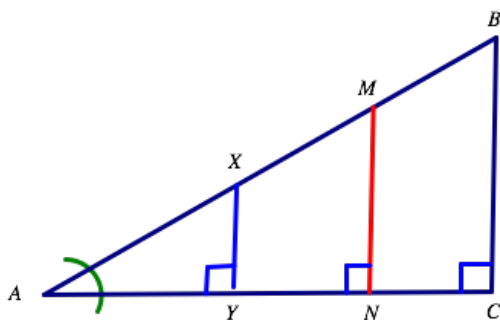


## Ch. Y Intro Right Triangle Trig

In our work with similar polygons, we learned that, by definition, the angles of similar polygons were congruent and their sides were in proportion - which means their ratios are the same. We also learned the Angle-Angle Postulate. That is, if two angles of one triangle were congruent to two angles of another triangle, the triangles were similar.

As we look at the ratios of the sides of similar triangles, we are going to use that information and name some of those ratios - an introduction to right triangle trigonometry. Look at the three triangles below -  $\triangle ABC$ ,  $\triangle AMN$  and  $\triangle AXY$ . They are similar because of the AA Postulate. Each triangle has a right angle and each triangle contains  $\angle A$ .



That means  $BC:MN$  have the same ratio as  $MN:XY$ . In fact, I could continue the relationships by saying  $BC:AB$  as  $MN:AM$  as  $XY:AX$ .

Let me write some of those relationships in words: the side opposite  $\angle A$  to the hypotenuse of each of the triangles,  $\triangle ABC$ ,  $\triangle AMN$  and  $\triangle AXY$  have the same ratio.

$$\frac{BC}{AB} = \frac{MN}{AM} = \frac{XY}{AX}$$

This is kind of important, so we are going to name that ratio. The side opposite an angle over the hypotenuse will be called the sine ratio, abbreviated sin, of that angle.

So,  $\sin A$  is equal to any of those ratios;  $\frac{BC}{AB} = \frac{MN}{AM} = \frac{XY}{AX}$

If we were to look at other ratios based on the above triangles, we would also realize because of similar triangles, these ratios are also equal;  $\frac{AC}{AB} = \frac{AN}{AM} = \frac{AY}{AX}$

Putting that ratio in words, the side adjacent to the angle over the hypotenuse will be called the cosine, abbreviated cos, of that angle.

So, the  $\cos A$  is equal to any of those ratios;  $\frac{AC}{AB} = \frac{AN}{AM} = \frac{AY}{AX}$

And one more set of ratios that would be equal are the ratio of the side opposite that angle over that adjacent side. That ratio is called the tangent, abbreviated tan.

Summarizing those names, we have

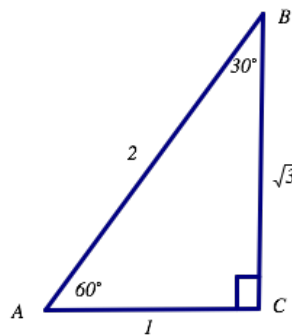
$$\sin A = \frac{\textit{opposite}}{\textit{hypotenuse}} \qquad \cos A = \frac{\textit{adjacent}}{\textit{hypotenuse}} \qquad \tan A = \frac{\textit{opposite}}{\textit{adjacent}}$$

A popular way to remember those ratios is by memorizing **SOHCAHTOA**.

The **S** stands for sine, **O** for opposite side, **H** for hypotenuse, **C** for cosine, **A** for adjacent side and **T** for tangent.

**Example 1** Using a 30-60-90° special right triangle, find the sine, cosine and tangent of 30°

Using **SOHCAHTOA**



$$\begin{aligned} \sin 30^\circ &= \frac{\textit{opposite}}{\textit{hypotenuse}} = \frac{1}{2} \\ \cos 30^\circ &= \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{\sqrt{3}}{2} \\ \tan 30^\circ &= \frac{\textit{opposite}}{\textit{adjacent}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \end{aligned}$$

**Example 2** Using the same 30-60-90° special right triangle, find the tan 60° and the cos 60°.

Using **SOHCAHTOA**

$$\tan 60^\circ = \frac{\textit{opposite}}{\textit{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\cos 60^\circ = \frac{\textit{adjacent}}{\textit{hypotenuse}} = \frac{1}{2}$$

Using your knowledge of special right triangles, you should be able to find the sine, cosine and tangent ratios for 30°, 45° and 60° angles. The ratios for other angles would have to be looked up in your book or found on your calculator.

We just said, based on our knowledge of special right triangles, the  $\cos 60^\circ = 1/2$ . If you were to use a table or your calculator, you would find the  $\cos 60^\circ = .500$ .

To find the  $\tan 42^\circ$ , use either your calculator or book and you will find that ratio of the opposite side to the adjacent side is always 0.9004. Because of similar triangles and the sides being in proportion, the ratio of the opposite side : adjacent side is called the tangent and the tangent of  $42^\circ$  will always be equal to 0.9004.

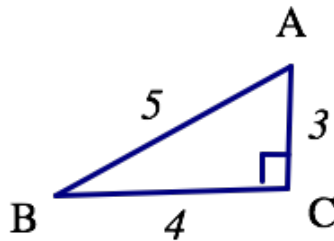
**Example 3** Use the tables in your calculator or in your book to find the sine, cosine and tangent of the following angles.

1. 30°
2. 60°
3. 40°
4. 63°

Find the measure of the angle with the following values (ratios).

5.  $\sin A = .5150$
6.  $\cos B = .7071$
- 7,  $\tan C = 1.3270$

**Example 4** Let's find the sin B, cos B, and the tan B



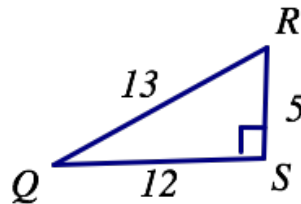
Remember **SOHCAHTOA**.

$$\sin B = 3/5 \quad \cos B = 4/5 \quad \tan B = 3/4$$

Was that hard? Of course not. Using the same triangle, find the sin A, cos A, and tan A.

$$\sin A = 4/5 \quad \cos A = 3/5 \quad \tan A = 4/3$$

**Example 5** Find the sin, cos, and tan for angles Q and R.



Using **SOHCAHTOA**

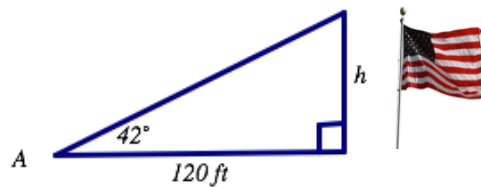
$$\begin{array}{ll} \sin Q = 5/13 & \sin R = 12/13 \\ \cos Q = 12/13 & \cos R = 5/13 \\ \tan Q = 5/12 & \tan R = 12/5 \end{array}$$

In these examples, we have just asked you to find the ratios given a specific angle. To solve problems, you will have to determine which trig ratio is most appropriate based on the information provided.

**Example 6**

A boy standing on level ground notices (or can measure) that the angle he has to look up (angle of elevation) to see the top of a flagpole is 42 degrees. He can also measure the distance he is from the pole and finds it to be 120 feet. How high is the flagpole?

We'll call the height of the flagpole " $h$ ." Let's look at a picture and fill in any information we have



Now, how can we determine the height? Well, since we are studying trig, let's hope we use one of the trig ratios. Which one do we use? That's the question.

The sine is the opposite over the hypotenuse; we don't know the opposite or the hypotenuse. So, we won't use the sine. Cosine is the adjacent over the hypotenuse; that does not give the height. So, I won't use that. Tangent is the opposite, which is  $h$ , over the adjacent. There, I know the adjacent is 120 and I'm looking for the opposite (height). Therefore, I will use the tangent ratio.

I will have to look up the  $\tan 42$  on my trig table.  
The  $\tan 42^\circ = .9004$

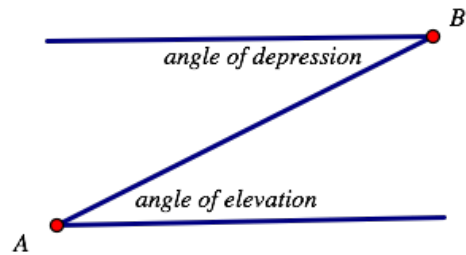
$$\tan 42^\circ = \frac{\textit{opposite}}{\textit{adjacent}} = \frac{h}{120}$$

$$.9004 = \frac{h}{120}$$

$$108 \approx h$$

So, the height of the flagpole is approximately 108 feet.

In a number of applications, professionals use terms like “angle of elevation” or “angle of depression”

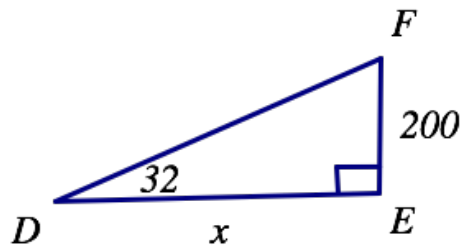


Line segment AB is the line of sight, the other two lines are horizontals. Notice how those two angles are defined with the horizontals. If you are looking **up**, that’s an angle of elevation. Looking **down** is an angle of depression.

To solve problems involving the trigonometric ratios, it’s always wise to think first, then set up a ratio. As we have said before, the choices we make can either make math easier or more cumbersome.

So, we have to ask, given certain information, am I going to use the sine cosine or tangent ratios and from what angle? So, thinking first is important.

**Example 7** Find the measure of DE in the diagram.



Using the trig ratios, we are working with 2 sides and an angle. To solve problems, we need to know 2 of those 3 and find the unknown.

In this problem, we have 2 sides and an angle. Notice that the **hypotenuse** is NOT one of the sides given. For me, that eliminates using the sine or cosine ratios.

So, that leaves me with the tangent ratio, do I want to use angle D or F? Remember, if  $\angle D = 32^\circ$ , then  $\angle F = 58^\circ$ . The sum must be  $90^\circ$ .

So now decide, which is an easier equation to solve, A or B?

$$A. \quad \tan 32^\circ = \frac{200}{x} \quad \text{or} \quad B. \quad \tan 58^\circ = \frac{x}{200}$$

In either case, I have to use a table or calculator to find the respective tangent values. However, since  $x$  is in the numerator of the B, I would choose to multiply, rather than divide.

$$\tan 58^\circ = \frac{x}{200}$$

$\tan 58^\circ \approx 1.6003$  (using a calculator)

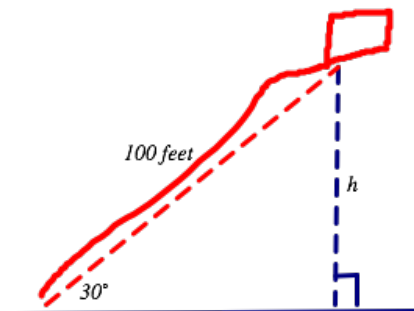
$$1.6003 = \frac{x}{200}$$

$$x \approx 200(1.6003) \approx 320$$

### Example 8

The length of a kite string is 100 feet long. How high is the kite if the kite make a  $30^\circ$  angle with the ground?

The first best thing to do is draw a picture and label what we can.



Drawing the picture, you can see I straightened out the string and the height of anything is always denoted by perpendicular lines to form a right triangle.

Now, the question is; which trig ratio is best to use? To solve a problem with a trig ratio, I need to know 2 of 3 things; sides and angles.

In this case, the  $\sin 30^\circ$  is something I know from applications of special triangles or I could look it up.

$$\sin 30^\circ = \frac{\textit{opposite}}{\textit{hypotenuse}}$$

Substituting  $\sin 30^\circ = 1/2$  and the hypotenuse is 100 ft.

$$\frac{1}{2} = \frac{h}{100}$$

$$2h = 100$$

$$h = 50$$

The tip of the kite string is 50 feet above the ground.

Let's examine what we know. All trig is in the study of the ratios of the sides of the right triangle. Those ratios are given names; sine, cosine, tangent, cosecant, secant, and cotangent. To know which ratio we are talking about, we stand on the vertex of the angle and use the acronym **SOHCAHTOA**. By looking at different right triangles and knowing the ratios are the same for equal angles (using SIMILAR TRIANGLES) we are able to create trig tables for all the angles that will give us the ratio of those sides. Those ratios are written in decimal form.

Don't you just love this stuff? You are probably thinking "trig is my life." Try these next two problems. In the first problem, the angle of depression is given. You need to know how to draw that - see page 6. We should not need tables because we are working with a  $60^\circ$  angle - use the special right triangle relationships.

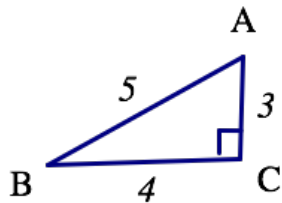
1. From the top of a building 300 feet high, you see a parked car on a bridge, if the angle of depression is  $60^\circ$ , how far is the car from the building?

In this problem, we don't know the ratio of sides for an angle measuring  $20^\circ$ , so we will have to use the book or calculator to determine that value based on the trig ratio you chose.

2. A pilot reads the angle of depression from his position 5000 feet in the air is  $20^\circ$ , how far is the plane from the airport?

## Sec. 2 Angle Relationships

In the previous section, we identified the trig values of angles when solving problems. In example 4, we had a right triangle shown below with the following relationships.



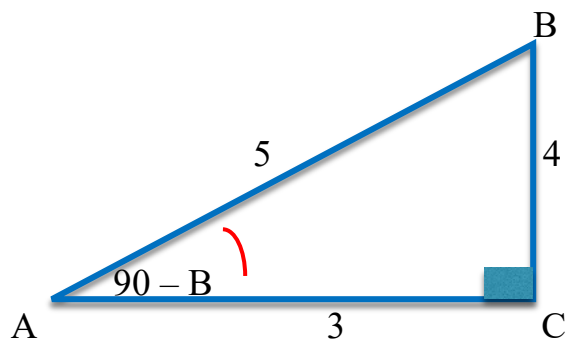
$$\sin B = 3/5 \qquad \cos B = 4/5 \qquad \tan B = 3/4$$

$$\sin A = 4/5 \qquad \cos A = 3/5 \qquad \tan A = 4/3$$

Notice the color coding. The  $\sin A = 4/5$  and the  $\cos B = 4/5$  and the  $\sin B = 3/5$  and the  $\cos A$  also equals  $3/5$ . Interesting?

Also notice, since  $\triangle ACB$  is a right triangle, then  $\angle A$  and  $\angle B$  are complementary – their sum is  $90^\circ$ .

That means if we know one of the acute angles is  $50^\circ$ , then the other angle, the cofunction, must be  $90 - 50$  or  $40^\circ$ .

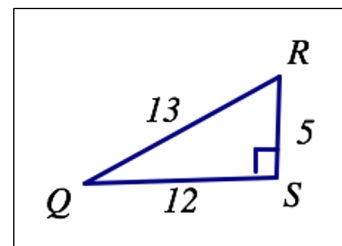


Now, let's combine those two ideas.  $\sin B = 3/5$  and the  $\cos A = 3/5$ . Using substitution and letting  $\angle A = 90 - B$ , we have  $\sin B = \cos (90 - B)$ .

Using the same reasoning, we have  $\sin A = \cos (90 - A)$ .

From Example 5, we had the following relationships:

$\sin Q = 5/13$	$\sin R = 12/13$
$\cos Q = 12/13$	$\cos R = 5/13$
$\tan Q = 5/12$	$\tan R = 12/5$



Generalizing those **Cofunction Relationships**, we have

$\sin Q = \cos (90 - Q)$	$\cos Q = \sin (90 - Q)$
$\tan Q = \cot (90 - Q)$	$\cot Q = \tan (90 - Q)$
$\sec Q = \csc (90 - Q)$	$\csc Q = \sec (90 - Q)$

If we looked for further relationships, we would also see other patterns. For instance,

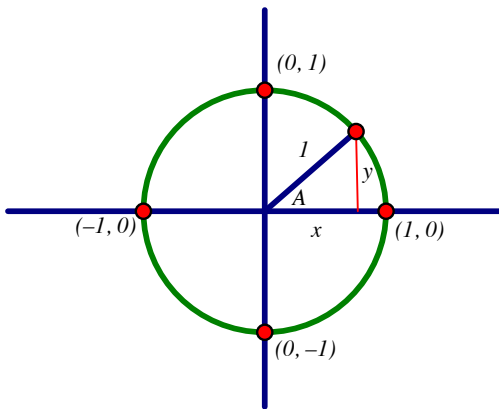
if we placed the value of the sine over the value of the cosine, **Quotient Relationships**, we get the value of the tangent. Let's look.

$\tan B = \frac{3}{4}$ , placing the  $\frac{\sin B}{\cos B} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$ , the same as the  $\tan B$ . Therefore, we can have the identity:

$$\tan B = \frac{\sin B}{\cos B}$$

Using the same thinking, the  $\cot B = \frac{\cos B}{\sin B}$

Let's extend these trig ratios to the rectangular coordinate system and draw a circle of radius 1 around the origin. This will lead us to a very important identity.



If I pick a point on the circle,  $(x, y)$  in the first quadrant and draw a line straight down to the  $x$ -axis, a right triangle is formed.

The length of the side opposite  $\angle A$  is  $y$ , adjacent side has measure  $x$ .

$$\sin A = \frac{y}{1} \text{ or } y = \sin A \text{ and}$$

$$\cos A = \frac{x}{1} \text{ or } x = \cos A$$

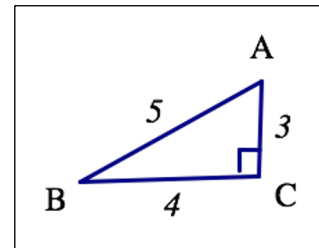
Using the Pythagorean Theorem, we know in the right triangle,  $x^2 + y^2 = 1$

Substituting  $x = \cos A$  and  $y = \sin A$ , we have  $\cos^2 A + \sin^2 A = 1$

If we continued looking at the other ratios from previous examples, we would see the same pattern. This is a **Pythagorean Relationship**.

Using the triangle on the right, we said the  $\sin B = \frac{3}{5}$  and the  $\cos B = \frac{4}{5}$ . If I find the sum of those trig ratios squared, we have

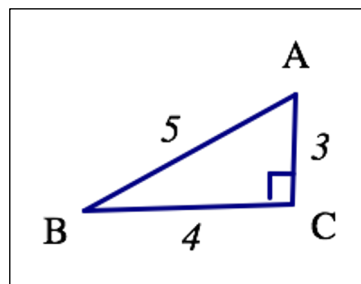
$$\sin^2 B + \cos^2 B = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1.$$



$$\cos^2 x + \sin^2 x = 1$$

Moving on, we will find more patterns – **Reciprocal Relationships**

And, using the same triangle we have been using, we can see there are a total of 6 ratios that can be made formed by the sides. The second set of three ratios are the **reciprocals** of the first three.



$$\frac{3}{5} \quad \frac{3}{5} \quad \frac{3}{4}$$

$$\frac{5}{3} \quad \frac{5}{3} \quad \frac{4}{3}$$

As we defined the sine, cosine and tangent ratios, we will define the reciprocals as the cosecant, secant and cotangent, respectively.

In other words,

The cosecant  $x$ , abbreviated  $\csc x$ , is equal to the reciprocal of the sine  $x$ .

The secant  $x$ , abbreviated  $\sec x$ , is equal to the reciprocal of the cosine  $x$ .

The cotangent  $x$ , abbreviated  $\cot x$ , is equal to the reciprocal of the tangent.

Mathematically, we write

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

So, if the  $\sin 30^\circ = \frac{1}{2}$ , then the  $\csc 30^\circ = \frac{2}{1}$  or 2. Piece of cake, right?

Now, using the **Pythagorean** and the **Quotient Relationships**,  
 $\sin^2 x + \cos^2 x = 1$ , dividing both sides by  $\cos^2 x$ , we have

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

And, if I divided both sides by  $\sin^2 x$ , we'd have

$$1 + \cot^2 x = \csc^2 x$$

These three formulas are called **Pythagorean Relationships**:

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

So, in a nutshell, all we are doing is looking at similar triangles and noticing some patterns based on how we name angles – SOHCAHTOA.

$$\sin A = \frac{\textit{opposite}}{\textit{hypotenuse}}$$

$$\cos A = \frac{\textit{adjacent}}{\textit{hypotenuse}}$$

$$\tan A = \frac{\textit{opposite}}{\textit{adjacent}}$$

From there, we saw some patterns for Cofunctions

$$\sin Q = \cos (90 - Q)$$

$$\cos Q = \sin (90 - Q)$$

$$\tan Q = \cot (90 - Q)$$

$$\cot Q = \tan (90 - Q)$$

$$\sec Q = \csc (90 - Q)$$

$$\csc Q = \sec (90 - Q)$$

then Reciprocal Relations

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

And we continued and “discovered” Quotient Relationships

$$\tan B = \frac{\sin B}{\cos B}$$

$$\cot B = \frac{\cos B}{\sin B}$$

which all led to the Pythagorean Relationships

$$\sin^2 x + \cos^2 x = 1 \quad \tan^2 x + 1 = \sec^2 x \quad 1 + \cot^2 x = \csc^2 x$$

## Laws of Sines & Cosines

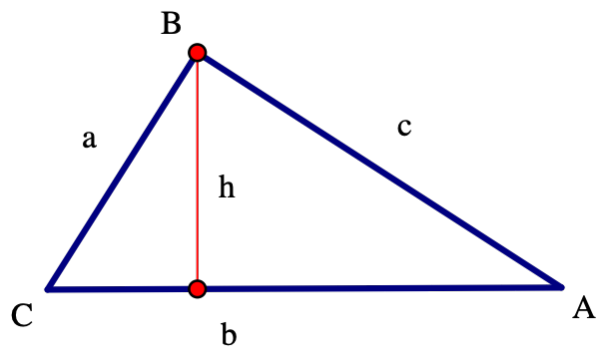
We have studied right triangle trigonometry and learned how to find sides and angles of right triangles. Well, a question that one might ask is what happens when we want to find sides or angles of triangles that are not right triangles. An oblique triangle is a triangle that does not contain a right triangle.

Good news, we can always divide an oblique triangle into right triangles. But, that can be a little cumbersome. So, we will use our knowledge of right triangles and develop some relationships that will work for oblique triangles.

### Law of Sines

If **two angles and a side of a triangle** are known, or if **two sides and an angle** opposite one of them is known, then the remaining parts can be found.

Given  $\triangle ABC$ , construct an altitude ( $h$ ), from  $\angle B$  to  $\overline{AC}$



Drawing the altitude forms 2 right triangles. Using  $\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$ ,

we have

$$\sin C = \frac{h}{a} \quad \text{and the} \quad \sin A = \frac{h}{c}$$

Solving both equations for  $h$ ,

$$h = a \sin C \quad \text{and} \quad h = c \sin A$$

Using substitution

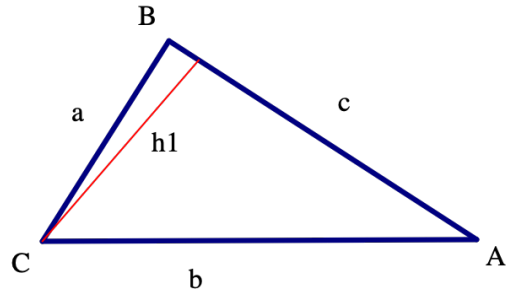
$$a \sin C = c \sin A$$

Divide both sides by  $(\sin C)(\sin A)$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Using the same  $\triangle ABC$ , construct an altitude  $h_1$  from  $\angle C$  to  $\overline{AB}$

Go through the same process for the triangles formed. Note  $h \neq h_1$



right

$$\sin A = \frac{h_1}{b} \quad \sin B = \frac{h_1}{a}$$

Solving for  $h_1$                        $h_1 = b \sin A$     and     $h_1 = a \sin B$

Using substitution                       $b \sin A = a \sin B$

Divide both sides by  $(\sin A)(\sin B)$

We have already shown that  $\frac{b}{\sin B} = \frac{a}{\sin A}$ , so using the Transitive Property we have

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

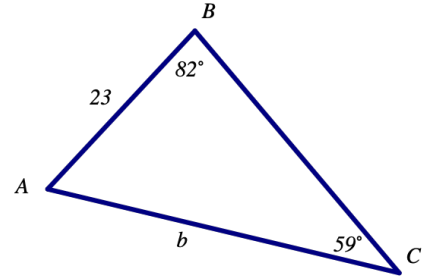
That gives us the **Law of Sines**:  $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

The ratio of any side of a triangle to the sine of the opposite angle is a constant. As we said earlier, we use the Law of Sines under a couple of conditions

- A) If two angles and a side of a triangle are known
- B) If two sides and an angle opposite one of them are known

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**Example 1** Given the  $\triangle ABC$ , find  $b$ .



Using the Law of Sines

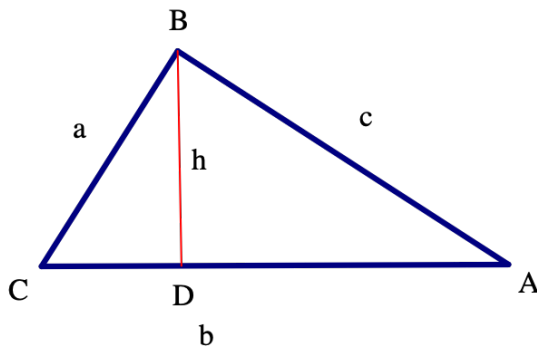
$$\frac{23}{\sin 59^\circ} = \frac{b}{\sin 82^\circ}$$

$$\begin{aligned} b \sin 59^\circ &= 23 \sin 82^\circ && \text{(look up values for sine)} \\ b (.86) &\approx 23 (.99) \\ b &\approx 26.5 \end{aligned}$$

## Law of Cosines

If two sides and the included angle or if three sides of a triangle are given, the Law of Sines can not be applied directly.

Given  $\triangle ABC$ , construct an altitude ( $h$ ), from  $\angle B$  to  $\overline{AC}$ . That altitude forms two right triangles that allows us to use the trig.



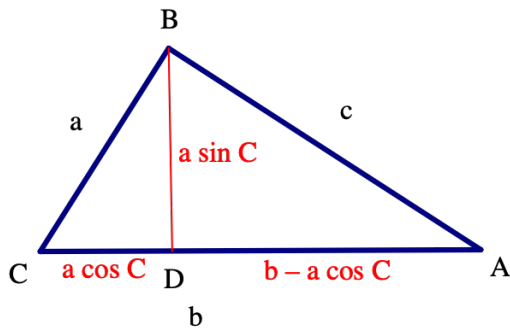
$$\begin{aligned} \sin C &= \frac{h}{a} && \text{or } h = a \sin C \\ \cos C &= \frac{\text{adj}}{\text{hyp}} \rightarrow \cos C = \frac{CD}{a} \\ \therefore CD &= a \cos C \end{aligned}$$

By the diagram  $AC = b$  and we just found that  $CD = a \cos C$ , then

$$AD = b - a \cos C$$

**We will use that information to relabel our triangle  $\triangle ABC$**

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$\triangle BDA$  is a right triangle,  
we can use the  
Pythagorean Theorem to  
write an equation

$$(a \sin C)^2 + \underline{(b - a \cos C)^2} = c^2$$

Squaring the binomial - underlined

$$a^2 \sin^2 C + \underline{b^2 - 2ab \cos C + a^2 \cos^2 C} = c^2$$

Rewriting the equation using the Commutative and associative properties that will result in a trig identity

$$a^2 \sin^2 C + a^2 \cos^2 C + b^2 - 2ab \cos C = c^2$$

Factoring  $a^2$  out of the first two terms

$$a^2 (\sin^2 C + \cos^2 C) + b^2 - 2ab \cos C = c^2$$

Substituting 1 for  $\sin^2 C + \cos^2 C$

$$a^2 (1) + b^2 - 2ab \cos C = c^2$$

$$a^2 + b^2 - 2ab \cos C = c^2$$

OR

### Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cos C$$

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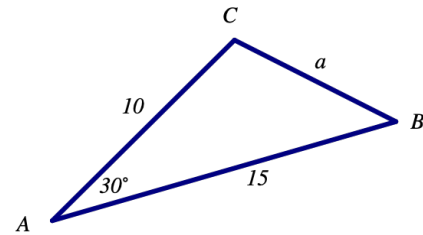
In any triangle the square of a side equals the sum of the squares of the other two sides decreased by twice the product of those sides and the cosine of the included angle.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

**Example 2**

Given  $\triangle ABC$ , find the value of  $a$ .



Using the Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a^2 = 10^2 + 15^2 - 2(10)(15)\frac{\sqrt{3}}{2}$$

$$a^2 \approx 100 + 225 - 300(.86)$$

$$a^2 \approx 325 - 258$$

$$a^2 \approx 67$$

$$a \approx 8.2$$

**Example 3**

Given three sides of a  $\Delta$ ,  $a = 15$ ,  $b = 17$ , and  $c = 19$ , find  $m\angle C$ , the side opposite side  $c$ .

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$19^2 = 15^2 + 17^2 - 2(15)(17) \cos C$$

$$361 = 225 + 289 - 510 \cos C$$

$$361 = 514 - 510 \cos C$$

$$-153 = -510 \cos C$$

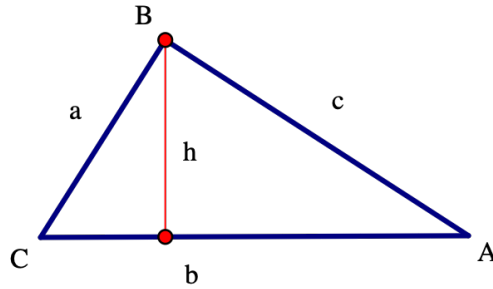
$$-153/-510 = \cos C$$

$$0.3 \approx \cos C$$

$$73^\circ \approx \angle C$$

## Area of a Triangle

Let's use  $\triangle ABC$ , and the area formula for a triangle;  $A = \frac{1}{2} bh$



$$\sin C = \frac{h}{a} \rightarrow h = a \sin C$$

$$\text{Area}_{\Delta} = \frac{1}{2}(AC)h$$

$$\text{Area}_{\Delta} = \frac{1}{2}(b)a \sin C$$

$$\text{Area}_{\Delta} = \frac{1}{2}ab \sin C$$

