Chapter 13 Rational Expressions, Equations & Inequalities

Sec. 1 Simplifying Rational Expressions

To be successful in this chapter, you have to be able to factor polynomials. If you can recognize the different patterns and factor, then this chapter will be easy. If you cannot **immediately** recognize when to use the Distributive Property, Difference of 2 Squares, Addition Method, Trial & Error (ac Method), Grouping or the Sum and Difference of 2 Cubes, you need to review that chapter on factoring.

We simplify rational expressions the same way we simplified fractions. When we first began to simplify fractions, we factored the numerator and denominator, then divided out common factors.

Example 1 Simplify $\frac{9}{12}$

Factoring the numerator and denominator, we have. $\frac{3x3}{3x4}$

If we divide out the 3's , we have $\frac{3}{4}$

Example 2 Simplify $\frac{28}{36}$

Factoring the numerator and denominator, we have $\frac{4x7}{4x9}$

Dividing out the 4's, we have 7/9

Simplifying algebraic fractions is done the very same way. We factor the numerator and denominator, then divide out common factors.

Example 3 Simplify $\frac{x^2 - 4}{x^2 + 5x + 6}$

The numerator is the Difference of 2 Squares, the denominator is factored using the Addition Method

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Factoring, we have $\frac{(x+2)(x-2)}{(x+2)(x+3)}$ The factor (x + 2)

divides out leaving us with $\frac{x-2}{x+3}$

I cannot make these more difficult if you know how to factor, we can only make the problem longer – not harder.

Example 4 Reduce $\frac{x^3-9x}{x^3+x^2-6x}$

We can use the Distributive Property in both numerator and denominator. We then see we can further the numerator further using the Difference of 2 Squares and the denominator by the Addition Method.

Factoring, we have

$$\frac{x(x^2-9)}{x(x^2+x-6)} = \frac{x(x+3)(x-3)}{x(x+3)(x-2)}$$

The x's cancel as do the (x + 3)'s, we are left with

$$= \frac{x-3}{x-2}$$

What could be easier? You can clearly see the importance in your ability to factor polynomials quickly. If you are still having some difficulty with these, I'm sure the issue is factoring. Determine what methods of factoring are causing you the difficulty, and address them so you know which method to use with a given polynomial. Your future success depends on your ability to factor.

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Sec 2. Multiplying & Dividing Fractions

Now, if you can factor and understand how to simplify rational expressions, then multiplying and dividing polynomials is just an extension of reducing.

To multiply polynomials, we factor the polynomials and divide out common factors. It's that simple. Can I make it more difficult? I wish. All you need to know is how to factor and cancel – just like reducing fractions. The problems are not more difficult – just longer.

Example 1 $\dfrac{x^2-3x-18}{x+7} \cdot \dfrac{x-3}{x-6}$

Factoring $\dfrac{(x-6)(x+3)}{x+7} \cdot \dfrac{x-3}{x-6}$

The only factor that divides (cancels) is the (x–6)

$$\frac{(x+3)(x-3)}{x+7}$$

Example 2 $\dfrac{x^2+4x+3}{x^2-9} \cdot \dfrac{x^2-6x+9}{x+3} \cdot \dfrac{x+3}{x-3}$

Factoring $\dfrac{(x+3)(x+1)}{(x+3)(x-3)} \cdot \dfrac{(x-3)(x-3)}{x+3} \cdot \dfrac{x+3}{x-3}$

Divide out common factors

$\dfrac{(x+3)(x+1)}{(x+3)(x-3)} \cdot \dfrac{(x-3)(x-3)}{x+3} \cdot \dfrac{x+3}{x-3}$

$= x + 1$

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Remember how you divided fractions? You inverted the divisor and multiplied. Well, to divide polynomials, we do the exact same thing, invert, factor and cancel out common factors.
Piece of cake, right?

Example 3 $\dfrac{x^2-81}{x^2-16} \div \dfrac{x^2+5x-36}{x^2-x-20}$

I am going to do two steps at once. I'm going to factor each of these polynomials and invert the divisor in the same step.

$$\frac{(x+9)(x-9)}{(x+4)(x-4)} \cdot \frac{(x-5)(x+4)}{(x+9)(x-4)}$$

Dividing out, we have $\dfrac{(x-9)(x+5)}{(x-4)(x-4)}$

Sec. 3 Adding Subtracting Rational Expressions

Again, the procedure used to add and subtract rational expressions in algebra are the same used in adding and subtracting fractions in 5th and 6th grades. That is;

1. Find a common denominator
2. Make equivalent fractions
3. Add/subtract the numerators
4. Bring down the denominator
5. Reduce

The biggest difference in adding algebraic expressions in algebra and adding fractions is, like most of algebra, we do the problems horizontally.

While with fractions, we used our knowledge of arithmetic facts to find the CD. In algebra, we need to remember a Common denominator, CD, is a denominator that contains all the factors of each denominator – a common multiple – the least common multiple – the LCM.

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To find a common denominator, several methods are available to students. The method chosen should be based on the given denominators. If the denominators are relatively prime (no common factors), then most will simply multiply the denominators together to find the lowest common denominator (LCD).

Example 1 Find the common denominator for 2/5 and 1/4

These denominators are relatively prime, so the LCD is

5 x 4 = 20.

In algebra, if the denominators are (x + 1) and (x – 5), then the LCD is found the same way, by multiplying the denominators we have (x + 1)(x – 5) as the LCD.

When the denominators have common factors, not relatively prime, we used the Reducing Method to find the lowest common denominator.

In the Reducing Method, we placed the denominators over each other and reduced, then when you cross multiply, that is the lowest common denominator.

Example 2 Find the common denominator for 18 and 24.

Placing 18 over 24 and reducing, $\frac{18}{24} = \frac{3}{4}$

By cross multiplying, we find the common denominator of 72.

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We will do exactly the same thing to find common denominators in algebra that have a common factor. That is, we will use the Reducing Method. We will place the denominators over each other, reduce, then cross multiply.

Example 3 Find the common denominator $(x^2 - 1)$ and $(x^2 + 2x + 1)$

To find the LCD, write the denominators over each other,

$$\frac{(x+1)(x-1)}{(x+1)(x+1)} = \frac{x-1}{x+1}$$

Now, cross multiply to find the LCD. The product of the means is equal to the product of the extremes.

$(x + 1)(x + 1)(x - 1) = (x + 1)(x - 1)(x + 1).$

The LCD is $(x + 1)(x + 1)(x - 1)$.

Now, to add the rational expressions where the denominators have common factors. To help insure we see don't overload, we will use the denominators from Example 3

Example 4 $\dfrac{5}{x^2-1} + \dfrac{7}{x^2+2x+1}$

$\dfrac{5}{(x+1)(x-1)} + \dfrac{7}{(x+1)(x+1)}$ the LCD is $(x+1)(x+1)(x-1)$

In the fraction on the left, the $(x+1)(x-1)$ divides out with the LCD, so it goes n $(x+1)$ times.

In the fraction on the right, the $(x+1)(x+1)$ divides out with the LCD $(x-1)$ times, resulting in the following equivalent fractions.

$$\frac{5(x+1)+7(x-1)}{(x+1)(x+1)(x-1)} = \frac{12x-2}{(x+1)(x+1)(x-1)}$$

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The point I want to make is the last problem got a little longer, it did not get more difficult. We found the LCD by the reducing method or logic. We made equivalent fractions, then added the numerators.

Now, if we looked at enough problems, we would be able to find patterns that would allow us to add/subtract fractions in our heads. Look at the following addition problems and their respective answers, see if you can identify a pattern that would allow you to add fractions whose denominators are relatively prime and have numerators of one.

$$\frac{1}{5}+\frac{1}{2}=\frac{7}{10} \qquad \frac{1}{3}+\frac{1}{5}=\frac{8}{15} \qquad \frac{1}{2}+\frac{1}{3}=\frac{5}{6}$$

Look at the numbers in the problems, look at the answers. See anything interesting? Use that pattern to add this mentally.

$$\frac{1}{3}+\frac{1}{4}$$

All those fractions above had a numerator of 1, and the denominators were relatively prime (no common factors) What happens if the numerators are not 1?

In the following examples, all the numerators will not be one.

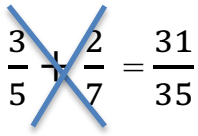
$$\frac{1}{5}+\frac{2}{7}=\frac{17}{35} \qquad \frac{3}{10}+\frac{1}{3}=\frac{19}{30} \qquad \frac{3}{4}+\frac{1}{5}=\frac{19}{20}$$

From these examples, it's easy to see where the common denominators come from, can you manipulate the numbers in the problems that would suggest where the numerators are coming from? It doesn't just jump out at you, you have to play with the numbers.

Well, if you played long enough you would see you get the common denominator by just multiplying the denominators. The numerator is obtained by multiplying the addends diagonally, then adding those products.

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Example 5 $\frac{3}{5}+\frac{2}{7}$



The CD is 7x5 = 35, the numerator is determined by multiplying diagonally and adding those products.

3x7 + 5x2 = 31

Generalizing that, we have

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

That pattern or formula would allow me to add or subtract rational expressions mentally. Also notice this pattern were for the original fractions with numerators of one that we were adding mentally.

Example 6 $\frac{2}{x}+\frac{3}{y}$

$$\frac{2}{x}+\frac{3}{y}=\frac{2y+3x}{xy}$$

You add and subtract rational expressions the same way you added and subtracted fractions. That is; 1. find the least common denominator, 2. make equivalent fractions, 3. add or subtract the numerators, 4. bring down the denominator and 5. reduce.

That's one of the real treats about math, once you've learned something, the concepts and strategies remain the same. Let's do this next example using that algorithm.

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Example 7 $\frac{x-2}{3}+\frac{x+5}{2}$

First, I find the least common denominator. In this case, its 6. I then make equivalent fractions.

$$\frac{2(x-2)}{6}+\frac{3(x+5)}{6}$$

I add my numerators; $\frac{2(x-2)+3(x+5)}{6}$

$=\frac{2x-4+3x+15}{6}$ Using the D-Prop

$=\frac{5x+11}{6}$ Combining terms

Now, rather than doing the steps

Typically, instead of doing those in separate steps, we can now do the problem using the pattern we saw before – almost by inspection

Example 8 $\frac{x-2}{3}+\frac{x+5}{2}=\frac{2(x-2)+3(x+5)}{6}$

$=\frac{2x-4+3x+15}{6}$ D-Prop

$=\frac{5x+11}{6}$ Combine terms

The least common denominator (LCD) is the smallest denominator that all other denominators will divide into. Another way of saying that is the LCD will be must be made up of all the factors of the other denominators.

Let's take a look at another example.

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Example 9 $\dfrac{5}{x+2}+\dfrac{3}{x+1}$

The LCD must have all the factors of those two denominators, therefore the LCD must have (x + 2)(x + 1) as factors. We got that by multiplying the two denominators together. Using the pattern of multiplying diagonally to find the numerator, we have

$$= \frac{5(x+1)+3(x+2)}{(x+2)(x+1)}$$

$$= \frac{5x+5+3x+6}{(x+2)(x+1)}$$

$$= \frac{8x+11}{(x+2)(x+1)}$$

Being able to operate with rational expressions allows us t be able to solve fractional equations.

Sec. 4 Fractional Equations

Your ability to solve fractional equations depends greatly on whether you can add or subtract rational expressions.

You might remember, to add or subtract rational expression; you find a common denominator, make equivalent fractions, add or subtract the numerators, bring down the denominator and reduce.

Have you ever noticed most people prefer not to work with fractions? We are going to adopt that same preference. Our strategy when we see a fractional equation is to get an equivalent equation that does not contain fractions.

How do we do that? Well, just like when we added rational expressions we found a common denominator. But rather than making equivalent fractions, we are going to multiply both sides of the equation by the least common

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denominator. That gets rid of the denominators, which means we don't have fractions any more.

Remember, the least common denominator must have all the factors of the other denominators. So again, you get to factor some polynomial expressions.

Example 1 Find the solution set $\frac{x}{x-6} = 3$

Since there is only one fraction, the LCD must be x–6. Now, multiplying both sides b the LCD, we have

$$(x-6)\frac{x}{x-6} = (x-6)3$$

$$x = 3x - 18$$
$$-2x = -18$$
$$x = 9$$

Can I make those problems more difficult? Absolutely not. All I can do is make them longer. The strategy is very straight forward, multiply both sides of the equation by the common denominator to get rid of the fractions, and then solve the resulting equation.

Example 2 Find the solution set; $\frac{x}{x-2} + \frac{2}{x+2} = \frac{x^2+4}{x^2-4}$

Factor $x^2 - 4$. The LCD is $(x + 2)(x – 2)$, so let's multiply both sides by the LCD to get rid of the fractions.

$$(x + 2)(x – 2)\left[\frac{x}{x-2} + \frac{2}{x+2}\right] = (x + 2)(x – 2)\frac{x^2+4}{x^2-4}$$

Now, using the Distributive Property and dividing out,

we have:

$x(x + 2) + 2(x – 2) = x^2 + 4$ Solving

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$$x^2 + 2x + 2x - 4 = x^2 + 4$$
$$x^2 + 4x - 4 = x^2 + 4$$
$$4x - 4 = 4$$
$$4x = 8$$
$$x = 2 \ \#$$

$\therefore \rightarrow \varnothing$ Why?

Piece of cake, you multiply by the LCD getting rid of the fractions and solve the resulting equation using previously learned strategies. But I have a minor glitch. The answer, the solution, is the value of the variable that makes the open sentence (original equation) true. If we were to substitute 2 back into the original equation, we would get a zero in the denominator. In math, we can't let this happen, that's undefined.

That means 2 does not work. It is not a value of the variable that makes the equation true. So we end up with a problem that does not have an answer. The way we say that mathematically, we say the answer is the empty set, the null set, mathematically expressed as $\emptyset$.

That is the first time something like that has happened to us. The question is, when does such a thing happen. To answer that we have to know what caused the problem. The 2 made the denominator zero. The problem occurs when you have fractions and the solution you find makes a denominator zero.

From this point on, anytime you solve a fractional equation, you are going to have to check your answer(s) to make sure they don't make denominators zero. That's real important!

Let's do another problem.

Example 3: Find the solution set; $\frac{2x+1}{x-1} - \frac{3x}{x+2} = \frac{5x-2}{x^2+x-2}$

Factor the denominators; $x^2 + x - 2 = (x + 2)(x - 1)$. The LCD is $(x + 2)(x - 1)$. Now multiply both sides by the LCD.

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$$(x-1)(x+2)\left(\frac{2x+1}{x-1}-\frac{3x}{x+2}\right)=(x-1)(x+2)\left(\frac{5x-2}{x^2+x-2}\right)$$

$$(2x+1)(x+2)-3x(x-1)=5x-2$$
$$2x^2+5x-2-3x^2+3x=5x-2$$
$$0=x^2-3x-4$$
$$0=(x-4)(x+1)$$

Therefore, the two solutions are $x = 4$ or $x = -1$.

Don't forget to check your answer!

In summary, this chapter depends upon your ability to factor polynomials. If you can factor, then simplifying, multiplying and dividing rational expressions is simple – factor and divide out common factors.

Adding rational expressions is done just like adding fractions, finding the common denominator, making equivalent fractions, then adding the numerators and simplifying.

Solving fractional equations is actually, in my opinion, easier. Again, using the strategy of getting rid of what we don't like. We get rid of the fractions by multiplying both sides by the common denominator and solve the resulting equations.

Some students experience difficulty differentiating between expressions and equations. Simply stated, fractional equations have an "=" sign. Once you multiply a rational equation by the LCD, the next step does not have fractions.

Rational expressions do not have "=" signs. Generally, the answer will also be written as a fraction.

This is an expression – there is no "=" sign

$$\frac{2x+1}{x-1}-\frac{3x}{x+2}$$

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This is an equation, it has an "=" sign.

$$\frac{x}{x-2} + \frac{2}{x+2} = \frac{x^2+4}{x^2-4}$$

Sec. 5 Fractional Inequalities

Fractional inequalities are solved just a little differently than fractional equations. With fractional equations, we found the common denominator and multiplied both sides of the equation by it as we were just doing.

With inequalities, we have to be careful because if we multiply or divide by a negative number, we have to reverse the order of the inequality. The issue is we don't know if the factors are positive or negative – that means we don't know whether or not to reverse the order of the inequality.

The solution to that issue is to not put ourselves in situation. To do that, we will rewrite our inequality with everything on one side of the inequality and zero on the side.

Once we do that, we simplify the fractions and factor the numerator and denominator. If we set the factors of the numerator equal to zero, we find the zeros, the roots, where the graph crosses the x-axis. Since the denominator cannot be equal to zero, we set those factors equal to zero as well, but those indicate vales of x that cannot be used as a solution because they make the denominator zero.

The zeros in the numerator and the zeros in the denominator are called critical points. Those critical points are used to divide the number lines into intervals.

Having said all that, let's come up with a procedure to solve rational inequalities.

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Procedure

1. Write the inequality as a single quotient on the left and zero on the right.
2. Determine the critical points, where the rational expression will be zero or undefined.
3. Use the critical points to divide the number line into intervals.
4. Test a value in each interval.(that's a lot of arithmetic so pick nice numbers)
5. Determine the intervals where the inequality is correct.

The answer is normally written in interval notation.

Let's rewrite the fractional equalities we have used in the previous section and make them inequalities for our examples

Example 1 Find the solution set: $\frac{4x}{x-6} - 1 \leq 0$

Step 1 $\frac{4x}{x-6} - \frac{x-6}{x-6} \leq 0$

$\frac{3x+6}{x-6} \leq 0$

Step 2 **Critical points: – 2 and 6, x ≠ 6**

Step 3



Step 4 **Section A, pick –10, not true**
Section B, pick 0, true
Section C, pick 10, not true

Step 5 **Solution is numbers in the interval B, (–2, 6)**

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Example 2 Find the solution set $\frac{x}{x-6} > 3$

Step 1 $\frac{x}{x-6} - 3 > 0$

$$\frac{x}{x-6} - \frac{3(x-6)}{x-6} > 0$$

$$\frac{x-3(x-6)}{x-6} > 0$$

$$\frac{-2x+18}{x-6} > 0$$

Step 2 critical points are 6 and 9 : $-2x + 18 = 0$, $\mathbf{x = 9}$

$x - 6 \neq 0$, $\therefore$ $\mathbf{x \neq 6}$

Step 3



Step 4 **Section A, pick 0, not true**
Section B, pick 8, true
Section C, pick 10, not true

Step 5 **Solution, numbers in the interval (–6, 9)**

Now, we can make these longer, not harder. So let's look at a dew more examples.

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Example 3 $\dfrac{2x+1}{x-1} - \dfrac{3x}{x+2} \leq \dfrac{5x-2}{x^2+x-2}$

Step 1 $\dfrac{2x+1}{x-1} - \dfrac{3x}{x+2} - \dfrac{5x-2}{x^2+x-2} \leq 0$

$$\frac{(2x+1)(x+2)-3x(x-1)-(5x-2)}{(x-1)(x+2)} \leq 0$$

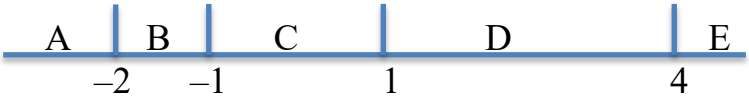
$$\frac{2x^2+5x+2-3x^2+3x-5x+2}{(x-1)(x+2)} \leq 0$$

$$\frac{-x^2+3x+4}{(x-1)(x+2)} \leq 0$$

$$\frac{-1(x-4)(x+1)}{(x-1)(x+2)} \leq 0$$

Step 2 critical points: x = 4, x = –1 x ≠ 1 and x ≠ –2

Step 3



Step 4
Section A, pick -10, true
Section B, pick – 1 ½, not true
Section C, pick 0, true
Section D, pick 3, not true
Section E, pick 10, true

Step 5 **The solution $(-\infty, -2)\ U\ [-1, 1)\ U\ [4, \infty)$**

Remember, the square bracket "[" is used when the number is included, the round bracket "(" is used when the number is not included.

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Example 4 $\frac{x}{x-2} + \frac{2}{x+2} \geq \frac{x^2+4}{x^2-4}$

Step 1 $\frac{x}{x-2} + \frac{2}{x+2} - \frac{x^2+4}{x^2-4} \geq 0$

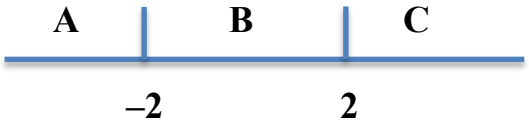
$\frac{x(x+2)+2(x-2)-(x^2+4)}{(x-2)(x+2)} \geq 0$

$\frac{x^2+2x+2x-4-x^2-4}{(x-2)(x+2)}$ **0**

$\frac{4x-8}{(x+2)(x-2)} \geq 0$

Step 2 critical points: $x \neq 2$ $x \neq -2$

Step 3



Step 4 **Section A, pick –5, not true**
Section B, pick 0, true
Section C, pick 5, true

Step 5 **(-2, 2) U (2, ∞)**

x = ±2 cannot be part of the solution, so we have to have open brackets.

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Solve the following rational inequalities

1. $\frac{4x}{x-6} < 1$

2. $\frac{3x}{x-3} < 1$

3. $\frac{3x}{x-4} - 2 < 0$

4. $\frac{5}{x^2-2x-15} > 0$

5. $\frac{1}{x^2+2x-8} > 0$

6. $\frac{1}{3} - \frac{2}{x^2} < \frac{5}{3x}$

7. $\frac{4}{x^2} + \frac{1}{2} < \frac{3}{x}$

8. $\frac{1}{3} + \frac{6}{x^2} < \frac{3}{x}$

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