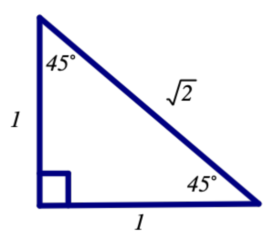
Graphing Trig Functions - Sine & Cosine

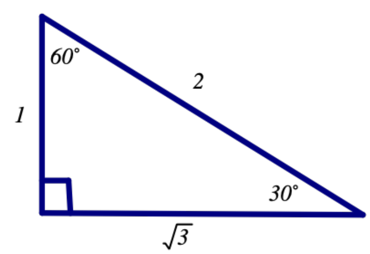
Sec. 1 Graphing the Sine and Cosine Functions

Up to this point, we have learned how the trigonometric ratios have been defined in right triangles. Most of us learned SOHCAHTOA as a memory aid. We then used that information to find the those ratios on a unit circle by using reference angles. Now, we will use that same information to graph trigonometric functions on the Cartesian Coordinate System.

You have to love how all this information ties together. So, let's use the angles in special triangles to help construct my graph and remembering the x-coordinate is the cosine and the y-coordinate is the sine.

From that information, I know these values in degrees and radians.
$\sin 0^\circ = 0$, $\sin 30^\circ = 1/2$, $\sin 45^\circ = \sqrt{2}/2$, $\sin 60^\circ = \sqrt{3}/2$, $\sin 90^\circ = 1$
$\sin 0 = 0$, $\sin \pi/6 = 1/2$, $\sin \pi/4 = \sqrt{2}/2$, $\sin \pi/3 = \sqrt{3}/2$, $\sin \pi/2 = 1$





Graphing Sine and cosine functions by an x-y chart

Just as we have seen to graph linear and quadratic equations initially, we will do the same thing using the general algorithm, that is:

1. Solve for y
2. Pick convenient values of x (special angles)
3. Find values of y, use SOHCAHTOA)
4. Graph the points,
5. Connect points

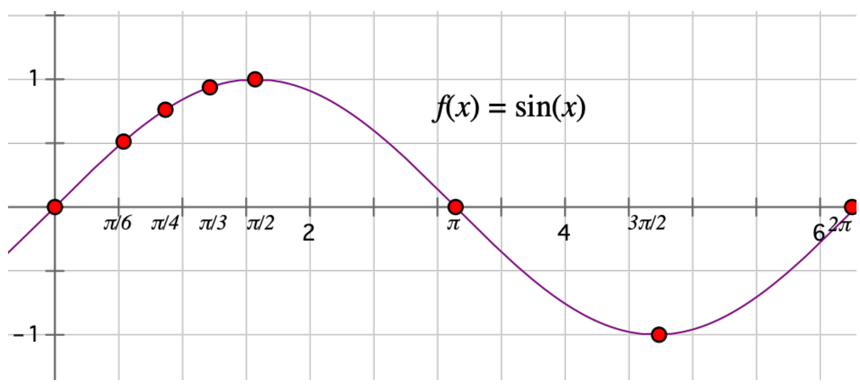
Example 1 Graph y = sin x for all x such that $0 \leq x \leq 2\pi$

x	y
0	0
$\pi/6$	$\frac{1}{2}$
$\pi/4$	$\frac{\sqrt{2}}{2}$
$\pi/3$	$\frac{\sqrt{3}}{2}$
$\pi/2$	1

I graphed only in the first quadrant with picking convenient values of x and finding corresponding values of y.

I could continue picking points or I can use my knowledge of reference angles and graph those points.

But, as you will see from below, I could use those reflections and determine the sign by the quadrant.



Notice, I labeled my x-axis in terms of radian measure ~ π. Once I relabeled in terms of π, I graphed the first 5 points in the first quadrant ($0 \leq x \leq \pi/2$). Since the y-coordinates are symmetric in the second quadrant, I can use symmetry to find the graph for $\pi/2 \leq x \leq \pi$. That gives me the graph in the $0 \leq x \leq \pi$.

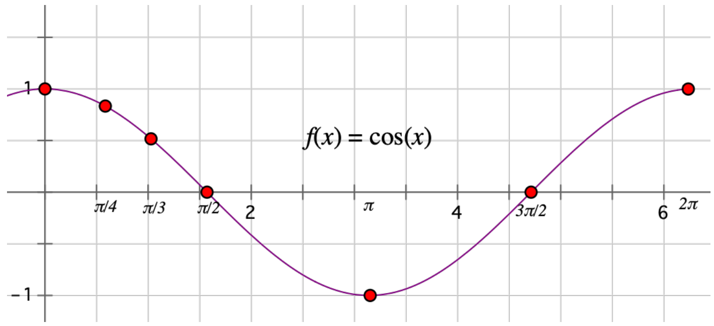
Angles greater than 180 and less than 360˚ fall in the third and fourth quadrants. In the third and fourth quadrants, the ratios are the same but the y-coordinates are negative. In the graph, we can see those points for the special angles fall below the x-axis.

My point, once I label the first few points in the first quadrant, $0 \leq x \leq \pi/2$, the rest is easy because the only difference in the y- coordinates is the "sign".

Using my special right triangles, let's find the values of the cosine, just like we did for the sine.

$\cos 0 = 1$, $\cos \pi/6 = \sqrt{3}/2$, $\cos \pi/4 = \sqrt{2}/2$, $\cos \pi/3 = 1/2$, $\cos \pi/2 = 0$

Example 2 Graph y = cos x, $0 \le x \le 2\pi$

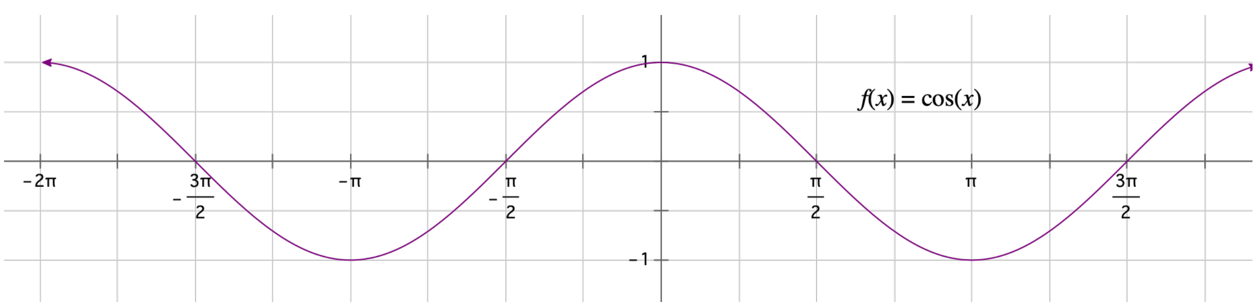


I labeled my x-axis in terms of π and used my special right triangles to find their corresponding values. Again, I only used values in the first quadrant because of symmetry. I know the cosine values (x-coordinates) are negative in the second and third quadrants so the graph should be below the line for $\pi/2 \le x \le 3\pi/2$.

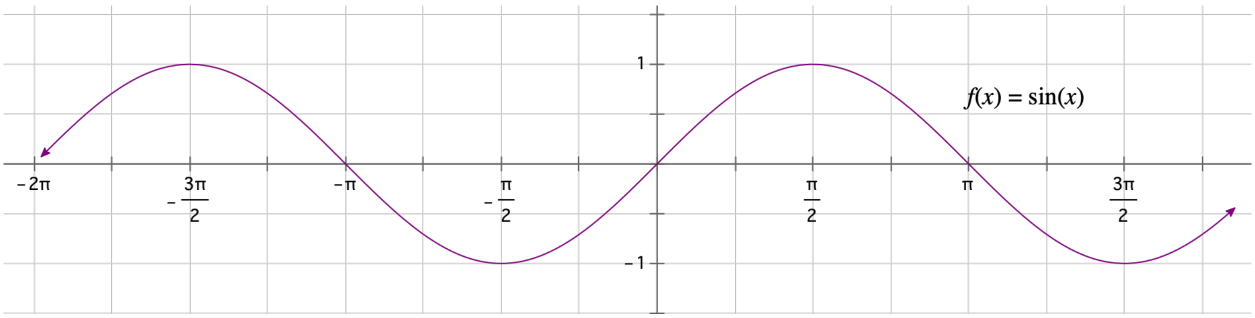
I also know the cos π = –1, so I graphed that point, the cos 3π/2 = 0, and the cos 2π = 1

You might notice that both graphs look like waves. If I started at x = 0 to graph, I see the sine curve's wave goes through the origin. At x = 0, cosine curve's wave passes through (0, 1). If I was able to shift one of the graphs, they would coincide.

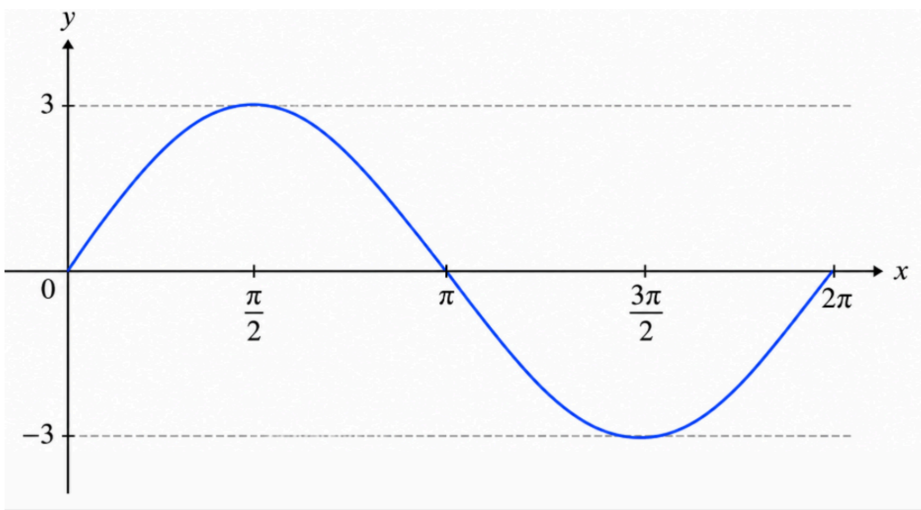
Remember when we looked at the unit circle and discussed reference angles. We indicated that an angle whose measure was 405˚ had a reference angle of 45˚. We got that by subtracting 360˚. Let me point out a couple of things; one, since we could make more than one rotation, the graph of the trig functions can go on infinitely, two, we said an angle whose measure is –210˚ has a a reference angle of 30˚ and that suggests the graph goes out infinitely in the negative direction, and three using the coordinates derived from the special right triangles, the waves keep the same size and shape.



So, the graph of the sine and cosine functions are continuous and go out to positive and negative infinity. And, just like on the unit circle, we see the repetition of the values.



Let's see what occurs if I have a coefficient in front of the sine, that is y = 3 sin x



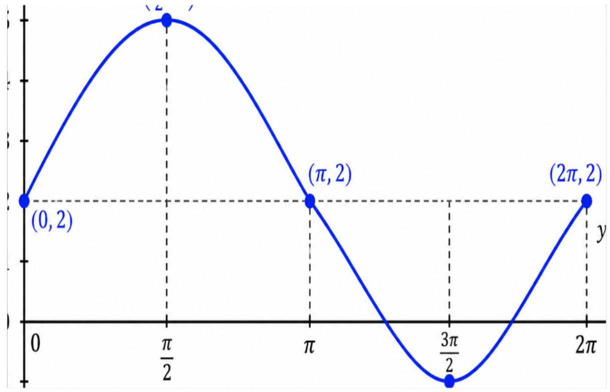
Notice, the height of the graph changed. Instead of going from a maximum of 1 to a minimum of –1, the height has gone from –3 to 3.

In the equation y = ***a*** sin x, the ***a*** is called the **amplitude.**

Also notice, the x-axis is in the middle of the graph, it's called the **midline**

Both graphs repeat at multiples of 2π. We say their period is 2π. As we saw on the unit circle and on these graphs, the maximum and minimum heights (y-values) are between one and negative one. We say their amplitude is 1 on the first graph and three and negative 3 on the second graph.

Now, let's add to the equation, y = 3 sin x and add 2, resulting in y = 3sin x + 2 and see what that does to the graph.



Notice the "+2" shifted the whole graph up 2 units. The midline is now 2, the amplitude is still 3, but the maximums and minimums went up and down 3 units from the midline 5 and –1.

Sec. 2 Transformations

And, the really good news, these graphs can be moved around the coordinate system just as we have done with our other graphs; parabolas, circles, absolute value, etc., using the same types of rules and notation.

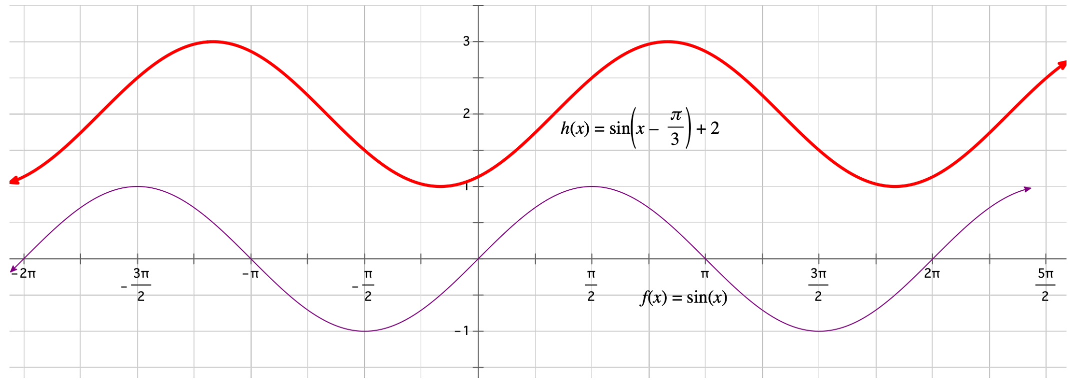
So, recall graphing a parabola, we know what $y = x^2$, looks like a U . If I changed the equation to $y = x^2 + 1$, the graph of the parabola is moved up one unit. The same is true for sine and cosine graphs, we know what the graph of y = sin x looks like. The graph of y = sin (x) + 1 moves the entire graph up one unit.

To move the graph horizontally, we looked at the value inside the parentheses. So, $y = (x – 2)^2$ moved the parabola over 2 units to the right. Similarly, the graph of y = cos (x – π/2) moves the graph of the cosine over π/2 units to the right.

And finally, if $y = 5x^2$ stretches the graph of the parabola. The same thing occurs with the sine and cosine graphs. That is, y = 5 sin x stretches the sine graph.

So, let's look at an example that translates the graph up 2 and over π/3 and see that graph looks exactly the same but is shifted.

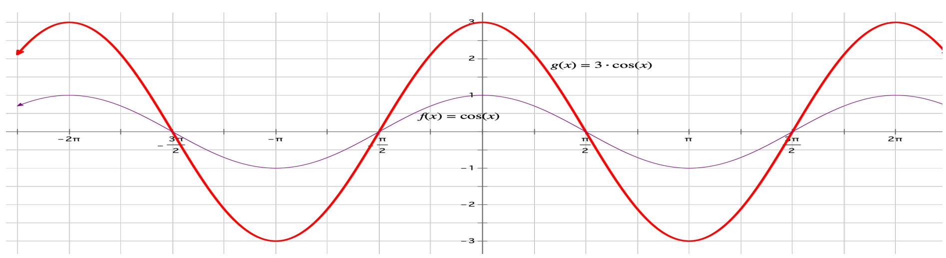
Example 3 Graph y = sin(x – π/3) + 2



Notice the graph of the curve is the same, but all the points were moved up 2 and over π/3.

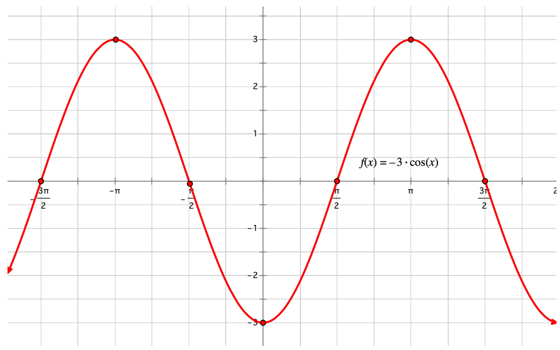
Example Graph y = 3 cos x

In this example, we can still recognize the graph of the cosine, but we can also see it has been vertically stretched by a factor of 3, the amplitude is 3.



What do think might happen to the graph if we multiplied the cosine by negative 3? y = – 3cos x. If you are not sure, try a couple of convenient values of x and see what happens.

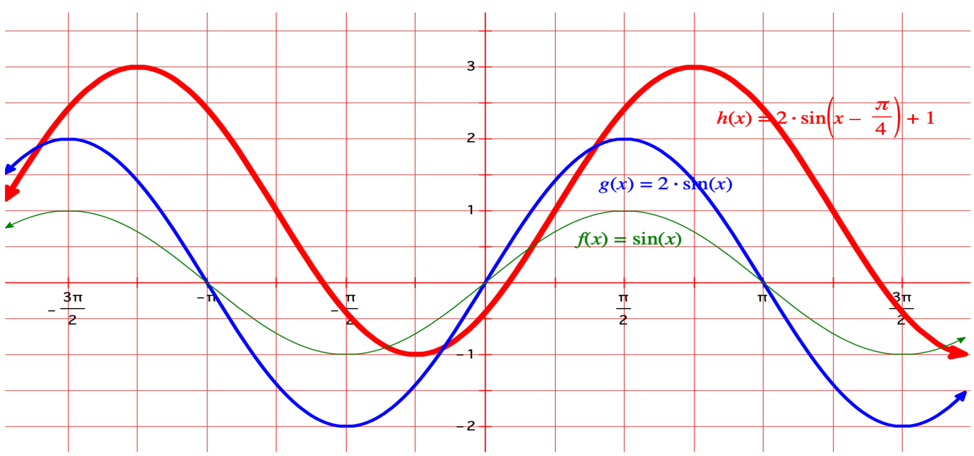
I graphed that by knowing it was going to be a wave that would typically start at (0, 1). But since the cosine was being multiplies by (–3), that my new graph should start at (0, –3). I then recalled my reference angles, at π/2 and 3π/2, x = 0, and at π, x = –1, but I was multiplying the cosine by (–3), so that would result in +3 on the graph. The rest was just symmetry



To graph these trig functions, you should be very aware of what the parent functions look like, then apply the stretch, and horizontal and vertical translations

Let's graph the next equation in pieces so we can see how the added components impact the parent function.

Example 5 Graph y = 2 sin (x – π/4) + 1

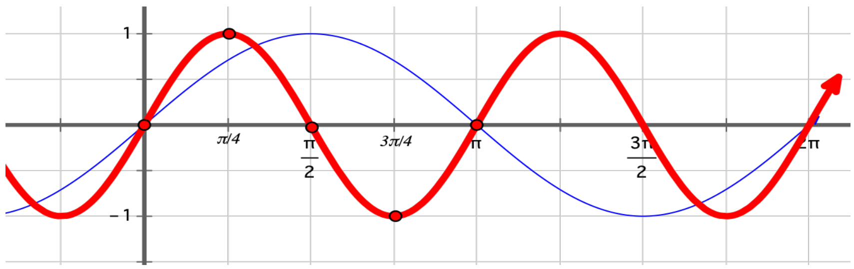


1. Graph the parent function in green, f(x) = sin(x)
2. Graph the vertical stretch, g(x) by multiplying all the y values by 2, the blue graph. Basically making the waves larger
3. Finally moving g(x), blue graph, over π/4 and up 1

All the waves we studied up to this point repeat every 2π. Can we make them repeat more or less often? Look at the following 2 equations;

$f(x) = \sin(x)$ and $h(x) = \sin(2x)$.

What is physically different in those two equations?



Now, we should be able to sketch f(x) pretty easily. Now, graph h(x) for values of 0, π/4, π/2 and π. Look at the red graph, what is the period, how often does the wave repeat?

What we have already discovered is if $f(x) = \sin bx$, $b > 0$ ranges between 0 and 2π we get one complete sine wave with amplitude 1. When $b = 1$, the period (one complete sine wave) is 2π. As we can see above, when $b = 2$, sin 2x, we got one complete sine wave at 2π/2 = π.

This suggests that the period of a function $f(x) = \boldsymbol{a}\sin(\boldsymbol{b}x + \boldsymbol{c}) + \boldsymbol{d}$ would be given by $2\pi/b$. This same argument could be used for graphing the cosine and later tangent functions.

And the phase shift (horizontal translation) would be determined by letting $bx + c = 0$ because the graph was defined between $0 \le x \le 2\pi$, so bx ranges between $-c$ and $2\pi - c$. That results in $x = -c/b$. Let's put this together in a procedure to make graphing these a whole lot easier.

General Form of Trig Graphs

Graph $y = a\sin(bx + c) + d$;

Procedure:

1. **Identify the midline** d
2. **Identify the amplitude** $|a|$
3. **Find the period** $\frac{2\pi}{b}$
4. **Identity the phase shift (bx + c = 0);** $\frac{-c}{b}$

Using that procedure, we can graph:

Example 5 Graph f(x) = 3sin(2x – π/2)

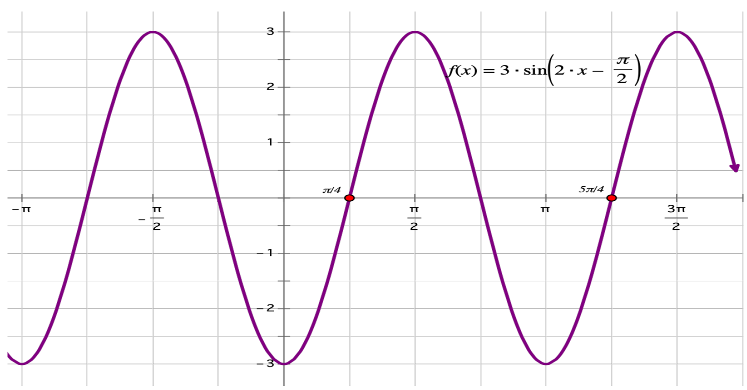
1. By inspection, the midline is 0,
2. Amplitude is 3
3. The period is 2π/b; 2π/2 = π
4. The phase shift is 2x – π/2 = 0, setting 2x = π/2, so the phase shift is π/4

That results in a complete sine wave between π/4 and 5π/4. Rather than just using that information, let's walk through our understanding.

To obtain the interval containing the sine wave, we let 2x – π/2 = 0 range from 0 to 2π. Solving, we have

2x – π/2 = 0	and	2x – π/2 = 2π
x = π/4		2x = 5π/2
		x = 5π/4

The amplitude is 3 and the period is 5π/4 – π/4 = π

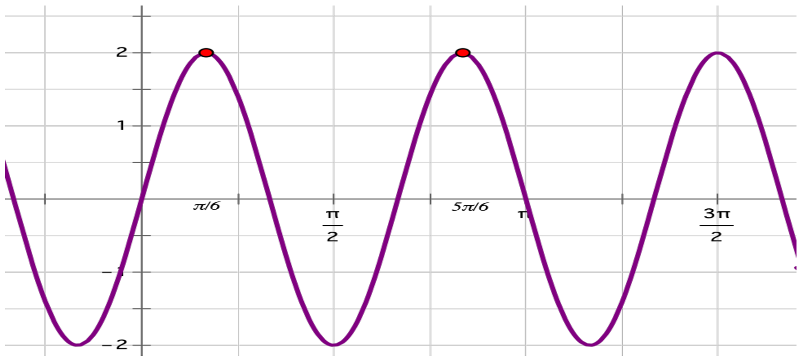


Notice: the one complete period began at π/4 and extended to 5π/4

Example 6 Graph y = 2 cos (3x – π/2)

Using our procedure,

1. the midline is 0
2. the amplitude is 2,
3. period, 3x = 2π results in 2π/3
4. and the phase shift is 3x – π/2 = π/6.



So, from the graph you can see the period went from $\pi/6$ to $5\pi/6$, so the period is $4\pi/6 = 2\pi/3$ and the shift was $\pi/6$ to the right.

Graph the following without plotting points – use the procedure.

1. $y = \sin x$	2. $y = 2 \sin x$
3. $y = \sin 2x$	4. $y = (1/3) \sin (x - \pi/4)$
5. $y = 3 \cos x$	6. $y = -2 \cos 2x$
7. $y = 2 \sin (2x - \pi/3)$	8. $y = 3 \cos (3x - \pi/2)$
9. $y = 2 \sin (2x - \pi/3) + 1$	10. $y = 3 \cos (3x - \pi/2) - 1$

Let's see all this helps us with periodic functions by seeing a few word problems solved. Like most everything in life, after doing a couple, they seem to get easier.

WORD Problems

1. **Using a periodic function, model a person's height on a ferris wheel using y = sin x where x measures time in seconds and y measures the height above ground.**

If rider on a Ferris wheel starts at the **midpoint height** of 50 feet and rises to a maximum of 90 feet and the wheel completes one rotation every 60 seconds. Write an equation to model a rider's height.

This is a periodic function because its repeating, round and round, Both the sine and cosine functions are periodic – they repeat. In this problem the person is starting at 50 feet and goes to a height of 90 feet. That means they go as low as 10 feet. S

Since they are starting at 50 feet and going around from there, 50 is the midline, the amplitude, going up 40 feet is the amplitude. The wheel goes around every 60 seconds, the period is defined as 2π/B, which means B = π/3. To determine to use the sine or cosine, we look where the ride starts, it starts at the midline, so we use the sine function and substitute the numbers.

Amplitude = $\frac{90-10}{2} = 40$
Midline = 50
Period = 60 → $B = \frac{2\pi}{60} = \frac{\pi}{30}$

$$y = \boldsymbol{a} \sin Bx + \boldsymbol{d}$$

$$h(t) = 40 \sin (\frac{\pi}{3} t) + 50$$

2. **At a beach, the water level is highest at 8 feet and lowest at 2 feet. The time between two high tides is 12 hours. Write an equation to model the water height.**

Since we are starting at high tide, a maximum height, we will use the cosine function. The midline (midpoint) is given by (8+2)/2 = 5, and goes up and down 3 feet, the amplitude is 3. It repeats every 12 hours, so the period is 3 resulting B = π/6

Amplitude = $\frac{8-2}{2} = 3$
Midline = 5
Period = 12 → $B = \frac{2\pi}{12} = \frac{\pi}{6}$

$$h(t) = 3\cos\left(\frac{\pi}{6}t\right) + 5$$

3. **The temperature varies from 60°F to 80°F over a 24-hour period. At midnight, the temperature is at the average and increasing. Write an equation to model the change in temperature.**

The average temperature, the midline, occurs at midnight and is increase. That suggests the sine function. Since the temperature range from 60˚ to 80˚, the midline is 70˚. The amplitude then is 10.

Amplitude = $\frac{80-60}{2} = 10$
Midline = 70
Period = 24 → $B = \frac{2\pi}{24} = \frac{\pi}{12}$

Filling in the information into the equations, we have: T(t) = 10 sin $(\frac{\pi}{12} t)$ + 70

4. **A pendulum swings with a maximum displacement of 5 cm from the center. It completes one full swing every 4 seconds. Write an equation for displacement.**

Since we begin the swing of the pendulum from 5 feet from the center, we are not starting at the midline, so we will use y = cos x. One complete swings takes 4 seconds, so the period is 4.

Amplitude = 5
Midline = 0
Period = 4 → $B = \frac{2\pi}{4} = \frac{\pi}{2}$

Filling the information into the cosine equation, we have D(t) = 5 cos $\frac{\pi}{2}$ t

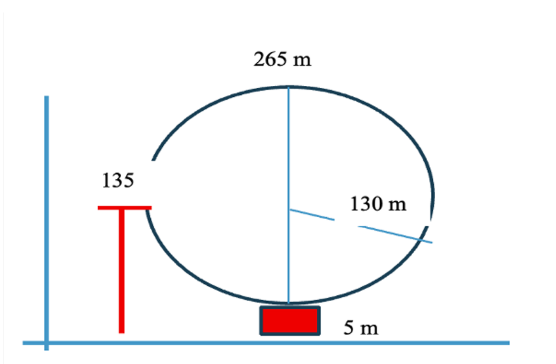
5. **A rotating beacon light varies in intensity from 0 to 100 units. It completes a full rotation every 10 seconds. At time $t = 0$, the intensity is at the midpoint and increasing. Write an equation for intensity.**

Amplitude = 50
Midline = 50
Period = 10 → $B = \frac{2\pi}{10} = \frac{\pi}{5}$

$$I(t) = 50\sin\left(\frac{\pi}{5}t\right) + 50$$

6. **A ferris wheel rotates every 30 minutes. If the diameter of the wheel is 260 meters and a person boards from a platform 5 feet above round, write an equation for height.**

It might be helpful to draw a picture. The period is 30 minutes, so B = 2π/30 = π/15. The amplitude, the radius is 130 meters and we are starting at the bottom of the ferris wheel to get on, that suggests the cosine curve. Here's where the picture will help; if the wheel starts at ground level, we would be going from zero to a height of 260 meters with a midline of 130meters. But, we are starting from a platform 5 feet above ground, so our midline is 135 meters and our radius, the amplitude, is 130 meters



Since we are starting at the bottom and going up, we will use the negative cosine.
Amplitude is 130
Period is 30 minutes, so B = $\pi/15$
Midline is 135

$$h(t) = -130 \cos\left(\frac{\pi}{15} t\right) + 135$$

Now, using that equation, like in the other problems, I could ask a question that you would use the equation. Such as, how high off the ground would you be in 10 minutes? Of course, you would substitute 10 for t in your equation.

Doing that, we have: $h(10) = -130 \cos\left(\frac{10\pi}{15}\right) + 135$ ----------→ $\frac{10\pi}{15} = \frac{2\pi}{3}$

$2\pi/3$ is in the 2^{nd} quadrant, so the cosine is negative: $-\frac{1}{2}$

$$h(10) = -130\left(-\frac{1}{2}\right) + 135$$
$$h(10) = 65 + 135$$
$$h(10) = 200$$

In 10 minutes, you would be 200 feet above ground