

## RATIO & PROPORTION

Solving problems using proportions is one of the most powerful problem solving tools in mathematics. Being comfortable in this chapter will make solving problems in all of math just a whole lot easier. So, let's get started with the basics, which always begins with definitions.

And the good news, if you can make equivalent fractions, then solving proportion will be just as easy. Let's get started.

### Sec 1. Defining Ratio & Proportion

A **RATIO** is a comparison between two quantities.

We use ratios every day and never think of it as math; one Pepsi costs 50 cents describes a ratio. On a map, the legend might tell us one inch is equivalent to 50 miles or we might notice one hand has five fingers. Those are all examples of comparisons – ratios.

#### 3 Ways to Write Ratios

A ratio can be written three different ways. If we wanted to show the comparison of one inch representing 50 miles on a map, we could write that as; *inches to miles*. Mathematically, we could write that three ways.

using “to”	1 to 50	or
using a colon	1 : 50	or
using a fraction bar	$\frac{1}{50}$	

No matter which way you write it, the way you say it is *1 to 50*.

Because we are going to learn to solve problems, it's easier to solve problems involving proportions using the fraction bar. If we looked at the ratio of one inch representing 50 miles on a map,  $\frac{1}{50}$ , we might determine 2 inches represents 100 miles, 3 inches represents 150 miles by using equivalent ratios.

That is, for every inch we add, we add 50 miles.

$$\frac{1}{50} = \frac{2}{100} = \frac{3}{150} = \frac{4}{200} = \frac{5}{250} \text{ and so on.}$$

That just seems to make sense. Look at that from a mathematical standpoint, it appears that we might also be able to simplify ratios: That is the ratio  $\frac{3}{150}$  can be simplified to  $\frac{1}{50}$ .

Does  $\frac{3}{150}$  represent the same comparison as  $\frac{1}{50}$ ?

The answer is yes – and if we looked at other ratios, we would see that simplifying ratios does not affect those comparisons. Those are equivalent ratios. As an example, travelling 300 miles in 5 hours could be described as travelling 60 miles per hour.

Now, we have some good news. We not only discovered how to write ratios, we also learned they can be simplified. And the really good news is we simplify them exactly the same way we simplified fractions. Using those linkages in math makes math so much easier to learn.

Now back to the example using a map. We noticed that  $\frac{3}{150}$ , 3 inches represents 150 miles, could be simplified to  $\frac{1}{50}$  meaning 1 inch represents 50 miles.

Mathematically, by setting those ratios equal, we could write

$$\frac{1}{50} = \frac{3}{150}$$

**Example 1** If 7 bars of candy cost \$1.40, we write that using fractional notation and simplify.  
*Candy bars to cost.*

$$\frac{7 \text{ bars}}{\$1.40} = \frac{1 \text{ bar}}{\$0.20}$$

**Example 2** If seven inches on a map represented 175 miles, what would one inch represent? First, write the ratio out in words, then set up your proportion

$$\frac{\text{inches}}{\text{miles}}; \frac{7}{175} = \frac{1}{25}$$

so one inch represents 25 miles on a map.

To write ratios, it is very important that the ratio describes what is being asked. So, if I say the ratio of boys to girls is 10 to 9 in a sentence, the ratio should be written as boys to girls and you should write the words;  $\frac{\text{boys}}{\text{girls}}$ , so you know how to write the actual ratio  $\frac{10}{9}$ .

If you were told there were 7 adult tickets for every 12 tickets sold, the ratio in words would be  $\frac{\text{adult tickets}}{\text{tickets sold}} = \frac{7}{12}$ .

I can't make writing ratios difficult. You write the word description, then using that, write the numbers in that ratio.

**Example 3** If you were given the following pictographs; ☆☆☆♥♥♥♦♦♦, I could ask for a number of different ratios; stars to hearts, stars to diamonds, hearts to

diamonds, diamonds to hearts, stars to total number of figures. To do those, write the words to the ratios before filling in the numbers in the ratio.

Notice or count, there are 3 stars, there are 5 hearts, and 4 diamonds. That's important!

So, the ratio of stars to hearts;  $\frac{\text{stars}}{\text{hearts}} = \frac{3}{5}$

The ratio of stars to diamonds;  $\frac{\text{stars}}{\text{diamonds}} = \frac{3}{4}$

The ratio of hearts to diamonds;  $\frac{\text{hearts}}{\text{diamonds}} = \frac{5}{4}$

Diamonds to hearts;  $\frac{\text{diamonds}}{\text{hearts}} = \frac{4}{5}$

Notice the **hearts to diamonds** ratio is **different** from the **diamonds to hearts**. It is important that you write the words in a ratio before you write the ratio.

The ratio of stars to total figures;  $\frac{\text{stars}}{\text{total figures}} = \frac{3}{12}$  which can be simplified to  $\frac{1}{4}$ .

Write a ratio and simplify the following ratios:

1. 12 boys for every 18 girls
2. 24 for every 6 dollars
3. 72 hours for every 3 days
4. 18 drinks cost \$27
5. 7 inches is 175 miles
6. Express each ratio as a fraction.
  - a. 3 to 4
  - b. 8 to 5
  - c. 9:13
  - d. 15:7
7. Express each ratio in simplest form
  - a. 12 to 10
  - b. 24:36
  - c.  $\frac{15}{18}$
8. A certain math test has 50 questions. The first 10 are true – false and the rest are matching. Find:
  - a. The ratio of true-false questions to matching questions.
  - b. The ratio of true-false questions to the total number of questions.
  - c. The ratio of the total number of questions to matching questions.

**But before we move on, let me make a point. While ratios can be written like fractions and simplified like fractions, a ratio is NOT a fraction! A fraction is part of a unit. A ratio represents a comparison.**

For instance;

$$\frac{1}{3} + \frac{1}{2} = \frac{5}{6} \text{ as a fraction}$$

$$\frac{1}{3} + \frac{1}{2} = \frac{2}{5} \text{ as a ratio}$$

What??? On the first example we had fractions, we found the common denominator, made equal fractions, added the numerators and brought down the denominator because you were determining what part of a unit we had.

In the second example we had ratios, let's say you got 1 hit out of 3 times at bat last week and 1 hit out of 2 times at bat this week. That results in 2 hits out of 5 times at bat.

What we want clear here is this ratio is a comparison between hits and times at bat. We cannot get one-third a hit. We either get a hit or we don't. Clearly using the fraction bar; “-”, for both fractions and ratios, the notation, can be misleading to students first learning about ratios.

Now that we cleared that up, let's get back to simplifying these ratios.

If we know that we can simplify ratios such as:  $\frac{3}{12} = \frac{1}{4}$ , that does mimic what we can do with simplifying fractions.

Another way to look at the equivalence is to recognize it as an equation. Since we notice the equality, that leads us to a new definition:

## Proportion

A **PROPORTION** is a statement of equality between 2 ratios.

Looking at a proportion like  $\frac{1}{2} = \frac{3}{6}$ , we might see some other relationships that exist if we take time and manipulate the numbers.

For instance, what would happen if we tipped both ratios **up-side down** in the proportion  $\frac{1}{2} = \frac{3}{6}$ ? In other words, if the ratio given was 3 boys for every 6 girls,  $\frac{\text{boys}}{\text{girls}}$ , which resulted in a simplified ratio 1 to 2, could we say that 6 girls for every 3 boys,  $\frac{\text{girls}}{\text{boys}}$  would simplify to 2 girls for every 1 boy and that proportion still be true?

$$\frac{2}{1} \text{ and } \frac{6}{3}, \text{ notice they are also equal, so } \frac{2}{1} = \frac{6}{3}$$

What that suggests is how you write the ratio, boys to girls, or girls to boys will not make a difference. But, what is very important is, once you choose how to express the ratio, it must be used on the other side of the equation. That is,  $\frac{\text{boys}}{\text{girls}} = \frac{\text{boys}}{\text{girls}}$  or  $\frac{\text{girls}}{\text{boys}} = \frac{\text{girls}}{\text{boys}}$ . In a nutshell, all that means is the ratio on the left should match the ratio on the right!

## Sec 2. Properties of Proportion

Using a little math, we can manipulate proportions that result in equalities which can be useful in problem solving; there are 5 properties of proportion that allows us to manipulate information to describe proportions.

I will use words to describe those properties to help in remembering. Depending on grade level, I'm also going to prove why they work and place those proofs at the end of the chapter so we don't get bogged down here.

**Cross Multiply** If we continued looking at the original proportion  $\frac{1}{2} = \frac{3}{6}$ , we might also notice we **could cross multiply** and retain an equality.

In other words,  $1 \times 6 = 2 \times 3$ . This idea of manipulating numbers is pretty interesting stuff, don't you think?

**Upside Down** Now, let's go back and look at these proportions. We found that we could write the

Proportion  $\frac{1}{2} = \frac{3}{6}$  upside down and maintain the equality.

$$\frac{1}{2} = \frac{3}{6} \text{ or } \frac{2}{1} = \frac{6}{3}$$

**Sideways** How about writing the original proportion *sideways*, will we maintain the equality?

$$\text{If } \frac{1}{2} = \frac{3}{6}, \text{ will } \frac{1}{3} = \frac{2}{6} \text{ still work?}$$

There are two more properties, we will introduce later.

Those properties can and will be used to help us solve problems. While we made these properties by observation, later we will prove them.

Knowing those properties, let's see how to use them to solve problems.

### Sec. 3 Ways to Solve a Proportion

#### 2 Best Ways of Solving Proportions

1. Equivalent Fractions
2. Cross Multiplying

Since we have seen that ratios can be simplified like fractions and manipulated, that will help us solve proportions very quickly.

If the numbers are nice, the easiest way to solve for a variable in a proportion is by using what we learned in fractions. That is, make equivalent fractions.

**Example 4** Find the value of  $x$  in this proportion.

$$\frac{2}{5} = \frac{x}{30}$$

Using equivalent fractions, multiply both numerator and denominator by 6. That results in  $x = 12$

**Example 5** Find the value of  $y$ .  $\frac{7}{10} = \frac{56}{y}$

Using equivalent fractions, multiply the numerator and denominator by 8.  $y = 80$

Too easy, right? If the numbers are not convenient, the other method to use is cross multiplying.

Let's redo **Example 4** by cross multiplying. Find the value of  $x$ .

$$\frac{2}{5} = \frac{x}{30}, \text{ that results in } 5x = 2 \cdot 30$$
$$5x = 60, \text{ so } x = 12$$

This problem could have been done by either equivalent fractions or by cross multiplying.

Some problems, equivalent fractions just don't work well. So, when that occurs, use cross multiplying.

**Example 6** Find the value of  $n$ .

$$\frac{3}{5} = \frac{n}{22}$$

We notice immediately that 5 does not go into 22, so using equivalent fractions is not the way to solve the problem. That leaves us with cross multiplying.

$$\begin{aligned}5n &= 3(22) \\5n &= 66 \\n &= \frac{66}{5} \quad \text{or} \quad 13\frac{1}{5}\end{aligned}$$

In many instances, it might appear at first glance that using equivalent fractions might not work, so we go immediately to cross multiplying. That's okay, but ... it might be advantageous to simplify a ratio first for two reasons; 1) you then might be able to solve the proportion by equivalent fractions, and 2) you will end up the smaller numbers.

So, look at the first problem below, since 18 is not a factor of 45, it appears we cannot use equivalent fractions to solve it, so we solve it by cross multiplying. Easy. But, if we simplified the ratio on the left to  $\frac{5}{9}$ , now 9 is a factor of 45 and we can solve it by equivalent fractions pretty easily. Or we could have used  $\frac{5}{9}$  and cross multiplied. Using smaller numbers is nice.

The point is, think before you do problems, those 3 seconds could make the arithmetic a lot easier.

In exercises 1-8, find the value of the variable that makes the proportion true.

1.  $\frac{10}{18} = \frac{x}{45}$

2.  $\frac{5}{8} = \frac{25}{y}$

3.  $\frac{120}{200} = \frac{36}{w}$

4.  $\frac{1}{8} = \frac{t}{6}$

5.  $\frac{x}{10} = \frac{200}{30}$

6.  $\frac{500}{25} = \frac{100}{y}$

7.  $\frac{7}{5} = \frac{P}{100}$

Note: If you simplified the ratios, two things could have happened. 1) you may have been able to do the problem by equivalent fractions and 2) the numbers would have been smaller which might make the arithmetic much easier.

#### Sec 4. Problem Solving Using Proportions

To solve problems, most people use either equivalent fractions or cross multiplying to solve proportions. Either way, start by writing the ratio in words, then fill in the numbers using the same ratio on both sides of the equation to solve the proportions

**Example** If a turtle travels 5 inches every 10 seconds, how far will it travel in 50 seconds?

What we are going to do is set up a proportion. The way we'll do this is to identify the comparison we are making. In this case we are saying 5 inches every 10 seconds. Therefore, and this is very important, we are going to set up our proportion by saying **inches is to seconds**  $\frac{\text{inches}}{\text{seconds}}$ .

On one side, we have  $\frac{5}{10}$  describing inches to seconds. On the other side we have to again use the same comparison, inches to seconds. We don't know the inches, so we'll call it "n". Where will the 50 go in the ratio, top or bottom? Bottom, because it describes seconds – good deal. So now we have.

$$\frac{\text{inches}}{\text{seconds}} \quad \frac{5}{10} = \frac{n}{50}$$

Now, we can find n by equivalent fractions or we could cross multiply.

$$\frac{5}{10} = \frac{n}{50}$$

$$10n = 5 \times 50$$

$$10n = 250$$

$$n = 25 \quad \text{The turtle will travel 25 inches in 50 seconds}$$

It makes it easier to understand initially to write the same comparisons on both sides of the equal signs. In other words, if we had a ratio on one side comparing inches to seconds, then we write inches to seconds on the other side.

If we compared the number of boys to girls on one side, we would have to write the same comparison on the other side, boys to girls. We could also write it as girls to boys on one side as

long as we wrote girls to boys on the other side. The Property of Proportion, tipping the ratios upside down, permits this to happen.

Solve these problems by setting up a proportion.

1. If there were 7 males for every 12 females at the dance, how many females were there if there were 21 males at the dance?  
*Ask yourself; is there a ratio, a comparison in that problem? What's being compared?*
2. David read 40 pages of a book in 5 minutes. How many pages will he read in 80 minutes if he reads at a constant rate?
3. On a map, one inch represents 150 miles. If Las Vegas and Reno are five inches apart on the map, what is the actual distance between them?
4. Bob had 21 problems correct on a math test that had a total of 25 questions, what percent grade did he earn? (In other words, how many questions would we expect him to get correct if there were 100 questions on the test?)
5. If there should be three calculators for every 4 students in an elementary school, how many calculators should be in a classroom that has 44 students? If a new school is scheduled to open with 600 students, how many calculators should be ordered?
6. If your car can go 350 miles on 20 gallons of gas, at that rate, how much gas would you have to purchase to take a cross country trip that was 3000 miles long?
7. A candle is 30 inches long. After burning 12 minutes, the candle is 25 inches long. How long would it take the whole candle to burn?

Solving proportions is interesting, because it leads us to something we might do in our daily lives, like checking prices. In math, we call it unit ratios or unit rates.

## Sec. 5 Unit Ratios – Unit Rates

Unit ratios are ratios that are expressed as a quantity of one, such as 2 feet per second, 7 miles per hour, \$5.00 per case; where the number written on the bottom describes **ONE** unit, hence the word unit. Unit ratios are often expressed using the word “per”.

So, from **Example 1**, if 7 bars of candy cost \$1.40, we wrote that using fractional notation and simplified.

$$\frac{\text{bars}}{\$}; \frac{7}{1.40} = \frac{1}{.20} \text{ one bar for 20 cents.}$$

And that was great. But we just discovered that we can tip proportions upside down and maintain equalities.

That is

$$\frac{\$}{\text{bars}}; \frac{1.40}{7} = \frac{.20}{1}$$

is the same as the first proportion we wrote. And, that allows us to write the ratio as a **unit ratio**, the denominator being ONE. The cost is \$0.20 per candy bar.

**Note, that when working with unit ratios, dollars, money, is typically written in the numerator as in the second example.**

Not all ratios can be reduced to a unit ratio, so there are times we might use an alternative method for finding unit ratios.

## 2 Ways to Determine Unit Ratios

1. Equivalent fractions
2. Dividing

**Example 2.** 24 cases of pens sell for \$16, find the unit price per case.

Writing that as a proportion, I want to know the cost of one case, so I am going to write the ratio with dollars on top and cases in the denominator. \$16 for 24 cases.

Setting up the ratio and proportion, we have  $\frac{\$}{\text{cases}}; \frac{16}{24} = \frac{2}{3}$

As we can see, that does not simplify to having “one” in the denominator using equivalent fractions. The best answer we get by simplifying the ratio is \$2 for every 3 cases. That is not a unit ratio.

**Another way, and in this case, a better way to handle unit rates is to simply divide the numerator by the denominator.** In other words, once we decide how to write the ratio to get the unit we are looking for in the denominator – just divide.

So, in that problem, the unit ratio would be  $\frac{16}{24}$  or  $\frac{2}{3}$  simplified, then divide.

$$24 \overline{)16} \quad \text{or} \quad 3 \overline{)2}$$

Either way, the answer is \$0.6666 or \$0.67 per case – a unit ratio

Usually, we write the ratio the way it was given to us, and simplify it. To find unit pricing, the best approach is to determine how you want to write the ratio first by asking yourself how you want to compare the quantities so you will have the unit written in the denominator.

Find the following unit rates – round your answers to the nearest hundredths.

Money goes on top

1. \$14 for 4 books.
2. 13 chocolate bars for \$18.
3. 95 miles on 9 gallons of gas.
4. 7 movie tickets cost \$40.
5. 5 calculators cost \$140.
6. Mowed 5 lawns for \$35.

All the proportion problems up to this point provided a ratio, then more information was given in terms of those ratios. That doesn't always happen.

### Sec 6. More Information Not Given in Terms of Ratio

The ratio and proportions problems we have done up to this point have expressed a ratio, then given more information in terms of the ratio previously expressed.

In other words, if the ratio expressed was male to female, then more information was given to you in terms of males or females and you set up the proportion. Piece of cake, right?

Well, what happens if you were given a ratio, like males to females, but then the additional information you received was not given in the terms of the original ratio, males or females? Where would you place that number? Maybe the additional information told you how many students were in a class altogether?

**Example** If there are 3 boys for every 7 girls at school, how many boys attend the school if the total student enrollment consists of 440 students?

The first comparison given is boys to girls. Knowing this, we would like to set up a proportion that looks like this:

$\frac{\text{boys}}{\text{girls}} = \frac{\text{boys}}{\text{girls}}$  just as we have done before. But ...where does the 440 go?

$$\frac{3}{7} = \frac{?}{?}$$

You wouldn't be able to set up a proportion based on what we know now.

But, don't you love it when somebody says "but"? It normally means more is to come. So, hold on to your chair to contain the excitement, we get to learn more by seeing patterns develop.

Up to this point, we know if 2 ratios are equal, then I can cross multiply, tip them upside down, write them sideways and we will continue to have an equality.

Remember, the way we developed those properties was by looking at equal fractions and looking for patterns. If we continued to look at equal ratios, we might come up with even more patterns.

Let's look, we know  $\frac{1}{3} = \frac{2}{6}$ . Now if we played with this proportion, then looked to see if the same things we noticed with other proportions, then we might be able to make some generalizations.

For instance, if I were to keep the numerator 1 on the left side, and add the numerator and denominator together to make a new denominator on the left side, would I still have an equality if I did the same thing to the right side? Let's peek.

We have  $\frac{1}{3} = \frac{2}{6}$ , keeping the same numerators, then adding the numerator and denominator together for a new denominator, we get

$$\frac{1}{1+3} \text{ and } \frac{2}{2+6}. \quad \text{Are they equal? Does } \frac{1}{4} = \frac{2}{8}?$$

Oh boy, the answer is yes. Don't you wonder who stays up at night to play with patterns like this? If we looked at other examples, we would find this seems to be true. So, what we do is generalize this.

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a}{a+b} = \frac{c}{c+d}$$

And

So now we have discovered another pattern for 4 Properties of Proportion. Again, those are proven at the end of the chapter.

Now, going back to the problem we were describing earlier that gave a ratio, then gave additional information that was **not** in terms of the ratio, we'll be able to manipulate the properties of proportion to solve that problems. Say yes to math, this is really great stuff.

**Example** If there are 3 boys for every 7 girls at school, how many boys attend the school if the total student enrollment consists of 440 students?

The first comparison given is boys to girls. Knowing this, we would like to set up a proportion that looks like this:

$\frac{\text{boys}}{\text{girls}} = \frac{\text{boys}}{\text{girls}}$  just as we have done before. But ....

The problem that we encounter is while we can put the 3 and 7 on the left side to represent boys to girls, we have to ask ourselves, where does 440 go? It does not represent just boys or just girls, so we can't put it in either position on the right side. 440 represents the total number of boys and girls.

Remembering what we just did, see there was a reason for looking for more patterns, we noticed if we have a proportion like

$$\frac{\text{boys}}{\text{total}} = \frac{\text{boys}}{\text{total}}$$

In other words, If  $\frac{b}{g} = \frac{b}{g}$ , then  $\frac{b}{b+g} = \frac{b}{b+g}$  would also be true.

From our problem, we now can see  $b + g$  would represent the **total** of the boys and girls.

This leads us to the ratio of boys to total;  $\frac{\text{boys}}{\text{total}}$

$$\frac{b}{g} = \frac{b}{g}, \text{ then } \frac{b}{b+g} = \frac{b}{b+g} \quad \frac{3}{3+7} = \frac{b}{440} \quad \rightarrow \quad \frac{3}{10} = \frac{b}{440}$$

$$\text{Solving; } 10b = 3 \times 440$$

$$10b = 1320$$

$$b = 132$$

There would be 132 boys; to find the number of girls we could subtract 132 from 440.

The point being, if you were given a problem being described by a ratio, then additional information was given to you **not** using the descriptors in the original ratio, you could manipulate the information using one of the properties of proportion.

To be quite frank, that is not the way I would usually attack that sort of problem. What I would prefer doing is setting up the problem algebraically.

Before going on, let's step back and see how ratios work. Remember, we said you could simplify ratios. In other words, if I had the ratio of  $\frac{3}{150}$ , I could simplify it to  $\frac{1}{50}$ .

Visually, the way you simplify is by dividing out a common factor. To simplify  $\frac{3}{150}$ , we could rewrite it as  $\frac{1 \times 3}{50 \times 3}$ . Dividing out the 3's, we would have  $\frac{1}{50}$ .

Now going back to the previous example, we had 3 boys for every 7 girls, with a total enrollment of 440. The ratio of boys to girls was simplified by dividing out a common factor.

Using algebra, the ratio of boys to girls,  $\frac{b}{g}$  is  $\frac{3}{7}$ .

Does that mean we have exactly three boys and seven girls? No, that ratio comes from simplifying the actual numbers in the boys to girls ratio. Since we don't know the common factor we would have divided out, we'll call it X. Unbelievable concept 😊

Using the ratio;  $\frac{b}{g}$  or  $\frac{3}{7}$ , we now know we have 3X boys and 7X girls. So, the ratio of  $\frac{b}{g}$

looks like this;  $\frac{3X}{7X}$ .

But notice the sum of 3X and 7X would be the total number of boys and girls, the total number of students. The total number of students is 440, therefore we have

$$\text{boys} + \text{girls} = 440$$

$$3X + 7X = 440$$

$$10X = 440$$

$$X = 44$$

The ratio of boys to girls  $\frac{3X}{7X}$ , that means there are 3X boys or 3 (44) which is 132 boys.

That manipulation allowed us to solve a proportion problem where more information was not given in terms of the first ratio using algebra. And, in my opinion, that makes solving those problems a lot easier.

**Example 1.** The ratio of length to width of a rectangle is 8:3, find the dimensions of the rectangle if the perimeter is 88 in.

Immediately notice we are given ratios in terms of sides and more information is given in terms of perimeter. To me, that suggests doing this problem algebraically.

So, let's attack is problem algebraically. The ratio  $\frac{\text{length}}{\text{width}} = \frac{8x}{3x}$ .

Be careful, the perimeter is 88 inches,  $8x + 3x$  is just half the perimeter, so we have

$$8x + 3x = 44$$

$$11x = 44$$

$$x = 4$$

The length is  $8x$  or 32 inches and the width is  $3x$  12 inches.

From my standpoint, there are two types of **Ratio & Proportion** problems.

**Type I** A ratio is given, and then more information is given in terms of the descriptors of the first ratio. For this type of problem, you set the two ratios equal and solve by equivalent fractions or cross multiplying

**Type II** A ratio is given, and then more information is given to you that is not using the descriptors in the original ratio. This type of problem can be done algebraically as we did in the last example or we can manipulate the ratios to fit the problem.

So, like all of math, we have choices, decisions to make on how to attack ratio and proportion problems. We can use the properties or we can try using algebra.

**Example 2.** The 3 sides of a triangle are in a ratio 4:5:6. The perimeter is the sum of the sides of a polygon, in this case 75 inches. Find the measure of each side.

In this problem, we have ratios of sides to the perimeter. In other words, more information is given **not** in terms of the ratio. I'd use algebra. The sum of the 3 sides is 75 inches.

$$\text{So, } 4x + 5x + 6x = 75$$

$$15x = 75$$

$$x = 5$$

The sides were  $4x$ ,  $5x$  and  $6x$ , substituting, so the sides are 20, 25, and 30 inches.

### **Information Not Given in Terms of the Given Ratio**

1. If there are 3 boys for every 7 girls at school, how many boys attend the school if the total student enrollment consists of 440 students?
2. If our class is representative of the university and there are 2 males for every 12 females. How many men attend the university if the female population totals 15,000?
3. The ratio of the sides of a certain triangle is 2:7:8. If the longest side of the triangle is 40 cm, how long are the other two sides?
4. If the ratios of the sides of a triangle are 4:5:6 and the perimeter is 75 inches, how long is each side?

As you try some of these problems, first determine if they are Type I or Type II, then use the appropriate problem solving strategy.

1. There are 5 boys for every 3 girls in Biology, Out of the 56 students in the class, how many are girls?
2. Tom drove 320 miles in 5 hours. At this rate, how long would it take him to travel 600 miles?
3. If a 15 lb. ham is enough to serve 20 people, how many lbs. of ham would be needed to serve 50 people?
4. A 30 pound moonling weighs 180 pounds on the earth. How much does a 300 pound Earthling weigh on the moon?
5. The ratio of the sides of a certain triangle is 2:7:8. If the longest side of the triangle is 40 cm, how long are the other two sides?
6. If a quarterback completes 20 out of 45 passes in his first game, how many passes do you expect him to complete in his second game if he only throws 18 passes?
7. On a trip across the country Joe used 20 gallons of gas to go 300 miles. At this rate, how much gas must he use to go 3500 miles?
8. On a map 3 inches represents 10 miles. How many miles do 16 inches represent?
9. If our class is representative of the university and there are 2 males for every 12 females. How many men attend the university if the female population totals 15,000?
10. The ratio of length to width of a rectangle is 8:3, find the dimensions of the rectangle if the perimeter is 88 in.

We can describe proportion problems another way and continue to use the same methods we have used to this point. We call those proportions “direct variations”. We set up the proportions the very same way, with one exception, the “dependent variable is always written in the numerator.

## Sec 8. Direct Variation; $y = kx$

Earlier, we defined a ratio as a comparison between two numbers. That is, there are three boys for every four girls, resulting in a ratio of boys to girls is 3:4. Someone else may have described that relationship as girls to boys or 4:3. Whichever way we set up the proportion, the value of the ratios were equal and we solved them to determine the number of boys or girls. It didn't matter.

Oftentimes in using ratios and proportions, it's important to note how one thing **effects** another. In the problems we have solved up to this point, we didn't typically have a cause and an effect relationship. In other words, whether we wrote our ratio as boys to girls or girls to boys, it didn't matter as long as we described the ratio the same way on both sides of the equation.

There are some problems in which there are causes and effects, that the value of one is dependent on the value of something else – like pay is to hours worked. The value of pay is dependent on the number of hours worked. Always placing the dependent variable on top that ratio would tell

us how much a person makes an hour. It's a constant. We will call that ratio the *Constant of Variation*.

In algebra, we want that better defined so proportions can be seen as functions and we can extend our learning. So, we will define a direct variation that way and say *y varies directly as x*. OR *y is directly proportional to x*. Those translate mathematically to

$\frac{y}{x} = k$ , where  $k$  is the constant of variation OR the constant of proportionality. We can still set up a proportion;  $\frac{y}{x} = \frac{y}{x}$  and solve it. So solving the proportion is no different than what we have been doing, but the ratio is will always be composed of the dependent variable on top, resulting in what is known as the Constant of Variation

In algebra, we solve for  $y$ , it's value depends on  $x$ . So,  $y$  is the dependent variable. So  $y$  is the dependent. That ratio  $\frac{y}{x}$  always equals a constant, called  $k$ .

That ratio when solved for  $y$  is:  $\frac{y}{x} = k$ , so  $y = kx$

From that rule, it is easy to see in a direct variation problem, as  $x$  increases,  $y$  increases or as  $x$  gets smaller,  $y$  gets smaller. That's very easily seen if we graph the equation.

Now the good news, we have a choice, we can solve these direct variations problems as proportions, like we have been doing, or we can solve them algebraically, as long as we place the dependent variable on top.

If we solved them as proportions, we'd have the following proportion:

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} \quad \text{Both those ratios will equal } k, \text{ the constant of variation}$$

In the problem below, we previously solved that by setting the ratios equal, then using equivalent ratios or cross multiplying to solve for the amount of gas. In that problem, we knew the ratios were equal, a constant of variation existed, but we didn't care what it was. Other than setting the ratios equal, we didn't care and really did not need to know what that constant was. Let's do that same problem using direct variation - algebra;

$y = kx$ .

Why is that important? In previous problems, like the boy-girl problem, we could solve problems in terms of the ratio of boys to girls or girls to boys. The constant of variation, would be different depending upon how you chose to set up the ratio. In other words, it is not clear which would be the independent vs dependent variable and importantly, and we didn't care.

Let me write same problem a few different ways and change the language slightly.

1. If there are 3 boys for every 7 girls at school, how many boys attend the school if the number girls is 420?

- 1a. If the number of boys going to a school **depends** on the number of girls and the ratio of boys to girls is 3:7 and there are 420 girls, how many boys go that school?
- 1b. If the number of boys attending a school **varies directly** with the number of girls and the ratio of boys to girls to 3: 7 and there are 420 girls, how many boys attend the school?

When we did that problem #1 initially, we didn't have the number of boys attending "depending" on the number of girls, so how we set up the ratio did not matter.

We have already solved the following problem using ratio and proportion, let's see how we can solve that same problem if the number of boys invited depends on the number of girls attending using algebra:  $y = kx$ .

Let's look at another simple example:

**Example 1** Given that  $m$  varies directly as  $t$  and the value of  $m$  is 4 when the value of  $t$  is 8, find the constant of variation and the value of  $m$  when  $t = 30$ .

**First, find the constant of evaluation,  $k$ , knowing  $m$  and  $t$  by substituting.**

Using the word translation,  $m = kt$   
 Substituting,  $4 = k(8)$   
 Solving,  $\frac{1}{2} = k$  the constant of variation

Using  $k = \frac{1}{2}$  in the equation,  $m = \frac{1}{2}t$   
 Substitute  $t = 30$   $m = \frac{1}{2}(30)$   
 Solving,  $m = 15$

To solve using algebra, we first must find the value of  $k$ , the constant of variation.

**Example 2** Your car can travel 350 miles on 20 gallons of gas. Using  $y = kx$ , the miles ( $y$ ) your travel depends on the amount of gas ( $x$ ) you have in the car. So, how many miles per gallon does your car get – that would be the **constant of variation**.

We could find that 2 ways  $350/20$ , setting up the ratio as we did before

$\frac{\text{miles}}{\text{gallon}}, \frac{y}{x} = k$  or we could substitute those values into  $y = kx$ .

$$\begin{aligned} y &= kx \\ 350 &= k(20) \\ 350/20 &= k \\ 17.5 &= k \end{aligned}$$

Either way, we would get 17.5.

$k$ , the constant of variation is 17.5. That means you get 17.5 miles per gallon.

Now, to find out how much gas you would need to travel 3000 miles, you substitute the values of  $k$  and  $y$  into

$$\begin{aligned}y &= kx \\3000 &= 17.5x \\171.42 &= x, \text{ we'd need 171.42 gallons.}\end{aligned}$$

OR

We could have solved that using a proportion with ratio  $\frac{\text{miles}}{\text{gallon}}$ ,

$$\frac{350}{20} = \frac{n}{1}$$

So, you choose to find a common denominator when adding fractions, you have to choose which method makes the problem easier.

When you read  $y = kx$  mathematically, we say  $y$  **varies directly as  $x$** .

Also, while we call  $k$  the constant of variation,  $k$  is also turns out to be the **slope** of a line, the rate of change – normally written and recognized in algebra by using  $m$ .

The advantage to solving these algebraically is with the dependent variable written in the numerator, we are able to use  $y = kx$  to get more information easily. For instance, in the equation  $y = kx$ , that's the same as  $y = mx$  in algebra which is the graph of a line that passes through the origin. You gotta love these linkages. And that can be written as a function.

Ok, back to solving these direct variation (proportion) problems. Below is the algorithm for how we solved that problem algebraically.

*Algorithm*

### **Solving Direct Variation Problems & Finding the Constant of Variation**

1. Write the equation  $y = kx$
2. Find the value of  $k$  by substituting the given values for  $x$  and  $y$
3. Rewrite the equation  $y = kx$
4. Substitute the value of  $k$  and the other value given to find the missing variable.

**Example 3** The number of students passing the driving test varies directly with the number of students who enrolled in driving school. Of the 400 students enrolled, 80 passed. If 500 students passed the test, how many would have enrolled in the driving school?

The number of students passing ( $y$ ) depends on the number of students enrolled ( $x$ ).

1. Write  $y = kx$ ,
2.  $80 = k(400)$ ; therefore  $\frac{80}{400} = \frac{1}{5} = k$  (the constant of variation)
3.  $y = kx$
4.  $y = \frac{1}{5}(500)$ , therefore  $y = 100$

In these problems, the constant of variation provides us the relationship that exists between the ratios. In the problem above, rather than using algebra, I could have just as easily written the ratio;  $\frac{\text{passing}}{\text{enrolled}} = \frac{80}{400} = \frac{1}{5}$ .

Now, someone might ask, if we can solve these problems by proportion, as we did before, why are we solving them algebraically? There are a few reasons, we can now graph the relationships.  $y = kx$  can be written as  $y = mx$ , an equation of a line that passes through the origin with slope  $k$  or later as a function;  $f(x) = mx$

We can also see a set of ordered pairs that belong to that function, if the vales of  $\frac{y}{x}$  are a constant, then we know that the relationship being described is not only linear, but is proportional.

As we have previously noted in our studies, math content is linked. The greatest differences occur in *language, notation and pattern development* because the math is being used in different contexts.

We have been working with proportions, we set the ratio on the left equal to the ratio on the right and solved.

Then we saw “some” of these relationships had a cause effect relationship. So instead of writing any ratio, we write them so the dependent variable is on top. as  $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ . We write the same problem as  $\frac{y}{x} = k$ , then convert that to  $y = kx$  and solve algebraically.

Now, look at the original proportion;  $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ . **Remember, we said we could cross multiply.** If I crossed multiplied, we get this relationship:

$$x_1y_2 = x_2y_1$$

That is just another way to describe a direct variation that some authors use. My point is if you know how to solve proportions, you have choices. My point is there are different ways to look at

these and we don't need to get confused by them – always go back to the definition, then you can see where these different approaches come from.

And the best reason to use algebra to solve proportions is when problems get longer – not harder – algebra simplifies the problems. Up to this point, we have used simple proportions. But, in science, we see **joint** or **combined** relationships that are dependent on more than one variable. So, we could have  $y$  varies directly with  $a$ ,  $b$ , and  $c$ . Algebraically we would write  $y = kabc$ .

As always, I could make them even longer by saying  $y$  varies directly with  $a$ ,  $b$ , and  $c$  and inversely with  $d^2$ . That would translate to  $y = \frac{kabc}{d^2}$ .

As an example, the electrical resistance  $R$  of a wire varies directly as the length  $l$  of the wire and inversely as the square of the distance  $d$ . An equation expressing that is  $R = \frac{kl}{d^2}$

The next section deals with indirect variations, that will explain why the  $d^2$  was in the denominator.

## Sec. 9 Indirect Variation; $xy = k$

An indirect variation is a relationship between 2 variables in which the product is a constant. In an indirect variation, when the value of one of the variables increases, the other decreases in proportion so that the product does not change. We say  $y$  varies indirectly as  $x$ ;  $xy = k$ . Remember, in a direct variation, when one variable increases, the other variable also increases.

Let's examine a simple illustration like going on a trip. **Rate varies indirectly with time**. In other words, the faster you go, the less time it takes to get somewhere.

If you want to go 40 miles/hour for 3 hours, the constant of proportionality,  $k$  is  $40(3) = 120$ . If the rate increases to 60 miles/hour multiplied by the time should also be 120. So, the time would be 2 hours.

Setting the problem up like we did for a direct variation, we now have

$$\begin{aligned}xy &= k \\40(3) &= k, \text{ so } k = 120 \\60(x) &= 120 \\ \text{so } x &= 2\end{aligned}$$

From the definition,  $xy = k$ , we can solve for  $y$ ,  $y = \frac{k}{x}$  and graph it. That turns out to be a hyperbola and we can very quickly see when  $x$  increases the value of  $y$  decreases.

Definitions

1.\*\*\*. Ratio

2.\*\*\* Proportion

3.\*\*\* Unit ratio

4.\*\*\* Write the 5 Properties of Proportion

5.\*\*\* Write 2 methods of solving proportional problems

6.\*\* Express the ratio 5 to 7 as a fraction.

7.\*\* Express each ratio in simplest form.

a. 6 to 8

b. 24 to 32

8.\*\* Bob's club has 50 members. Ten are male and the remainders are females. Find:

a. The ratio of males to females.

b. The ratio of males to the total number of members in the club.

c. The ratio of the total number of people in the club to the number of females.

9.\*\* Solve for x.  $\frac{3}{4} = \frac{21}{x}$

10.\*\* Solve for x.  $\frac{15}{24} = \frac{75}{x}$

11.\*\* Solve for x.  $\frac{2}{3} = \frac{15}{x}$

12.\*\* If there are 12 inches in one foot, how many inches are there in five feet?

13.\*\* The legend on a map indicates one inch equals 35 miles. If the distance traveled on the map is four inches, how many miles would have to be traveled?

- 14.\*\* If a basketball player makes 25 out of 45 baskets in his first game, how many baskets would you expect him to make in the second game if he attempted 18 shots?
- 15.\*\* On a trip across country Bob used 25 gallons of gas to travel 300 miles. At this rate, how much gas will he use to travel 3000 miles?
- 16.\*\* If three pieces of candy cost 10¢, how much will 16 pieces cost?
- 17.\*\* If the ratio of boys to girls in art class is 5 to 7, how many girls are there in class if there are 60 students enrolled?
- 18.\* SBAC The ratio of length to width of a rectangle is 8 : 3. Find the dimensions of the rectangle if the perimeter is 44 inches.
- 19.\* SBAC Bob saves \$16 in 16 days. His sister Sarah saves \$49 in 7 weeks, are these rates equivalent. Explain your answer.
- 20.\* SBAC A candle is 30 inches long. After burning 12 minutes, the candle is 25 inches long. How long would it take the whole candle to burn?
- 21.\*\*\* Provide parent/guardian contact information; phone, cell, email, etc. (CHP)

## Proofs – 5 Properties of Proportion

If  $\frac{a}{b} = \frac{c}{d}$ , then  $ad = bc$

(Cross Multiplying)

1.  $\frac{a}{b} = \frac{c}{d}$  Given
2.  $\frac{a}{b} bd = \frac{c}{d} bd$  Mult. Prop. of Equality
3.  $ad = bc$  Mult. Inverse & Identity

If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{a}{c} = \frac{b}{d}$

(Sideways)

1.  $\frac{a}{b} = \frac{c}{d}$  Given
2.  $ad = bc$  Cross Multiply
3.  $\frac{ad}{cd} = \frac{bc}{cd}$  Div. Prop. of Equality
4.  $\frac{a}{c} = \frac{b}{d}$  Mult. Inverse & Identity

If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{b}{a} = \frac{d}{c}$

(Upside down)

1.  $\frac{a}{b} = \frac{c}{d}$  Given
2.  $ad = bc$  Cross Multiply
3.  $\frac{ad}{ac} = \frac{bc}{ac}$  Div. Prop. of Equality
4.  $\frac{d}{c} = \frac{b}{a}$  Mult. Inverse & Identity
5.  $\frac{b}{a} = \frac{d}{c}$  Symmetric Property

If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{a+b}{b} = \frac{c+d}{d}$

(Add N and D/D)

1.  $\frac{a}{b} = \frac{c}{d}$

Given

2.  $\frac{a}{b} + \frac{b}{b} = \frac{c}{d} + \frac{d}{d}$

Add Prop. of Equality (+1)

3.  $\frac{a+b}{b} = \frac{c+d}{d}$

Add Fractions

If  $\frac{a}{b} = \frac{c}{d}$ , then  $\frac{a}{b} = \frac{a+c}{b+d}$

(Add N, Add D)

1.  $\frac{a}{b} = \frac{c}{d}$

Given

2.  $ad = bc$

Cross mult.

3.  $\frac{a}{b} = \frac{a}{b} * \frac{b+d}{b+d}$

Mult. by 1

4.  $\frac{a(b+d)}{b(b+d)}$

Simplify Fractions

5.  $\frac{ab+ad}{b(b+d)}$

Distributive Prop.

6.  $\frac{ab+bc}{b(b+d)}$

Substitution; #2

7.  $\frac{b(a+c)}{b(b+d)}$

Factor – Distrib. Prop.

8.  $\frac{a+c}{b+d}$

Mult. Inverse & Identify

9.  $\frac{a}{b} = \frac{a+c}{b+d}$

Transitive Property