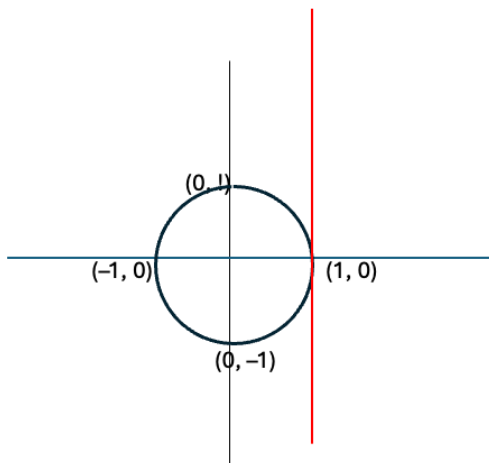


Circular Functions – Radian Measure

The good news, all of the information learned from right triangle trig will be applied to trig and circular functions.

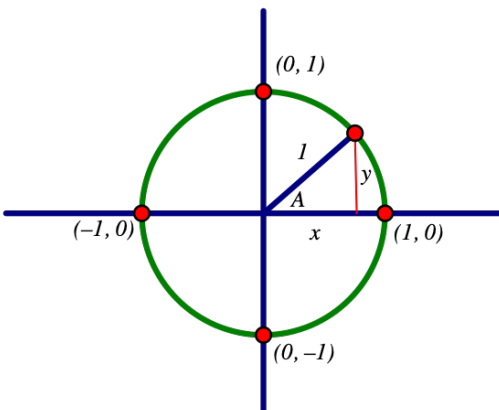
Let's look at a circle of radius 1 with center at the origin and a number line tangent to that circle at $(1, 0)$. Now, imagine the number line is flexible and can be wrapped around that unit circle in a counterclockwise direction.

That wrapping results is in each number on the number line being paired with points on the circle.



As we wrap the number line around the unit circle, more than one number from the number line will be paired with a point on a circle

Now, if I extend the trig ratios to the rectangular coordinate system and draw a circle of radius 1 around the origin, we will see how those coordinates are related to the special right triangles.



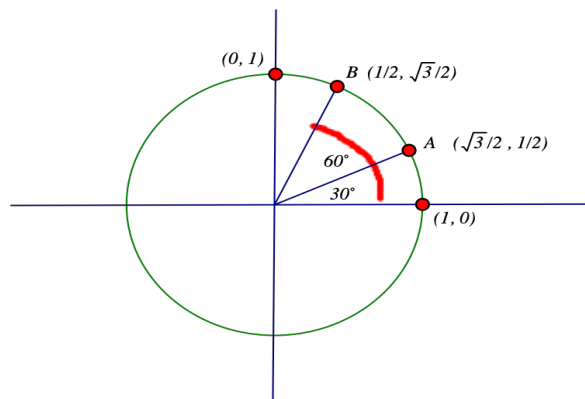
Putting this altogether, we have the trig ratios from the two special right triangles, the cosine and sine being defined on the unit circle ($r=1$) as the x and y coordinates respectively, and understanding that because the triangles that make up the trig ratios are similar, the ratios are equal.

Using that we see the $\cos A = \frac{x}{1}$, so the $\cos A = x$

And the $\sin A = \frac{y}{1}$, so $\sin A = y$

Keep this in mind, the ordered pairs (x, y) corresponds to **(cosine, sine)**

Using the 30-60-90° triangles above, we know the $\sin 30^\circ = 1/2$ and the $\cos 30^\circ = \sqrt{3}/2$. We also see the $\sin 60^\circ = \sqrt{3}/2$ and the $\cos 60^\circ = 1/2$.

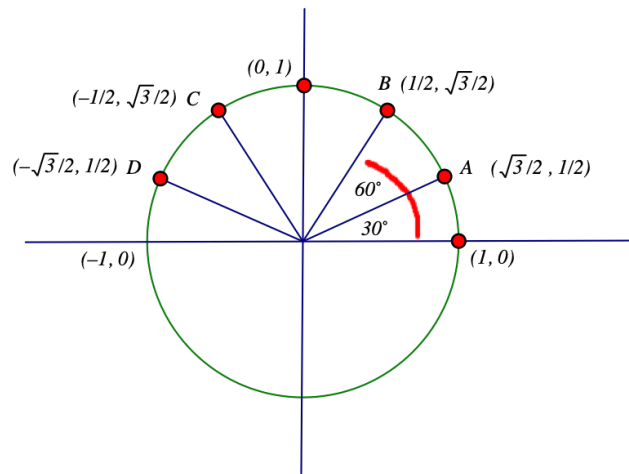


The terminal side of the 30° angle intersects the circle at A. How was I able to label the ordered pair $(\sqrt{3}/2, 1/2)$? Well, I looked at the special right triangle to come up with the ratios and based on the previous drawing, we **defined the cosine as the x-coordinate and the sine as the y-coordinate** using SOHCAHTOA.

The same is true for the 60° angle whose terminal side intersects the unit circle at B. Using the special right triangle, the $\cos 60^\circ = 1/2$, the $\sin 60^\circ = \sqrt{3}/2$ and I write them as an ordered pair. **To help you remember the ordered pairs, my suggestion is the “c” in cosine comes before the “s” in sine as the “x” comes before the “y”.**

Notice the graph of the unit circle intersects the y-axis at $(0, 1)$. Therefore, the $\cos 90^\circ$ (the x-coordinate) is 0, the $\sin 90^\circ$ (y-coordinate) is 1.

Either using “reflections” across the y-axis and our understanding of similar triangles, we can come up with more coordinates in the second quadrant.



Notice the ordered pairs in the second quadrant, the only thing that seems to change is the sign of the x-coordinate. Point C would represent 120° and point D would represent 150° . So, through symmetry we can find those ordered pairs.

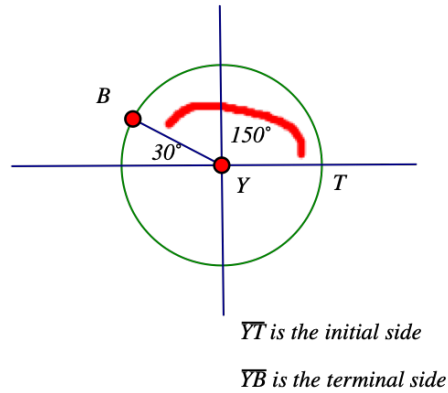
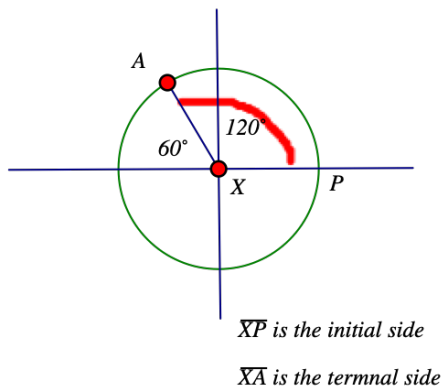
Reference Angles – Very Important

Unfortunately, we don't have special triangles that have 120° or 150° angles. But, if we use our knowledge of similar triangles, we can see an angle of 120° corresponds to a triangle of 60° . We can also see the 150° angle would correspond to a 30° angle in the similar triangles.

Because we see these relationships, we are able to find other angles based on the angles in the 1st quadrant. The angles in the 1st quadrant will be called **reference** angles. This definition of a reference angle is important and is used!

A **reference** angle is the acute angle formed by the terminal side of an angle in standard position and the x-axis.

Another way of saying this is the terminal side of a 150° angle is 30° above the x axis. The terminal side of the 120° angle is 60° above the x-axis. That would suggest that the coordinates of a 150° and 30° angle would be the same - except for the sign of the x-coordinate.



We can continue this process into the third and fourth quadrants and see the pattern of the coordinates with only sign changes. That means we can look at a 210° angle and realize that is 30° below the x-axis, a 240° angle is 60° below the x-axis, a 300° angle is 60° below the x-axis in the 4th quadrant and a 330° angle is 30° below the x-axis. Since all these coordinates can be found using angles in the first quadrant and similar triangles or reflections, we call them reference angles. You just need to remember the signs.

In the above problems, we used the special right triangle; $30\text{-}60\text{-}90^\circ$. We could also use the $45\text{-}45\text{-}90^\circ$ special triangles to find angles of 45° , 90° , 135° , 225° , 270° and 315° .

If you are not using the special triangle relationships, you will use a table or calculator and have to remember, as before, to assign the correct sign depending upon the quadrant that terminal side of the angle is in.

Example 1 Find the reference angle that corresponds to an angle that measures 170° .

I can draw a picture to make that determination, but I might be able to visualize that 170° is 10° above the x-axis. So the reference angle is 10° .

Example 2 Find the reference angle that corresponds to an angle that measures 140° .

The terminal side is 40° above the x-axis, therefore the reference angle is 40° .

Example 3

Find the reference angle that corresponds to an angle that measures 200° .

The terminal side lies in the third quadrant, 20° below the x-axis, therefore the reference angle is 20° .

Example 4

Find the reference angle that corresponds to an angle whose measure is 300° .

Since the terminal side of that angle lies in the fourth quadrant, 60° below the x-axis, the reference angle is 60° .

Example 5

Find the reference angle that corresponds to an angle whose measure of -200° .

The terminal side of an angle of -200° lies in the second quadrant, it is 20° above the x-axis, so the reference angle is 20° .

Example 6

Find the reference angle that corresponds to an angle whose measure is 400° .

A 400° angle does one complete rotation (360°) and goes another 40° . Therefore the reference angle is 40° .

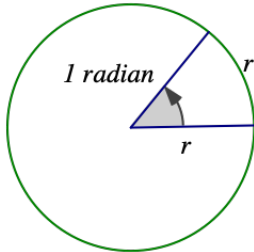
Example 7

Find the reference angle that corresponds to an angle whose measure is 510° .

A 510° angle does one couple rotation (360°) and results in an angle whose terminal side lies in the second quadrant at 150° . An angle of 150° is 30° above the x-axis, so the reference angle is 30° .

Radian Measure

When a central angle intercepts an arc that has the same length of the radius of the circle, the measure of the angle is defined to be one radian.



Because the circumference of a circle is $2\pi r$, there are 2π radians in any circle. Therefore, 2π radians = 360° or π radians = 180° .

That leads us to the following proportion:

$$\frac{d^\circ}{180^\circ} = \frac{r \text{ radians}}{\pi \text{ radians}}$$

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Example 1 Find the radian measure of an angle of 60° .

Using the proportion and filling in the degree measure, 60°

$$\frac{60^\circ}{180^\circ} = \frac{r \text{ radians}}{\pi \text{ radians}}$$

$$60\pi = 180r$$

$$\frac{60\pi}{180} = r$$

$$\frac{\pi}{3} = r$$

Example 2 Find the degree measure of $-3\pi/4$ radians.

$$\frac{d^\circ}{180^\circ} = \frac{-3\pi}{4} \frac{\text{radians}}{\pi \text{ radians}}$$

$$d\pi = 180 \frac{-3\pi}{4}$$

$$d = 45(-3)$$

$$d = -135^\circ$$

Rather than substituting values into the proportion, we can convert a set of units to another set by multiplying by 1 in terms of what is given and what we want to convert to.

For instance, to convert 3 miles to feet, we start with 3 miles and multiply by 1 in terms of feet and miles: $\frac{5,280 \text{ feet}}{1 \text{ mile}}$

$$3 \text{ miles} \times \frac{5,280 \text{ ft.}}{1 \text{ mi}} = 15,840 \text{ ft}$$

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To convert degrees to radians, multiply by $\frac{\pi \text{ radians}}{180^\circ}$

To convert radians to degrees, multiply by $\frac{180^\circ}{\pi \text{ radians}}$

Example 3 Convert 45° to radians.

Multiply 45° by 1; $45^\circ \cdot \frac{\pi \text{ radians}}{180^\circ}$

Simplifying, we have $\pi/4$ radians

To help you remember this, multiply by 1; if you want radians, then radians should be in the numerator. If you want degrees, then degrees should be in the numerator.

Example 4 Convert $5\pi/6$ to degrees.

Multiply $5\pi/6$ by 1

I want to convert to degrees, so I use the

factor that $\frac{180^\circ}{\pi \text{ radians}}$ has degrees in the numerator -

$$\frac{5\pi}{6} \frac{180^\circ}{\pi \text{ radians}} = 150^\circ$$

Now, let's look at our reference angles in terms of radian measure. We want to know our special triangles. In essence, from our formula, we multiply the degree measure by π and divide by 180.

Example 5 Convert 30° , 45° and 60° to radians.

$$30[\pi/180] = \pi/6$$

$$45[\pi/180] = \pi/4$$

$$60[\pi/180] = \pi/3$$

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It's that easy and you should be able to do those in your head.

The simplest way to convert radians to degrees is let $\pi = 180^\circ$

$$\pi/6 = 30^\circ$$

$$2\pi/6 = 60^\circ \quad \text{---} \quad 2\pi/6 = \pi/3 = 60^\circ$$

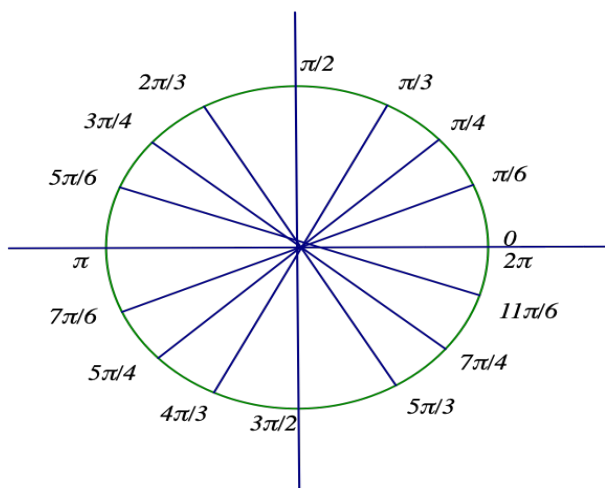
$$3\pi/6 = 90^\circ \quad \text{---} \quad 3\pi/6 = \pi/2 = 90^\circ$$

$$4\pi/6 = 120^\circ \quad \text{---} \quad 4\pi/6 = 2\pi/3 = 120^\circ$$

$$5\pi/6 = 150^\circ$$

$$6\pi/6 = 180^\circ \quad \text{---} \quad 6\pi/6 = \pi = 180^\circ$$

Using this information, we can draw the unit circle and label the angles in terms of radians.



By inspection, we can see

The reference angle of $5\pi/6$ is $\pi/6$

The reference angle of $2\pi/3$ is $\pi/3$

The reference angle of $3\pi/4$ is $\pi/4$

Example 6 Find the reference angle that corresponds to $11\pi/6$.

$11\pi/6$ is in the 4th quadrant and $\pi/6$ below the x-axis, the reference angle is $\pi/6$.

You should also know that $\pi/6 = 30^\circ$. So $11\pi/6 = 330^\circ$

Know these angles and their multiples!

$$\pi/3 = 60^\circ \quad \pi/4 = 45^\circ \quad 2\pi/3 = 120^\circ \quad \pi/2 = 90^\circ \quad \pi = 180^\circ \quad 2\pi = 360^\circ$$

The biggest difference between using angles in our special triangles and using SOHCAHTOA, we will be looking at angle measure using radians. **The hypotenuse becomes the terminal side of the triangle which turns to be the radius.**

That means, the notation we will use is because we use the x-axis

$$\begin{array}{ll} \sin \theta = y/r & \csc \theta = r/y \\ \cos \theta = x/r & \sec \theta = r/x \\ \tan \theta = y/x & \cot \theta = x/y \end{array}$$

My suggestion is to draw your special right triangles labeling the angles and sides and the rectangular coordinate system labeling the points on the axes as ready references for the quadrantal angles..

Example: Find the $\sin 120^\circ$ in terms of an acute angle.

The reference angle is 60° , 120° is in the 2nd quadrant, so the sine is positive. And using the 30-60-90 Δ , the $\sin 120^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}$

Example: Find the $\cos 225^\circ$ in terms of an acute angle.

The reference angle is 45° , 225° is in the 3rd quadrant, so the cosine is negative. Using the 45-45-45 Δ , the $\cos 225 = -\cos 45^\circ = -\frac{\sqrt{2}}{2}$

Example: Find the $\cos 3\pi/4$ in terms of an acute angle.

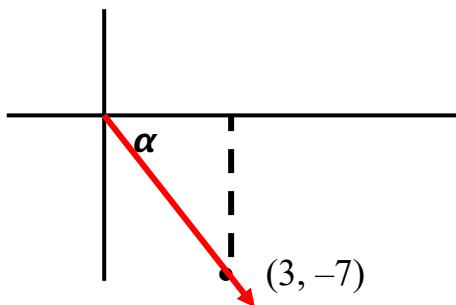
The reference angle is $\pi - 3\pi/4 = \pi/4$, it's in the 2nd quadrant, so the cosine is negative. Using the 45-45-45 Δ , the

$$\cos 3\pi/4 = -\cos \pi/4 = -\frac{\sqrt{2}}{2}$$

Because of circular functions relationship to triangles, it's very helpful to draw the angle on the x-y axes, construct a right triangle and use the Pythagorean Theorem to find trig values.

Example: If the terminal side of an angle contains the point $(3, -7)$, find the sine, cosine and tangent of that angle.

Draw the picture, draw the picture, draw the picture



Construct a rt Δ by connecting the point to the x-axis, label the two sides and use the Pythagorean Theorem to find terminal side, the radius (hypotenuse).

$x = 3, y = -7$, in 4th quadrant

$$\sin \alpha = y/r$$

$$\cos \alpha = x/r,$$

$$r^2 = 3^2 + 7^2$$

$$r^2 = 9 + 49$$

$$r = \sqrt{58}$$

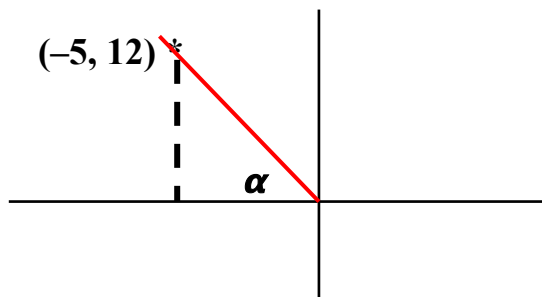
4th quadrant, sin is negative, cos is positive, therefore

$$\sin(\alpha) = \frac{-7}{\sqrt{58}}$$

$$\cos(\alpha) = \frac{3}{\sqrt{58}}$$

Example: Given that $(-5, 12)$ is on the terminal side of an angle α in standard position, find the sine, cosine and tangent.

Draw the picture, draw the picture, draw the picture



$$x = -5, y = 12$$

$$r^2 = 5^2 + 12^2$$

$$r = \sqrt{169}$$

$$r = 13$$

Second quadrant sine is positive, cosine is negative, tangent is negative

$$\sin \alpha = 12/13$$

$$\cos \alpha = -5/13$$

$$\tan \alpha = -12/5$$

Example: Given the $\sin \alpha = 5/13$ and the $\cos \alpha < 0$ find the other 5 trig values and the quadrant the angle is in.

Draw the picture, draw the picture, draw the picture

We can use the Pythagorean Theorem to find x, letting $y = 5$ and $r = 13$

OR

$$\cos^2 \alpha + \sin^2 \alpha = 1$$

$$\cos^2 \alpha + (5/13)^2 = 1$$

$$\cos^2 \alpha = 1 - 25/169 = 144/169$$

$$\cos \alpha = 12/13$$

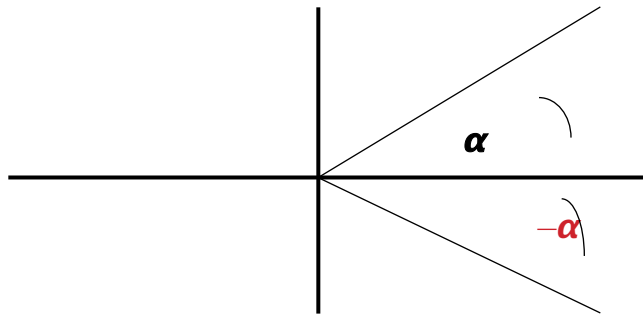
The sine is positive, the cosine is negative, therefore α is in quadrant 2.

$$\sin \alpha = 5/13, \quad \cos \alpha = -12/13 \quad \tan \alpha = -5/12$$

$$\csc \alpha = 15/5 \quad \sec \alpha = -13/12 \quad \cot \alpha = -12/5$$

Even and Odd Functions

To determine if a trig function is even or odd, we just look at their values on the rectangular coordinate system. We know by quadrant the signs of both sine and cosine.



The $\cos(\alpha)$ and the $\cos(-\alpha)$ are the x values and both positive, therefore the $\cos(-\alpha) = \cos \alpha$. These are called **even** functions. That means, anytime we take the $\cos(-\alpha)$ we can substitute $\cos(\alpha)$.

Let's use the same diagram for finding the $\sin(\alpha)$ and the $\sin(-\alpha)$. The sine represent the y values. Notice the values when α is positive lie above the x-axis and are therefore positive. When α is negative, the sign is negative. We can see the $\sin(-\alpha) = -\sin(\alpha)$. These are called **odd** functions.

Since the $\tan \alpha$ is the $\sin \alpha / \cos \alpha$, the tangent is an odd function. Using that logic, since the $\sec \alpha = 1 / \cos \alpha$, that's an even function.

Examples: $\cos(-45^\circ) = \cos 45^\circ = \frac{\sqrt{2}}{2}$

$$\sin(-30^\circ) = -\sin 30^\circ = -\frac{1}{2}$$

$$\tan(-\pi/3) = -\tan(\pi/3) = -\sqrt{3}$$