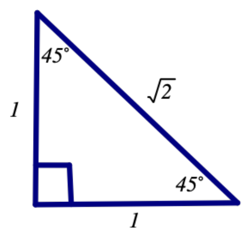
Angular Rotations

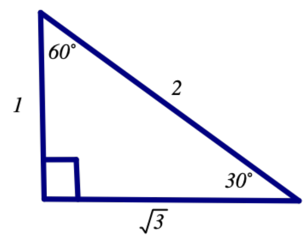
This unit is built upon your knowledge and understanding of the right triangle trigonometric ratios. A memory aid that is often used was SOHCAHTOA.

$$\sin x = \frac{opposite}{hypotenuse} \quad \cos x = \frac{adjacent}{hypotenuse} \quad \tan x = \frac{opposite}{adjacent}$$

In addition, we then learned about special right triangles. Those special right triangles allowed us to find the sides of triangles without using the Pythagorean Theorem. The two special right triangles were the 45-45-90 and the 30-60-90 degree triangles shown below.



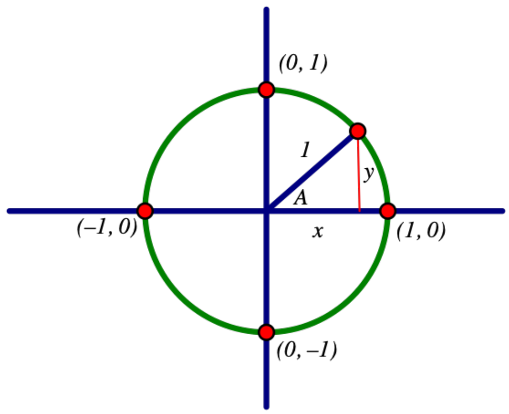
45-45-90˚ Δ
Hypotenuse = √2 leg



30-60-90˚ Δ
Hypotenuse = 2 short leg,
long leg = √3 short leg

Using SOHCAHTOA, we see the sin 30˚ = 1/2, sin 60˚ = √3/2 and sin 45˚ = 1/√2

Now, if I extend the trig ratios to the rectangular coordinate system and draw a circle of radius 1 around the origin, we will see how those coordinates are related to the special right triangles.



The terminal side of $\angle A$ intersects the circle at (x, y).

Using SOHCAHTOA on the unit circle

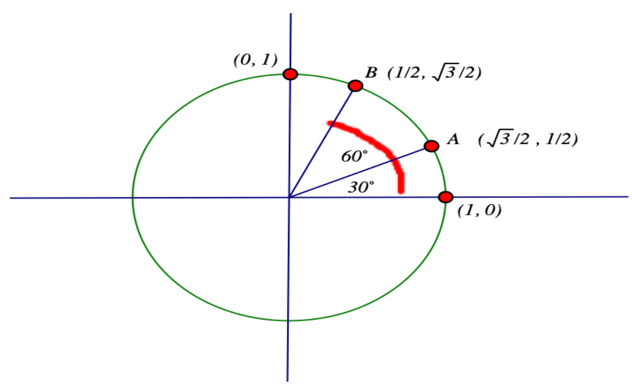
sin A = y/1 or sin A = y

cos A = x/1 or cos A = x

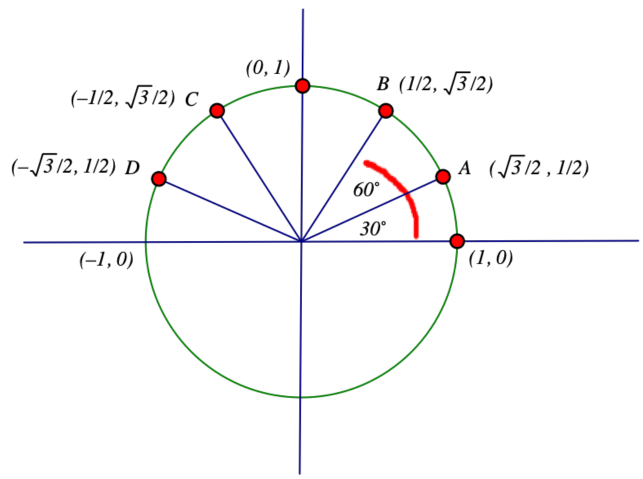
Putting this altogether, we have the trig ratios from the two special right triangles, the cosine and sine being defined on the unit circle (r =1) as the x and y coordinates respectively, and understanding that because the triangles that make up the trig ratios are similar, the ratios are equal.

(x, y) —-≥ (cosine, sine)

Using the 30-60-90˚ triangles above, we know the sin 30˚ = 1/2 and the cos 30˚ = √3/2. We also see the sin 60˚ = √3/2 and the cos 60˚ = 1/2.



The terminal side of the 30˚ angle intersects the circle at A. How was I able to label the ordered pair (√3/2, 1/2)? Well, I looked at the special right triangle to come up with the ratios and based on the previous drawing, we **defined the cosine as the x-coordinate and the sine as the y-coordinate** using SOHCAHTOA.



The same is true for the 60˚ angle whose terminal side interests the unit circle at B. Using the special right triangle, the cos 60˚ = 1/2, the sin 60˚ = √3/2 and I write them as an ordered pair. To help you remember the ordered pairs, my suggestion is the "c" in cosine comes before the "s" in sine as the "x" comes before the "y".

Notice the graph of the unit circle intersects the y-axis at (0, 1). Therefore, the cos 90˚ (the x-coordinate) is 0, the sin 90˚ (y-coordinate) is 1.

Using "reflections" across the y-axis and our understanding of similar triangles, we can come up with more coordinates in the second quadrant.

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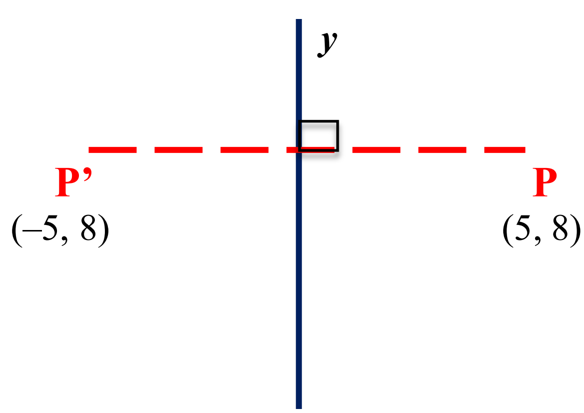
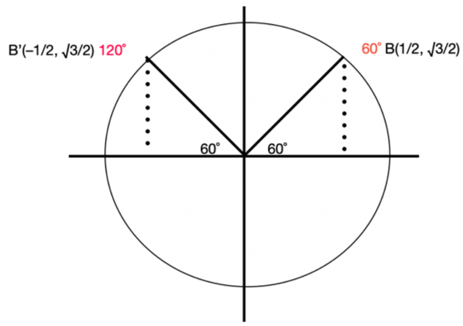
Notice the ordered pairs in the second quadrant, the only thing that seems to change is the sign of the x-coordinate. Point C would represent 120˚ and point D would represent 150˚. So, through symmetry we can find those ordered pairs.

By definition, a reflection in a line *j* maps every point P into a point P' so that *j* is the perpendicular bisector of the segment $\overline{PP'}$.



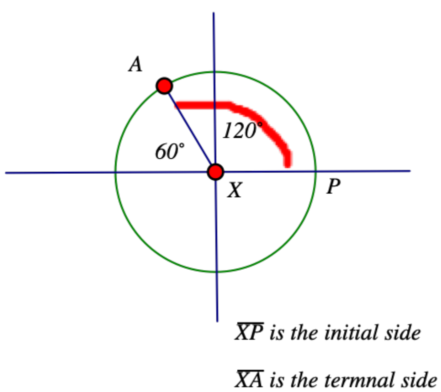
Reference Angles

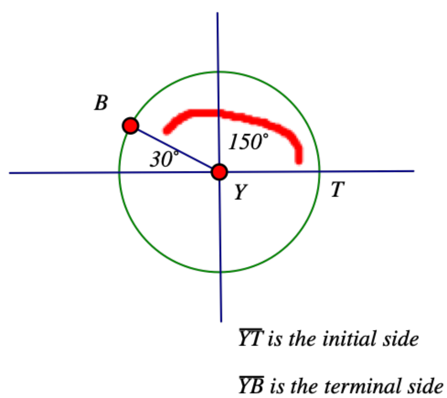
Unfortunately, we don't have special triangles that have 120 or 150˚ angles. But, if we use our knowledge of reflections and similar triangles, we can see an angle of 120˚, by drawing an altitude to the x-axis, corresponds to an angle of 60˚ in the triangle formed in the 1st quadrant.



The terminal side of the 120˚ angle is 60˚ above the x-axis. That would suggest that the coordinates of a 120˚ and 60˚ angle would be the same - except for the sign of the x-coordinate.

We could do the same with an 150˚ angle, by drawing an altitude, that would correspond to a 30˚ angle in the first quadrant. Another way of saying this is the terminal side of a 150˚ angle is 30˚ above the x axis.





That suggests that a way of finding angles greater then 90˚, angles whose terminal sides are in the 2nd, 3rd and 4th quadrants can be found by referencing them to angles in the 1st quadrant – we call them reference angles!

To determine the reference angles, and this is important, we find how far above or below the terminal side of the angles is from the x-axis. The only thing that changes is the "sign" of the coordinates based on the quadrant where the terminal side of the angle.

We can continue this process into the third and fourth quadrants and see the pattern of the coordinates with only sign changes. That means we can look at a 210˚ angle and realize that is 30˚ below the x-axis; a 240˚ angle is 60˚ below the x-axis, a 300˚ angle is 60˚ below the x-axis in the 4th quadrant and a 330˚ angle is 30˚ below the x-axis. Since all these coordinates can be found using angles in the first quadrant and similar triangles or reflections, we call them reference angles. You just need to remember the signs.

In the above problems, we used the special right triangle; 30-60-90˚. We could also use the 45-45-90˚ special triangles to find angles of 45˚, 90˚, 135˚, 225˚, 270˚ and 315˚.

If you are not using the special triangle relationships, you will use a table or calculator and have to remember, as before, to assign the correct sign depending upon the quadrant that terminal side of the angle is in.

Let's do a few examples so you are comfortable finding reference angles.

Example 1 Find the reference angle that corresponds to an angle that measures 170˚.

I can draw a picture to make that determination, but I might be able to visualize that 170˚ is 10˚ above the x-axis. So the reference angle is 10˚.

Example 2 Find the reference angle that corresponds to an angle that measures 140˚.

The terminal side is 40˚ above the x-axis, therefore the reference angle is 40˚.

Example 3 Find the reference angle that corresponds to an angle that measures 200˚.

The terminal side lies in the third quadrant, 20˚ below the x-axis, therefore the reference angle is 20˚.

Example 4 Find the reference angle that corresponds to an angle whose measure is 300˚.

Since the terminal side of that angle lies in the fourth quadrant, 60˚ below the x-axis, the reference angle is 60˚

Example 5 Find the reference angle that corresponds to an angle whose measure of –200˚.

The terminal side of an angle of –200˚ lies in the second quadrant, it is 20˚ above the x-axis, so the reference angle is 20˚.

Example 6 Find the reference angle that corresponds to an angle whose measure is 400˚.

A 400˚ angle does one complete rotation (360˚) and goes another 40˚. Therefore the reference angle is 40˚.

Example 7 Find the reference angle that corresponds to an angle whose measure is 510˚.

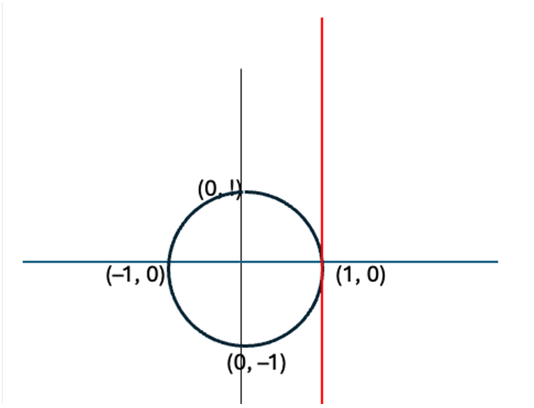
A 510˚ angle does one couple rotation (360˚) and results in an angle whose terminal side lies in the second quadrant at 150˚. An angle of 150˚ is 30˚ above the x-axis, so the reference angle is 30˚.

Radian Measure

Up to this point, all our angles were measured in degrees. What if we wanted to find the measure, the length, of the arc formed by a circle.

Let's look at a circle of radius 1 with center at the origin and a number line tangent to that circle at (1, 0). Now, imagine the number line is flexible and can be wrapped around that unit circle in a counterclockwise direction.

That wrapping results is in each number on the number line being paired with points on the circle.



As we wrap the number line around the unit circle, more than one number from the number line will be paired with a point on a circle

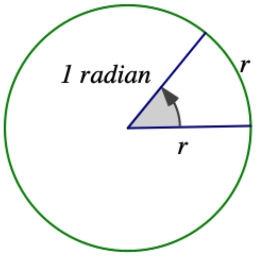
We learned from middle school, the arc around a circle is called the circumference and we find it by using the formula 2πr. So the circumference of a circle with radius 4 would be 2π4 or 8π units.

We typically approximate π as 3.14

You might recall we used a central angle of a circle to find the degree measure of an arc. We'll use the central angle again to measure radians. That introduces us to looking at angles in terms of arc length called radians.

A radian is defined as the measure of a central angle subtending (intersecting) and arc of equal length to the radius.

When a central angle intercepts an arc that has the same length of the radius of the circle, the measure of the angle is defined to be one radian.



Because the circumference of a circle is 2πr, there are 2π radians in any circle. Therefore, 2π radians = 360˚ or π radians = 180˚.

That leads us to the following proportion: $$\frac{d^\circ}{180^\circ} = \frac{r\ radians}{\pi\ radians}$$

Using that proportion, we can determine the measure of 1 radian in term of degrees.

$\frac{d}{180} = \frac{1}{\pi}$; $\frac{d}{180} = \frac{1}{3.14}$ so 1 radian approximates 57˚

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Example 1 Find the radian measure of an angle of 60˚.

Using the proportion and filling in the degree measure, 60˚

$$\frac{60^\circ}{180^\circ} = \frac{r\ radians}{\pi\ radians}$$

$$60\pi = 180r$$

$$\frac{60\pi}{180} = r$$

$$\frac{\pi}{3} = r$$

Example 2 Find the degree measure of –3π/4 radians.

$$\frac{d^\circ}{180^\circ} = \frac{\frac{-3\pi}{4}\ radians}{\pi\ radians}$$

$$d\pi = 180\frac{-3\pi}{4}$$
$$d = 45(-3)$$
$$d = -135^\circ$$

Rather than substituting values into the proportion, we can convert a set of units to another set by multiplying by 1 in terms of what is given and what we want to convert to.

For instance, to convert 3 miles to feet, we start with 3 miles and multiply by 1 in terms of feet and miles: $\frac{5{,}280\ feet}{1\ mile}$

3 miles x $\frac{5{,}280\ ft}{1\ mi}$ = 15, 840 ft

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To convert degrees to radians, multiply by $\frac{\pi\ radians}{180^\circ}$

To convert radians to degrees, multiply by $\frac{180^\circ}{\pi\ radians}$

Example 3 Convert 45˚ to radians.

Multiply 45˚ by 1; $45^\circ\ \frac{\pi\ raians}{180^\circ}$

Simplifying, we have $\frac{\pi}{4}$ radians

To help you remember this, multiply by 1; if you want radians, then radians should be in the numerator. If you want degrees, then degrees should be in the numerator.

Example 4 Convert 5π/6 to degrees.

Multiply 5π/6 by 1
I want to convert to degrees, so I use the factor that has degrees in the numerator - $\frac{180^\circ}{\pi\, radians}$

$$\frac{5\pi}{6}\frac{180^\circ}{\pi\ radians} = 150^\circ$$

Now, let's look at our reference angles in terms of radian measure. We want to know our special triangles. In essence, from our formula, we multiply the degree measure by π and divide by 180.

Example 5 Convert 30˚, 45˚ and 60˚ to radians.

$30[\frac{\pi}{180}] = \pi/6$

$45[\frac{\pi}{180}] = \pi/4$

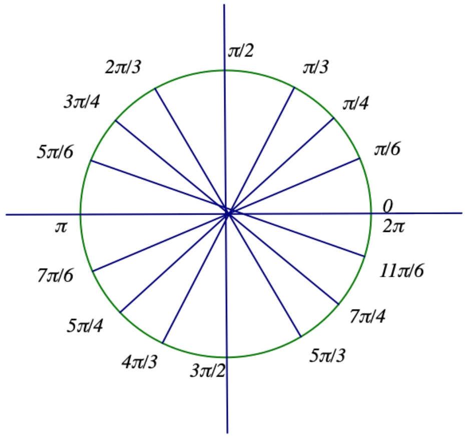
$60[\frac{\pi}{180}] = \pi/3$

It's that easy and you should be able to do those in your head.

The simplest way to convert radians to degrees is let π = 180˚

$\pi/6 = 30^\circ$
$2\pi/6 = 60^\circ$ — $2\pi/6 = \pi/3 = 60^\circ$
$3\pi/6 = 90^\circ$ — $3\pi/6 = \pi/2 = 90^\circ$
$4\pi/6 = 120^\circ$ — $4\pi/6 = 2\pi/3 = 120^\circ$
$5\pi/6 = 150^\circ$
$6\pi/6 = 180^\circ$ — $6\pi/6 = \pi = 180^\circ$

Using this information, we can draw the unit circle and label the angles in terms of radians.



By inspection, we can see

The reference angle of 5π/6 is π/6

The reference angle of 2π/3 is π/3

The reference angle of 3π/4 is π/4

Example 6 Find the reference angle that corresponds to 11π/6.

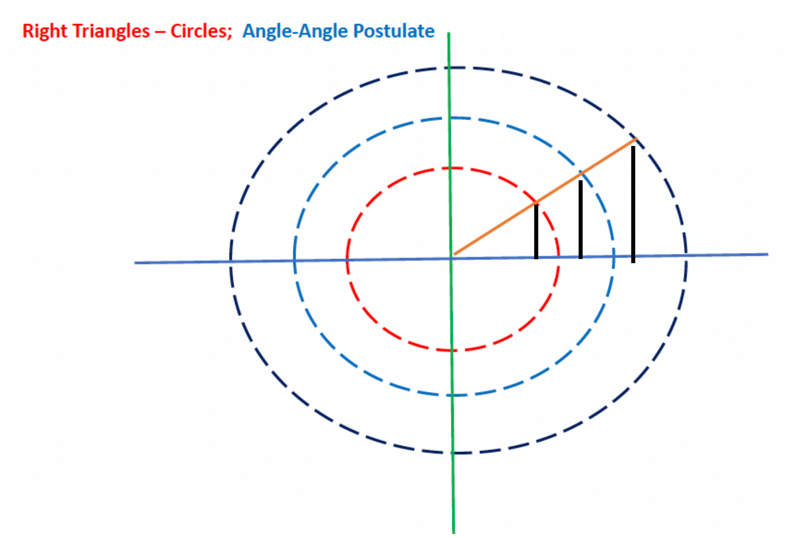
11π/6 is in the 4th quadrant and π/6 below the x-axis, the reference angle is π/6.

You should also know that π/6 = 30˚. So 11π/6 = 330˚

Know these angles and their multiples!

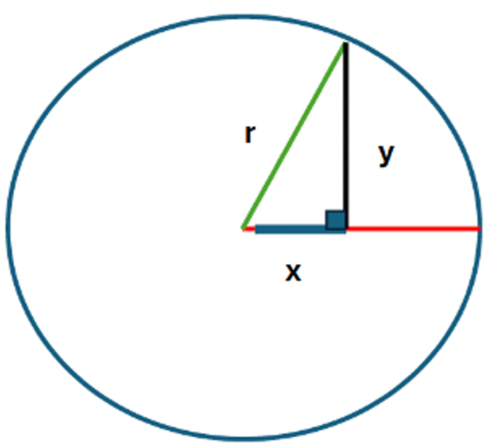
π/3 = 60˚ π/4 = 45˚ 2π/3 = 60˚ π/2 = 90˚ π = 180˚ 2π = 360˚

When the terminal side of an angle intersects concentric circles, right triangles can be formed by dropping perpendiculars to the x-axis. That results in all the triangles being formed being similar by the Angle-Angle Postulate – which indicates all the sides are proportional no matter the length of the radius.



So, let's look specifically at a unit circle, a circle of radius 1.

If r = 1, then the sin $\theta = \frac{y}{r}$ and the cos $\theta = \frac{x}{r}$, since r = 1,
then $\mathbf{x = cos}\ \theta$ and $\mathbf{y = sin}\ \theta$



As we have seen, that means an ordered pair (x, y) on a unit circle can now be written in terms of the ratio **(cos α, sin α)** no matter the length of the

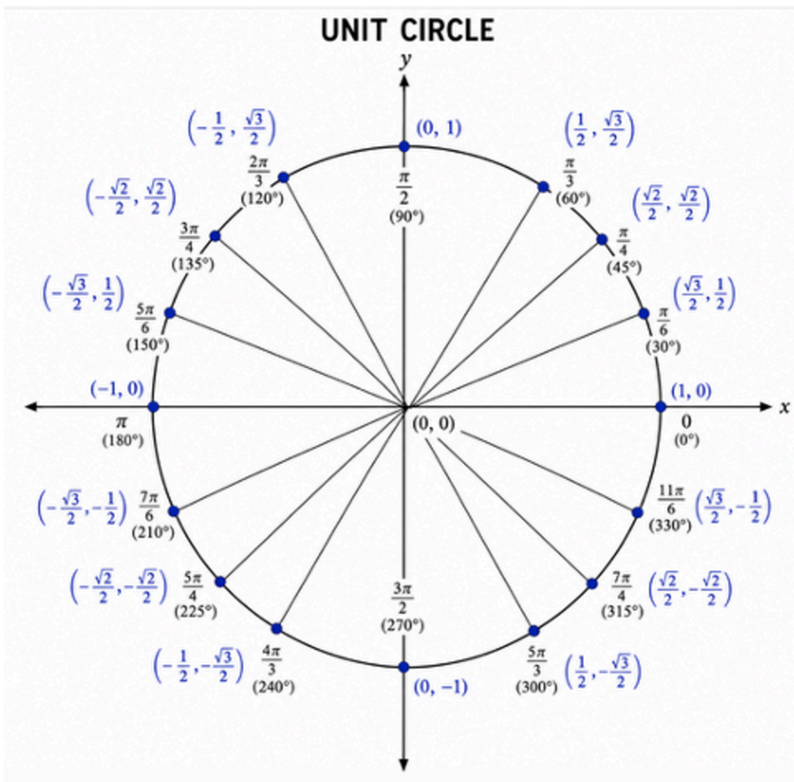
Now the ratios of the angle formed are the same, but if we look at the arcs formed by the concentric circles, they are clearly different sizes. Those lengths will be measured in radians. Now, keep in mind, you have used radian measure before. You found circumferences of circles as 2πr. The greater the radius, the greater the circumference. All we do now is measure the arc length – that measure changes with the length of the radius

The arc length, called **"S" = $\boldsymbol{\theta r}$**, θ is the angle, r is the radius measured in radians!!!

As an example, an angle of measure 45˚with circle of radius 1, will be $\frac{\pi}{4} \cdot 1$radians.

Example, an angle of measure 60˚ with a circle of radius 2, will be $\frac{\pi}{3} \cdot 2 = \frac{2\pi}{3}$ radians.

Knowing radians and the relationship between angles and radians, let's look at the coordinates of angles written in radian measure on a unit circle.



From here we can see the ordered pair for $\frac{\pi}{3}$ is $(\frac{1}{2}, \frac{\sqrt{3}}{2})$

The biggest difference between using angles in our special triangles and using SOHCAHTOA, we will be looking at angle measure using radians. The hypotenuse becomes the terminal side of the triangle which turns to be the radius.

That means, the notation we will use is because we use the x-axis

sin θ = y/r	**csc θ = r/y**
cos θ = x/r	**sec θ = r/x**
tan θ = y/x	**cot θ = x/y**

My suggestion is to draw your special right triangles labeling the angles and sides and the rectangular coordinate system labeling the points on the axes as ready references for the quadrantal angles..

Example: Find the sin 120˚ in terms of an acute angle.

The reference angle is 60˚, 120˚ is in the 2nd quadrant, so the sine is positive. And using the 30-60-90 Δ, the sin 120˚ = sin 60˚ = $\frac{\sqrt{3}}{2}$

Example: Find the cos 225˚ in terms of an acute angle.

The reference angle is 45˚, 225˚is in the 3rd quadrant, so the cosine is negative. Using the 45-45-45 Δ, the cos 225 = – cos 45˚ = – $\frac{\sqrt{2}}{2}$

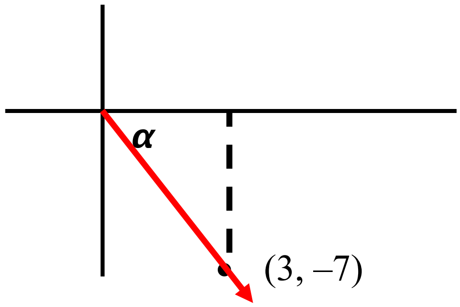
Example: Find the cos 3π/4 in terms of an acute angle.

The reference angle is π – 3π/4 = π/4, it's in the 2nd quadrant, so the cosine is negative. Using the 45-45-45 Δ, the
cos 3π/4 = – cos π/4 = – $\frac{\sqrt{2}}{2}$

Because of circular functions relationship to triangles, it's very helpful to draw the angle on the x-y axes , construct a right triangle and use the Pythagorean Theorem to find trig values.

Example: If the terminal side of an angle contains the point (3, –7), find the sine, cosine and tangent of that angle.

Draw the picture, draw the picture, draw the picture



Construct a rt Δ by connecting the point to the x-axis, label the two sides and use the Pythagorean Theorem to find terminal side, the radius (hypotenuse).
x = 3, y = 7, in 4th quadrant

sin $\boldsymbol{\alpha}$ = y/r   cos $\boldsymbol{\alpha}$ = x/r,

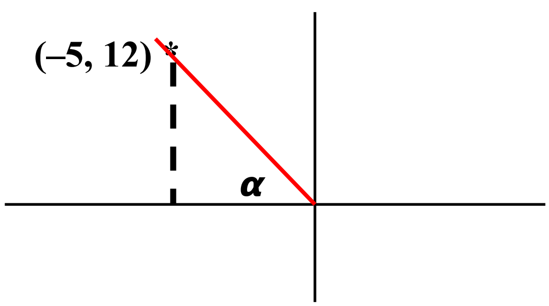
$r^2 = 3^2 + 7^2$
$r^2 = 9 + 49$
$r = \sqrt{58}$

4th quadrant, sin is negative, cos is positive, therefore

$\sin(\boldsymbol{\alpha}) = \frac{-7}{\sqrt{58}}$
$\cos(\boldsymbol{\alpha}) = \frac{3}{\sqrt{58}}$

Example: Given that (–5, 12) is on the terminal side of an angle $\boldsymbol{\alpha}$ in standard position, find the sine, cosine and tangent.

Draw the picture, draw the picture, draw the picture



x = 5, y =12
$r^2 = 5^2 + 12^2$
$r = \sqrt{169}$
r = 13

Second quadrant sine is positive, cosine is negative, tangent is negative

sin $\boldsymbol{\alpha}$ = 12/13   cos $\boldsymbol{\alpha}$ = – 5/13   tan $\boldsymbol{\alpha}$ = –12/5

Example: Given the sin α = 5/13 and the cos α < 0 find the other 5 trig values and the quadrant the angle is in.

Draw the picture, draw the picture.

We can use the Pythagorean Theorem to find x, letting y = 5 and r = 13

OR

$\cos^2\alpha + \sin^2\alpha = 1$
$\cos^2\alpha + (5/13)^2 = 1$
$\cos^2\alpha = 1 - 25/169 = 144/169$
$\cos\alpha = 12/13$

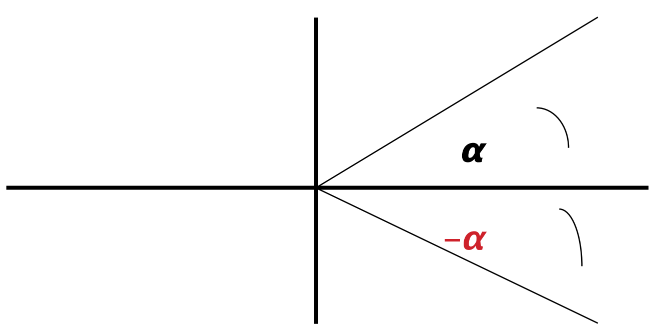
The sine is positive, the cosine is negative, therefore α is in quadrant 2.

sin α = 5/13, cos α = –12/13 tan α = –5/12

csc α = 15/5 sec α = – 13/12 cot α = –12/5

Even and Odd Functions

To determine if a trig function is even or odd, we just look at their values on the rectangular coordinate system. We know by quadrant the signs of both sine and cosine.



The cos(α) and the cos(–α) are the x values and both positive, therefore the cox (–α) = cos α. These are called **even** functions. That means, anytime we take the cos (–α) we can substitute cos (α).

Let's use the same diagram for finding the sin (α) and the sin ($-\alpha$). The sine represent the y values. Notice the values when α is positive lie above the x-axis and are therefore positive. When α is negative, the sign is negative. We can see the sin ($-\alpha$) = – sin (α). These are called **odd** functions.

Since the tan α is the sinα/cosα, the tangent is an odd function. Using that logic, since the sec α = 1/cos α, that's an even function.

Examples: $\cos(-45^\circ) = \cos 45^\circ = \frac{\sqrt{2}}{2}$

$\sin(-30^\circ) = -\sin 30^\circ = -\frac{1}{2}$

$\tan(-\pi/3) = -\tan(\pi/3) = -\sqrt{3}$