **Pick a convenient point, 0 if possible, to find a second point**
4. **Use symmetry to find 3rd point**
5. **Sketch the graph**

Example Graph $y = 3x^2 - 12x + 13$

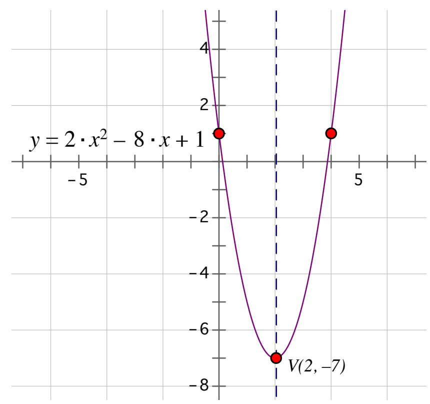


1. **Using –b/2a for x: –(–12)/6 = 2**
2. **Substitute to find y, $3(2)^2 - 12(2) + 13 = 1$ so V(2, 1)**
2. **Let x = 0, then y = 13; (0, 13)**
3. **Use symmetry, 3rd point is (4, 13)**
4. **Sketch the graph**

Example 9 Graph $y = 2x^2 - 8x + 1$

Using $\dfrac{-b}{2a}$, we have $\qquad \dfrac{-(-8)}{2(2)} = \dfrac{+8}{4} = 2$

So the vertex is at (2, plug 2 into original eqn.)$\longrightarrow$ (2, –7)

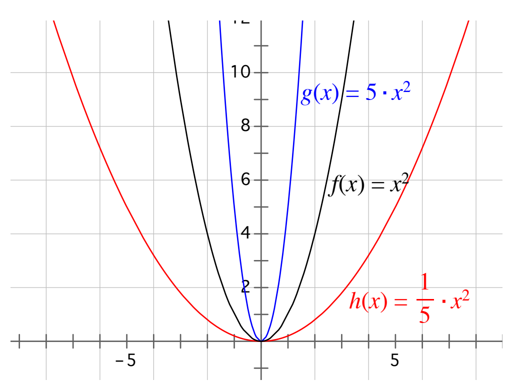


So, we can pick a convenient value of x, say 0, which is two units to the left of the vertex and find y, find the corresponding y, (0, 1) then use symmetry to find a third point (4,1) by going over 2 to the right of the vertex and up to the same height at 1.

After finding the vertex, we pick convenient values of x to find the respective y-coordinates. Personally, I like to let x = 0, finding the y-intercept, then use symmetry to find another point.

If we studied enough graphs of parabolas, we would see the value of ***a***, the coefficient of the quadratic term, tells us not only if the graph opens up or down, but if its stretched vertically or horizontally. We saw in Example 3, when ***a*** is negative, the graph opens down. If |***a***| is large, the graph is vertically (thinner) stretched. if |a| is small, the graph is horizontally (wider) stretched.

Let's look at the graphs of three equations on the same axes and see how the graphs looked different based only on the value of ***a***. Notice the graph in black is the parent function, $\mathbf{f(x) = x^2}$. When that was multiplied by 5, the blue graph got thinner, stretched vertically. When the parent graph was multiplied by 1/5, the graph got wider, horizontally stretched.



Finding Equations of Parabolas

Knowing the Vertex Form of a Parabola will also help us find an equation of parabola given some information.

$$y = a(x - h)^2 + k$$

Example 1 Find an equation of a parabola with vertex (3, 1) that passes through the point (5, 9).

Since we know the vertex, we know the values of *h* and *k* in the Vertex Form of a Parabola. We also know the graph passes through the point (5, 9) which means the value of y is 9 when x = 5. So the only thing we are missing is the value of ***a***.

Substituting those values into the Vertex Form of a Parabola, we have $9 = a(5 - 3)^2 + 1$

$$9 = a(2)^2 + 1$$
$$9 = 4a + 1$$
$$8 = 4a$$
$$2 = a$$

So, to write an equation of the parabola, I substitute the values of ***h, k,*** and ***a*** into the Vertex Form

$$y = 2(x - 3)^2 + 1$$

Problem Solving

When we look at parabola, we can see there is either a maximum or minimum value - point on the graph. When the value of the coefficient of the quadratic term, ***a***, is positive - we have a minimum. When ***a*** is negative, we have a maximum - look at the graphs!. Both of those occur, the maximum and minimum, occur at the vertex. We can use that information to solve problems.

Example 1 An object is thrown upward with an initial velocity of 128 feet per second. If the height above ground after t seconds is given by the formula $h = 128t - 16t^2$, when will it reach it's maximum height and what is the maximum height of the object?

The max or min on quadratic equations occurs at $-b/2a$ (vertex) in the equation $y = ax^2 + bx + c$
In this case, $b = 128$ and $a = -16$, substitute those numbers into $-b/2a$; $-128/-32 = +4$. So the max height occurs at 4 seconds.

Substitute 4 into the height equation; $h = +128t - 16t^2$

$$= 128(4) - 16(4)^2 = 256 \text{ feet}$$

The maximum height is 256 feet and occurs at 4 seconds.

Example 2 An object is thrown upward with an initial velocity of 128 feet per second from a height of 20 feet above ground. If the height above ground after t seconds is given by the formula $h = 20 + 128t - 16t^2$, when will it reach its maximum height and what is the maximum height of the object?

This problem is the same as problem 4 except the ball is thrown from a height of 20 feet. In the last problem it reached a max height of 256 feet, we just just added 20 to the height equation, final answer is 276 feet. Or we could work it out ;-)

Example 3 In a 110 volt circuit having a resistance of 11 ohms, the power W in watts when a current I is flowing is given by $W = 110I - 11I^2$. Determine the maximum power that can be delivered in this circuit.

We should recognize we have a quadratic equation, a parabola, so let's find the vertex.

Using $-b/2a$ to find when the max I occurs. $b = 110$, $a = -11$
$-b/2a = -110/-22 = 5$

Substitute $I = 5$ in the for power equation
$W = 110(5) - 11(5)^2 = 275$ watts

The maximum per is 275 watts.

Example 4 A building developer estimates the monthly profit **p** in dollars from a building **s** stories high is given by $p = -2s^2 + 88s$. To maximize his profit, how many stories should he build?

Using $-b/2a$, to find the max stories
$-b/2a = -88/-4 = 22$, to max his profit, he should build 22 stories.
The problem did not ask for his max profit, if it did, you would substitute 22 in the profit equation given.

As you can see, having good data, creating an equation, then using your knowledge of math makes good business sense. Just guessing at the number of stories might give you a profit - or even a loss - but the math allows you to maximize your profit.

Example 5 A shuttle operator charges a fare of $10 to the airport and carries 300 people per day. The owner of the shuttle service estimates for every dollar increase in fare, he will lose 15 passengers. Find the most profitable fare for him to charge.

Whoops - no equation was given. Let's see how we can address that.
Cost x people = fare
10 x 300 = $3000

For every increase of $1, he loses 15 people. Writing an equation, we have

$$(10 + x)(300 - 15x) = \text{fare}$$
$$3000 + 150x - 15x^2 = \text{fare}$$

Using $-b/2a$, $b = 150$, $a = -15$, we have $-150/-30 = 5$. The x coordinate of the vertex. So he should charge $15.

If he increased the fare by \$5 to \$15, he would maximize his profit. Substituting 5 into our equation, we have

$$(10 + 5)(300 - 15(5)) = \text{fare}$$
$$15\ (225) = \$3375$$

Knowing the math increased his income from fares from \$3000 to \$3,375. An increase of \$375.00

Summary

Graphing Parabolas

3 Ways to Graph a Quadratic – Parabola

a) Plotting points (not recommended)

b) Vertex Form, $y = \boldsymbol{a}(x - \boldsymbol{h})^2 + \boldsymbol{k}$, $\boldsymbol{a = 1}$ 1)*. Plot* **V*(h, k),*** 2. Use (h, k), the vertex, to translate the graph.

c) Using General Form, $y = \boldsymbol{ax^2 + bx + c}$, find vertex by placing ***–b/2a***, then substituting that value into the equation to find the y-coordinate of the vertex. Find a 2nd point by picking a convenient value of x and corresponding y, then find a 3rd point by symmetry.

Problem Solving

When problem solving, know the maximum and minimum values occur at the vertex. To find the vertex, use *–b/2a, to find the 1st coordinate, where the max or min occurs. To find the max value, substitute –b/2a into the equation.*